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Cayley graphs and G-graphs: Some applications

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ABSTRACT

This paper introduces some relations about Cayley graphs and G-graphs. We present a sufficient condition to recognize when a G-graph is a Cayley graph. The relation between G-graphs and Cayley graphs allows us to consider some applications to the hamiltonicity of Cayley graphs. In the last section we illustrate our results by showing that some new classes of Cayley graphs are hamiltonian.

1. Introduction

G-graphs have been introduced to study the graph isomorphism problem (Bretto and Faisant, 2005). These ones are constructed from a group and like Cayley graphs, have nice and highly regular properties; they may or may not be regular (Bretto et al., 2007, 2008). Most well-known graphs are in fact G-graphs: Hamming graphs, meshes of d-ary trees $MT(d, 1)$, and some star graphs, to name a few. Moreover the algorithm to construct G-graphs is simple. A popular representation of groups by graphs is the Cayley graph representation. These graphs were first used by A. Cayley in 1878 (Cayley, 1878, 1889) to construct pictorial representations of finite groups. To a group G and a set $S \subseteq G$ of generators a digraph called *Cayley graph* is associated. The set of vertices of this graph is the set of elements of G and two vertices x, y are adjacent if and only if there exists $s \in S$ such that $y = sx$. If $S = S^{-1}$ the graph is undirected. Cayley graphs are important in many areas of science, (for a survey, see Cooperman et al. (1991)). G-graphs can be used in many areas of science as well: they are a good tool in the construction of symmetric and semi-symmetric graphs, but also, they give a better upper bound, $(2p^2)$ for the $(p, 6)$-cage problem than the Sauer bound which is equal to $4(p-1)^3$.

This paper introduces a new application from G-graphs to the hamiltonicity of Cayley graphs.

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2. Basic definitions

In the sequel of this paper, groups will be finite. We shall denote the unit element by e. Let G be a group, and let $S = \{s_1, s_2, \ldots, s_k\}$ be a non-empty subset of G. The set S is a *set of generators* of G if any element $\theta \in G$ can be written as a product $\theta = s_{i_1}s_{i_2}s_{i_3}\ldots s_{i_t}$ with $i_1, i_2, \ldots i_t \in \{1, 2, \ldots, k\}$. We say that G is *generated* by $S = \{s_1, s_2, \ldots, s_k\}$ and we write $G = \langle s_1, s_2, \ldots, s_k\rangle$.

A group G acts in the left way on a space Ω when the following operation $(a, x) \longrightarrow a.x$ from $G \times \Omega$ to Ω verifies:

- $e.x = x$.
- $a.(b.x) = (a.b).x., a, b \in G$ and $x \in \Omega$.

Let H be a subgroup of G, we denote Hx instead of $H\{x\}$. The set Hx is called *right coset* of H in G. A subset T_H of G is said to be a *right transversal* for H if $\{Hx, x \in T_H\}$, is precisely the set of all cosets of H in G.

An S-group is a couple (G, S) where G is a finite group and S is a subset of G. A S-group morphism between (G_1, S_1) and (G_2, S_2) is a morphism f from G_1 to G_2 such that $f(S_1) \subseteq S_2$.

Let (G, S) be a group, the automorphism set f of G such that $f(S) = S$ we will be denoted by $Aut_S(G)$. It is subgroup of $Aut(G)$. A group G acts *regularly* on a finite set if for any couple (x, y) , $x, y \in X$ there is a unique $f \in Aut(G)$ such that $f(x) = y$. In this case we have $|G| = |X|$.

2.1. Graph definitions

We define a graph $\Gamma = (V; E; \epsilon)$ as follows:

- V is the set of vertices and E is the set of edges.
- ϵ is a map from E to $P_2(V)$, where $P_2(V)$ is the set of subsets of V having 1 or 2 elements.

In this paper graphs are finite, i.e., sets V and E have finite cardinalities. For each edge a, we denote $\epsilon(a) = [x; y]$ if $\epsilon(a) = \{x, y\}$ with $x \neq y$ or $\epsilon(a) = \{x\}$ if $x = y$. In this case a is a *loop*. The elements x, y are called extremities of a, and a is incident to x and y. The set $\{a \in E, \epsilon(a) = [x; y]\}$ is called *multi-edge* or *p-edge*, where p is the cardinality of the set. We define the *degree* of x by $d(x) = card(\{a \in E, x \in \epsilon(a)\})$.

Given a graph $\Gamma = (V; E; \epsilon)$, a *chain* is a non-empty graph $P = (V, E, \epsilon)$ with $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{a_1, a_2, \ldots, a_{k-1}a_k\}$, where $x_i, x_{i+1}, (i \mod k)$ are extremities of a_i. The elements of E must be distinct. The cardinality of E is the *length* of this chain. A graph is *connected* if, for all $x, y \in V$, there exists a chain from x to y.

$\Gamma' = (V'; E'; \epsilon')$ is a *subgraph* of Γ if it is a graph satisfying $V' \subseteq V, E' \subseteq E$ and ϵ' is the restriction from ϵ to E'. If $V' = V$ then Γ' is a *spanning subgraph*.

An *induced subgraph* generated by A, $\Gamma(A) = (A; U; \epsilon)$, with $A \subseteq V$ and $U \subseteq E$ is a subgraph such as $U = \{a \in E, \epsilon(a) = [x; y], x, y \in A\}$.

An induced subgraph such that any pair of vertices are adjacent is called a *clique*. Let $\Gamma = (V; E; \epsilon)$ be a graph, a *component* of Γ is a maximal connected induced subgraph.

Let $\Gamma_1 = (V_1; E_1; \epsilon_1)$ and $\Gamma_2 = (V_2; E_2; \epsilon_2)$ be two graphs, a *morphism* from $\Gamma_1 = (V_1; E_1; \epsilon_1)$ to $\Gamma_2 = (V_2; E_2; \epsilon_2)$ is a couple $(f, f^\#)$ where $f : V_1 \longrightarrow V_2$ is a map and $f^\# : E_1 \longrightarrow E_2$ is a map such that:

$$\text{if } \epsilon_1(a) = [x; y] \text{ then } \epsilon_2(f^\#(a)) = [f(x); f(y)].$$

So (id_V, id_E) is a morphism from $G = (V; E; \epsilon)$ to G.

The *product* of two morphisms $(f, f^\#)$ and $(g, g^\#)$ is defined by: $(f, f^\#) \circ (g, g^\#) := (f \circ g, f^\# \circ g^\#)$. $(f, f^\#)$ is an *isomorphism* if there exists a morphism $(g, g^\#)$ from $\Gamma_2 = (V_2; E_2; \epsilon_2)$ to $\Gamma_1 = (V_1; E_1; \epsilon_1)$ such that $(g, g^\#) \circ (f, f^\#) = (id_{V_1}, id^\#_{E_1})$ and $(f, f^\#) \circ (g, g^\#) = (id_{V_2}, id^\#_{E_2})$. In this case we will denote $(g, g^\#) = (f, f^\#)^{-1}$. So $(f, f^\#)$ is an isomorphism if and only if f is a bijection and $f^\#$ is a bijection. If there exists an isomorphism between Γ_1 and Γ_2 we will denote $\Gamma_1 \simeq \Gamma_2$ and we will say that Γ_1 is *isomorphic* to Γ_2.

3. Graph-group construction

Let (G, S) be a group with a set of generators $S = \{s_1, \ s_2, \ s_3 \dots \ s_k\}$, $k \geq 1$. For any $s \in S$, we consider the left action of the subgroup $H = \langle s \rangle$ on G. So we have a partition $G = \bigsqcup_{x \in T_s} \langle s \rangle x$, where T_s is a right transversal of $\langle s \rangle$. The cardinality of $\langle s \rangle$ is $o(s)$, the order of the element s.

Let us consider the cycles:

$$(s)x = (x, sx, s^2x, \dots, s^{o(s)-1}x)$$

of the permutation $g_s \colon x \longmapsto sx$ of Σ_G. Hence $\langle s \rangle x$ is the support of the cycle $(s)x$. Notice that just one cycle of g_s contains the unit element e, namely $(s)e = (e, s, s^2, \dots s^{o(s)-1})$. We define a graph denoted by $\Phi(G; S) = (V; E; \epsilon)$ in the following way:

- The vertices of $\Phi(G; S)$ are the cycles of g_s, $s \in S$, i.e., $V = \sqcup_{s \in S} V_s$ with $V_s = \{(s)x, x \in T_s\}$.
- For each $(s)x$, $(t)y \in V$, if $card(\langle s \rangle x \ \cap \langle t \rangle y) = p$, $p \geq 1$ then $\{\langle s \rangle x, \langle t \rangle y\}$ is a p-edge.

Thus, $\Phi(G; S)$ is a k-partite graph and any vertex has a $o(s)$-loop. We denote $\widetilde{\Phi}(G; S)$ the graph $\Phi(G; S)$ without loop. By construction, one edge stands for one element of G. We can remark that one element of G labels several edges. Both graphs $\Phi(G; S)$ and $\widetilde{\Phi}(G; S)$ are called *graphs from groups* or *G-graphs* and we can say that the graph is *generated* by the groups $(G; S)$. If $S = G$, the G-graph is called *canonic graph*.

3.1. Algorithmic procedure

The following algorithm constructs a G-graph from the list L of the cycles of a group:

```
Group_to_graph_G(L)
for all s in L
    Add s to V
    for all s’ in L
        for all x in s
            for all y in s’
                if x=y then  Add (s,s’) to E
```

For the construction of the cycles we use the following algorithm, written in the GAP programming language (The GAP Team, 2002):

```
InstallGlobalFunction (
c_cycles, function(G, ga)
local ls1,ls2,gs,k,x,oa,a,res,G2;
res:=[];
G2:=List(G);
for a in ga do
    gs:=[];
    oa:=Order(a)-1;
    ls2:=Set([]);
    for x in G do
        if not(x in ls2) then
            ls1:=[];
            for k in [0..oa] do;
                Add(ls1, Position(G2, (a^k)*x));
                AddSet(ls2, (a^k)*x);
            od;
            Add(gs, ls1);
        fi;
    od;
    Add(res, gs);
od;
return res; end);
```

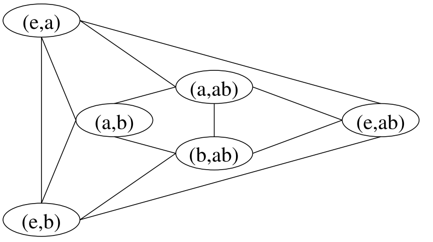


Fig. 1. The octahedral graph.

3.2. Complexity and examples

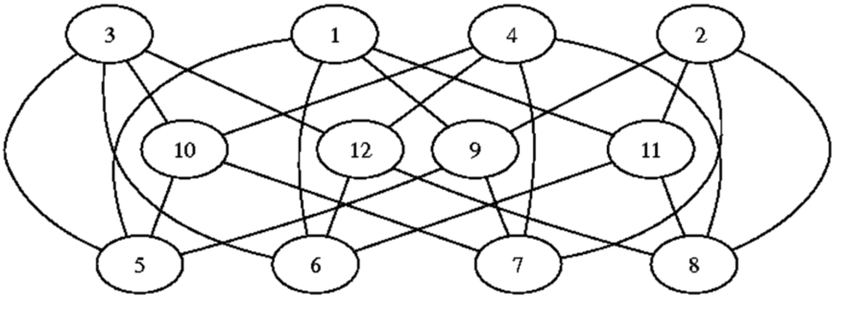
It is easy to see that the complexity of our implementation is $O(n^2 \times k^2)$ where n is the order of the group G and k is the cardinal of the family S.

Let G be the KLEIN's group, the product of two cyclic groups of order 2. So $G = \{e, a, b, ab\}$ with $o(a) = 2$, $o(b) = 2$ and $ab = ba$. The set $S = \{a, b, ab\}$ is a family of generators of G. Let us compute the graph $\widetilde{\Phi}(G; S)$.

The cycles of the permutation g_a are $(a)e = (e, ae) = (e, a)$; $(a)b = (b, ab)$.

The cycles of the permutation g_b are $(b)e = (e, be) = (e, b)$; $(b)a = (a, ba) = (a, ab)$. The cycles of the permutation g_{ab} are $(ab)e = (e, abe) = (e, ab)$; $(ab)a = (a, aba) = (a, b)$. The graph $\widetilde{\Phi}(G; S)$ is isomorphic to the octahedral graph (see Fig. 1). The octahedral graph is a 3-partite symmetric quartic graph.

Let $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be a group generated by $S = \{(1, 0, 0); (0, 1, 0); (0, 0, 1)\}$. The graph $\widetilde{\Phi}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}; S = \{(1, 0, 0); (0, 1, 0); (0, 0, 1)\})$ is isomorphic to the graph shown in the figure below.



4. Basic properties of *G*-graphs

Proposition 1. *Let $\Phi(G; S) = (V; E; \epsilon)$ be a $\mathbb{G}$-graph. Then the following properties are equivalent:*

(i) *$\Phi(G; S)$ has no multi-edges except loops.*
(ii) *for all $s, t \in S$, $\langle s > \cap \langle t >= e$.*

In particular it happens when for all $s, t \in S$, $\gcd(o(s), o(t)) = 1$.

Proof. (i) $\Rightarrow$ (ii) is easy
(ii) $\Rightarrow$ (i) : let $a = ([\langle s > x; \langle t > y], u)$ and $b = ([\langle s > x; \langle t > y], v)$ be two edges with $s \neq t$. From the construction of G-graphs there exist $i, j, k, l \in \mathbb{N}$ such that $s^i x = t^j y = u$ and $s^k x = t^l y = v$ with $0 \leq i, k < o(s)$ and $0 \leq j, l < o(t)$. So $yx^{-1} = s^i t^{-j} = s^k t^{-l}$ and $s^{i-k} = t^{j-l} \in \langle s\rangle \cap \langle t\rangle = e$. Hence $s^{i-k} = t^{j-l} = e$, which leads to $i = k$ and $j = l$, consequently $u = v$ and $a = b$. $\square$

We will need the following result (Bretto et al., 2007).

Proposition 2. *Let h be a morphism between (G_1, S_1) and (G_2, S_2), then there exists a morphism, $\Phi(h)$, between $\Phi(G_1; S_1)$ and $\Phi(G_2; S_2)$.*

In addition we have $\Phi(h \circ h') = \Phi(h) \circ \Phi(h')$, and $\Phi(id_{(G,S)}) = id_{\Phi(G,S)}$; hence if $(G_1; S_1) \simeq (G_2; S_2)$ then $\Phi(G_1; S_1) \simeq \Phi(G_2; S_2)$.

As for the Cayley graph $Cay(G; S)$ it is easy (see Bretto and Faisant (2005)) to prove that $\Phi(G; S)$ is connected iff $\langle S \rangle = G$.

Let $\widetilde{\Phi}(G; S) = (V; E, \epsilon)$ be a G-graph : for any $g \in G$ one can associate the map: $\delta_{g^{-1}} : V \longmapsto V$, defined by $\delta_{g^{-1}}((s)x) = (s)xg^{-1}$, and $\delta^{\#}_{g^{-1}} : E \longmapsto E$ $\delta^{\#}_{g^{-1}}(([\langle s \rangle x; \langle t \rangle y], u)) = ([\langle s \rangle xg^{-1}; \langle t \rangle yg^{-1}], ug^{-1})$.

We have the following theorem.

Theorem 3. *Let $\widetilde{\Phi}(G; S) = (V; E, \epsilon)$ be a $\mathbb{G}$-graph, where $|S| \geq 2$.*

(1) *For all $g \in G$, $(\delta_{g^{-1}}, \delta^{\#}_{g^{-1}}) \in \mathcal{A}ut_\pi(\widetilde{\Phi}(G; S))$.*

(2) *The map $\delta : G \longmapsto \mathcal{A}ut_\pi(\widetilde{\Phi}(G; S))$ defined by $\delta(g) = (\delta_{g^{-1}}, \delta^{\#}_{g^{-1}})$ is a morphism.*

(3) *$\delta(G)$ acts transitively on every V_s, $s \in S$*
$Stab_{\delta(G)}((s)) = \delta(\langle s \rangle)$ is a cyclic subgroup of $\delta(G)$ with an order $d_s | o(s)$.
$Stab_{\delta(G)}((s)x) = \delta(x^{-1}\langle s \rangle x)$ and the stabilizers of the vertices of a principal clique are pairwise distinct.

(4) *$Ker\delta = \bigcap_{s \in S; x \in G} x\langle s \rangle x^{-1}$.*
If there exists $s, t \in S$ such that $\langle s \rangle \cap \langle t \rangle = \{1\}$ then δ is injective.

(5) *If $S = \{s_1, s_2, \ldots s_k\}$ and if for all $i \in \{1, 2, \ldots k\}$ there is $\sigma_i \in Aut\widetilde{\Phi}(G; S)$ such that $\sigma_i((s_i)) = (s_{i+1})$, then $\widetilde{\Phi}(G; S)$ is vertex transitive.*

Proof. (1): We have $\delta_{g^{-1}}(V_s) \subseteq V_s$) and $\delta_{g^{-1}}|_{V_s}$ is a bijection since if $(s)xg^{-1} = (s)yg^{-1}$ then there exists i such that $xg^{-1} = s^i yg^{-1}$ and $(s)x = (s)y$, moreover $(s)x = \delta_{g^{-1}}((s)xg)$.

Let $e = \{([\langle s \rangle x; \langle t \rangle y], u)\}$, then $\delta^{\#}_{g^{-1}}(e) = ([\langle s \rangle xg^{-1}; \langle t \rangle yg^{-1}], ug^{-1}) \in E$ because $u \in \langle s \rangle x \cap \langle t \rangle y$, which implies that $ug^{-1} \in \langle s \rangle xg^{-1} \cap \langle t \rangle yg^{-1}$. So $\delta^{\#}_{g^{-1}}(e)$ is an edge between $(s)xg^{-1} = \delta_{g^{-1}}((s)x)$ and $(t)yg^{-1} = \delta_{g^{-1}}((t)y)$. The map $\delta^{\#}_{g^{-1}}|_E$ is an injection because $([\langle s \rangle xg^{-1}; \langle t \rangle yg^{-1}], ug^{-1}) = ([\langle s' \rangle x'g^{-1}; \langle r' \rangle y'g^{-1}], u'g^{-1})$, we have $u = u'$ and $[\langle s \rangle xg^{-1}; \langle t \rangle yg^{-1}] = [\langle s' \rangle x'g^{-1}; \langle r' \rangle y'g^{-1}]$. Consequently $([\langle s \rangle x; \langle t \rangle y], u) = ([\langle s' \rangle x'; \langle r' \rangle y'], u')$. Hence this map is a bijection and $\delta_{g^{-1}} \in \mathcal{A}ut_\pi(\widetilde{\Phi}(G; S))$. (2): It is easy to show that $\delta(g_1.g_2) = \delta(g_1).\delta(g_2)$; moreover the graph $\widetilde{\Phi}(G; S) = (V; E)$ being simple δ is injective.
(3): Let $(s)x, (s)y \in V_s$, we have $(s)y = (s)xx^{-1}y = \delta_{x^{-1}y}((s)x)$. So the action is transitive on V_s. Assume that there are two vertices, $(s_1)x$, and $(s_2)x$ of a principal clique, (see Section 5) such that $\delta(x^{-1}\langle s_1 \rangle x) = \delta(x^{-1}\langle s_2 \rangle x)$. Hence $\langle s_1 \rangle = \langle s_2 \rangle$. Contradiction
If $\delta(g)((s))x) = (s)x$ then the cycles $(s)xg^{-1}$ and $(s)x$ are the same (i.e. there is $i \geq 0$ such that $s^i x = xg^{-1}$), hence $g = x^{-1}s^{-i}x \in x^{-1}\langle s \rangle x$. Conversely if $g \in x^{-1}\langle s \rangle x$, there is $i \geq 0$ such that $g = x^{-1}s^{-i}x$, so $xg^{-1} = s^i x$. That leads to for all $k \geq 0$, such that $s^k xg^{-1} = s^{k+i}x$, consequently $(s)xg^{-1} = (s)x$. We have $\delta(g)((s))x) = (s)x$.

Since for all s^i, $0 \leq i \leq o(s)$ $\delta(s^i) \in \delta(\langle s \rangle)$, $Stab((s)) = \delta(\langle s \rangle)$ is a cyclic subgroup of $\delta(G)$ with an order $d_s | o(s)$.
(4): If $g \in Ker\delta$, then we have for all $x \in G$, $s \in S$, $(s)xg^{-1} = (s)x$. So there exists $i \geq 0$ such that $g = x^{-1}s^{-i}x$, so $xg^{-1} = s^i x \in x^{-1}\langle s \rangle x$. Conversely $x \in G$, $s \in S$,we have $g \in x^{-1}\langle s \rangle x$, hence there is $i \geq 0$ such that $g = x^{-1}s^{-i}x$. Following (3) we obtain $Ker\delta = \bigcap_{s \in S; x \in G} x\langle s \rangle x^{-1}$. From this result it is easy to see that if there exists $s, t \in S$ such that $\langle s \rangle \cap \langle t \rangle = e$, we have $Ker\delta = \{e\}$.
(5): Under these hypothesis and from (3) we have the result. □

5. Cayley graphs and G-graphs

Let $\Phi((G; S))$ be a G-graph with $\langle S \rangle = G$. Assume that $\Phi((G; S))$ is simple. We call *principal-clique* the clique K_x, $x \in G$ such that any edge of K_x is labeled by x, (K_x is the set of vertices of $\widetilde{\Phi}(G; S)$ which are extremities of edges of the form $a = ([\langle s \rangle y; \langle t \rangle z], x))$. Hence the principal-clique number is equal to $|G|$ and $|K_x| = |S|$.

The *clique graph* denoted by $\mathcal{K}((G,S))$ is the graph defined in the following way:

- the vertices of $\mathcal{K}((G,S))$ are the principal-cliques.
- $u=\{K_x,K_y\}$ is an edge iff $K_x\cap K_y\neq\emptyset$.

Because $\Phi((G;S))$ is simple it is easy to see that $|K_x\cap K_y|\leq 1$.

Let (G,S) be a group we denote $S^*=\bigcup_{s\in S}\langle s\rangle\setminus e$, ($e$ denotes the neutral of G).

Theorem 4. *The graph $\mathcal{K}((G,S))$ is isomorphic to $Cay((G,S^*))$.*

Proof. Define $\phi:\mathcal{K}((G,S))\longmapsto Cay((G,S^*))$ such that $\phi(K_x)=x$. It is easy to see that $K_x=K_y$ imply that $x=y$.

Assume that $u=\{K_x,K_y\}$ is an edge. So there is $\langle s\rangle.a$ vertex of K_x and there is $\langle t\rangle.b$ vertex of K_y such that $x\in\langle s\rangle.a$ and $y\in\langle t\rangle.b$ with $\langle s\rangle.a=\langle t\rangle.b$. Hence there is $1\leq i\leq o(s)$ such that $y=s^i.x$.

Conversely suppose that $\{x,y\}$ is an edge of $Cay((G,S^*))$, hence $y=s^i.x$, $1\leq i\leq o(s)$. Because $\langle s\rangle.x\in K_x$ we have $y\in K_x$. Hence $x,y\in K_x\cap K_y$ and $\{K_x,K_y\}$ is an edge of $\mathcal{K}((G,S))$. □

The following theorem gives a sufficient condition to recognize when a G-graph is a Cayley graph.

Theorem 5.1. *Let $\widetilde{\Phi}(G;S)$ be a G-graph where for all $s\in S$ $o(s)=k>0$ and $|G|=n$. Assume that:*

(i) *There exists a subgroup A of $Aut_S(G)$ which acts regularly on S.*
(ii) *There exists a subgroup K of G with $|K|=\frac{n}{k}$such that:*
  (a) $\forall\alpha\in A,\alpha(K)=K$.
  (b) $\forall s\in S, K\cap\langle s\rangle=\{e\}$.

Under these conditions $H=\delta(K)\rtimes A\leq Aut\widetilde{\Phi}(G;S)$ acts regularly on the set of vertices $V(\widetilde{\Phi}(G;S))$ and $\widetilde{\Phi}(G;S)$ is a Cayley graph $Cay(H;T)$. Moreover if $\alpha\in A\Longrightarrow\alpha_K=id_K$ then the product is direct.

Proof. The proof is constructed as follows:

(1) We show that $\forall s\in S,\ G=\bigsqcup_{k\in K}\langle s\rangle.k.$;
(2) we show $\langle\delta(K),A\rangle=\delta(K)\rtimes A$;
(3) finally we prove that $H=\delta(K)\rtimes A\leq Aut\widetilde{\Phi}(G;S)$ acts regularly on the set of vertices $V(\widetilde{\Phi}(G;S))$.

Show that:

$$\forall s\in S,\quad G=\bigsqcup_{k\in K}\langle s\rangle.k. \tag{1}$$

Assume that $\langle s\rangle.k\cap\langle s\rangle.l\neq\emptyset$, $k,l\in K$. There are $i;j\in\mathbb{N}$ such that $s^ik=s^jl$. So $kl^{-1}=s^{j-i}$. Consequently $kl^{-1}\in\langle s\rangle$, it leads to $kl^{-1}\in\langle s\rangle\cap K$ thus $kl^{-1}=1$ and $k=l$. Moreover $|\bigsqcup_{k\in K}\langle s\rangle.k|=|\langle s\rangle|.|K|$, (from ii)b)) $=o(s).\frac{n}{k}=n=|G|$. So $G=\bigsqcup_{k\in K}\langle s\rangle.k$.

Show that: $\langle\delta(K),A\rangle=\delta(K)\rtimes A$. $\langle\delta(K),A\rangle=\delta(K).A$, indeed: for $k\in K$, $\delta_k\circ\alpha((s)x)=\delta_k((\alpha(s))\alpha(x))=(\alpha(s))\alpha(x)k^{-1}$. There is $u^{-1}\in K$ such that $(\alpha(s))\alpha(x)k^{-1}=(\alpha(s))\alpha(x)\alpha(u^{-1})=\alpha((s)xu^{-1})=\alpha\circ\delta_u((s)x)$. So $\delta(K).A\subset A.\delta(K)$, in the same way one can show that $A.\delta(K)\subset\delta(K).A$ and $\delta(K).A=A.\delta(K)$, hence $A.\delta(K)$ is a subgroup of G and we have $\langle\delta(K),A\rangle=\delta(K).A$.

$\delta(K)\lhd\langle\delta(K),A\rangle$: Let $\alpha\in A$ and $k\in K$, then $\alpha^{-1}\circ\delta_k\circ\alpha((s)x)=\alpha^{-1}\circ\delta_k((\alpha(s))\alpha(x))=\alpha^{-1}[(\alpha(s))\alpha(x)k^{-1}]=(s)x\alpha^{-1}(k^{-1})=(s)x\alpha(k)=\delta_{\alpha^{-1}(k)}((s)x)$. Because there is $u\in K$ such that $\alpha^{-1}(k)=u$, we have $\delta_{\alpha^{-1}(k)}\in\delta(K)$.

$\delta(K)\cap A=\{e\}$: Assume that $\delta(K)\cap A\neq\{1\}$. Let $\alpha\in\delta(K)\cap A$, then $\alpha\neq 1$. There exists $k\in K$ such that $\alpha=\delta_k$. Because A acts regularly on S we have $|A|=|S|$. There are $s,s'\in S$ such that $\alpha(s)=s'$. Consequently $\alpha((s)x)=(s')\alpha(x)$. $\delta_k((s)x)=(s)xk^{-1}$, hence $(s)xk^{-1}=(s')\alpha(x)$ and $(s)=(s')\alpha(x)kx^{-1}$. So $\delta_{xk^{-1}}((s')\alpha(x))=(s)$, which is in contradiction from the definition of δ.

Now we prove that $H=\delta(K)\rtimes A\leq Aut\widetilde{\Phi}(G;S)$ acts regularly on the set of vertices $V(\widetilde{\Phi}(G;S))$.

Let $v\in V_s$ and $w\in V_{s'}$ be two vertices. We have $v=(x,sx,s^2x,\ldots)$. There is $k\in K$, (from Eq. (1)) such that $x=s^ik$, $i\in\mathbb{N}$, hence $v=(s^ik,s^{i+1}k,\ldots)=(k,sk,s^2k,\ldots)$; in the same way $w=(l,s'l,s'^2l,\ldots)$.

From (i) there is $\alpha\in A$ such that $\alpha(s)=s'$, we have : $\alpha(v)=(\alpha(k),s'\alpha(k),s'^2\alpha(k),\ldots)$. From Eq. (1) we have $\alpha(k)\in\langle s'\rangle.m$, $m\in K$, thus $\alpha(k)=s'^jm$, $j\in\mathbb{N}$. Consider $z=l^{-1}t^{-1}\alpha(k)=$

$l^{-1}m$. We have $\delta_z(\alpha(v)) = (\alpha(s))\alpha(k)z^{-1} = (\alpha(k)z^{-1}, s'\alpha(k)z^{-1}, s'^2\alpha(k)z^{-1}, \ldots)$, but $\alpha(k)z^{-1} = \alpha(k)\alpha^{-1}(k)s'^j l = s'^j l$. So $\delta_z(\alpha(v)) = s'^j l, s'^{j+1}l, \ldots) = (l, s'l, s'^2 l, \ldots) = (s')l = w$. Consequently $\delta_z \circ \alpha \in \delta(K) \rtimes A$ maps v on w.

Assume now that there are $z' \in K$ and $\alpha' \in A$ such that $\delta_{z'} \circ \alpha'(v) = w$. One has $\delta_{z'} \circ \alpha'(v) = (\alpha'(k)z'^{-1}, t\alpha'(k)z'^{-1}, t^2\alpha'(k)z'^{-1}, \ldots) = (\alpha(k)z^{-1}, s'\alpha(k)z^{-1}, s'^2\alpha(k)z^{-1}, \ldots)$. It implies that $s' = t$, so $\alpha'(s') = \alpha(s')$, hence $\alpha' = \alpha$, from (i). $\alpha'(k)z'^{-1} = \alpha(k)z'^{-1} = s'^u l$, so $z' = l^{-1}s'^{-u}\alpha(k)$, but $\alpha(k) = s'^j m$, hence $z' = l^{-1}s'^{j-u}m$, then $lz'm^{-1} = s'^{j-u} \in \underset{\sim}{K} \cap \langle s\rangle = \{1\}$, so $\underset{\sim}{z'} = l^{-1}m = z$.

We conclude that $H = \delta(K) \rtimes A$ acts regularly on $V(\widetilde{\Phi}(G; S))$ and that $\widetilde{\Phi}(G; S)$ is a Cayley graph $Cay(H; T)$.

Assume now that $\alpha_K = id_K$. $v = (k, sk, s^2k, \ldots) \in V_s$ and $k \in K$. For $u \in K$, $\delta_u \circ \alpha(v) = (\alpha(k)u^{-1}, \alpha(s)\alpha(k)u^{-1}, \alpha(s)^2\alpha(k)u^{-1}, \ldots) = (ku^{-1}, \alpha(s)ku^{-1}, \alpha(s)^2ku^{-1}, \ldots)$. In the same way $\alpha(v) \circ \delta_u(v) = (ku^{-1}, \alpha(s)ku^{-1}, \alpha(s)^2ku^{-1}, \ldots)$. Consequently $\delta_u \circ \alpha = \alpha(v) \circ \delta_u$. So $H = \delta(K) \times A$. □

6. Hypergraph, Cayley graphs and *G*-graphs

In this section we deal with groups generated by involutions, Example 3.2 is a group generated by involutions. Recall that a cycle is *Hamiltonian* if it goes through any vertex exactly once.

A *hypergraph H* on a finite set **S** is a family $(E_i)_{i\in I}$, $I = \{1, 2, \ldots, n\}$ $\quad n \in \mathbb{N}$ of non-empty subsets of **S** called hyperedges with:

$$\bigcup_{i\in I} E_i = \mathbf{S}.$$

Denote a hypergraph by: $H = (\mathbf{S}; (E_i)_{i\in I})$.

A hypergraph is simple if $E_i = E_j \Longrightarrow i = j$, i.e. there is no repeated hyperedge in H.

For $x \in S$, a *star* of H – with x as a center – is the set of hyperedges which contains x, and is called $\mathcal{H}(x)$. The *degree of* x is the cardinality of the star $\mathcal{H}(x)$. We will denote it by $deg(x)$.

A hypergraph is said *k-uniform* if $|E_i| = k$ for all $i \in I$.

A *chain of length k* in a hypergraph $\mathcal{H}$ is a sequence $x_1E_1x_2E_2 \ldots x_kE_kx_{k+1}$ where E_i are distinct hyperedges and the x_i are vertices such that for $1 \le i \le k$ $x_i, x_{i+1} \in E_i$. If $x_1 = x_{k+1}$ the chain is called *cycle*.

A hypergraph is *connected* if any two vertices are joined by a chain.

The *representative graph* (or *line-graph* but also *intersection graph*) of a hypergraph $\mathcal{H}$ is the graph $L(\mathcal{H})$ such that the vertices are the hyperedges of $\mathcal{H}$ and two distinct vertices x, y form an edge of $L(\mathcal{H})$ if the hyperedges standing for x and y have a non-empty intersection.

A hypergraph is *linear* if $|E_i \cap E_j| \le 1$ for $i \ne j$.

The *dual* of a hypergraph $\mathcal{H} = (E_1, E_2, \ldots, E_m)$ on S is a hypergraph $H^* = (X_1, X_2, \ldots, X_n)$ whose vertices $e_1, e_2, \ldots, e_m$ correspond to the hyperedges of $\mathcal{H}$, and with hyperedges such that

$$X_i = \{e_j, x_i \in E_j\}$$

In an equivalent way: $H^* = (E, (H(x))_{x\in S}$

Remark 5. We have $H^{**} = H$.

The *2-section* of a hypergraph $\mathcal{H}$ is the graph denoted by $[\mathcal{H}]_2$ such that the vertices of this graph are the vertices of $\mathcal{H}$ and two vertices form an edge if and only if they are in the same hyperedge of $\mathcal{H}$.

Remark 6. It is well known (Berge, 1989; Bretto, 2004) that the 2-section of a hypergraph H is isomorphic to the line graph of H^*.

Indeed, the set of vertices of $L(H)$ is E and the set of vertices of $[\mathcal{H}^*]_2$ is also E. Take the identity on E as a bijection on vertices of $L(H)$ to the vertices of $[\mathcal{H}^*]_2$.

If $\{e_i, e_j\}$ is an edge of $L(H)$ that is equivalent to say that $E_i \cap E_j \ne \emptyset$ that is equivalent to say that there is $x \in E_i \cap E_j$ that is equivalent to say $X \ni e_i, e_j$ that is equivalent to say that $\{e_i, e_j\}$ is an edge of $[\mathcal{H}^*]_2$.

Sometimes we will denote the set of vertices of H by $V(H)$ and the set of hyperedges by $E(H)$.

Lemma 7. *Let $H = (\mathbf{S}; (E_i)_{i\in I})$ be a hypergraph. The dual of H, $H^* = (E, (H(x))_{x\in S}$ is a graph if and only if no vertex lies in more than two hyperedges.*

Proof. Assume that the dual of H is a graph. The hyperedge of H^* being the stars of H, these ones have a cardinality equal to two at most. Consequently any vertex of H has a degree equal to either 1 or 2.

Conversely, assume that no vertex lies in more than two hyperedges. For all $x \in S$ $|H(x)| \leq 2$. Hence H^* is a graph. □

The graphs below could have some multi-edges and some loops; for simple graphs we have the following.

Proposition 8. *Let $H = (\mathbf{S}; (E_i)_{i\in I})$ be a hypergraph. Suppose that $H^* = (E, (H(x))_{x\in S}$ is a graph. H^* is simple if and only if H is linear and any vertex of H belong to exactly two hyperedges.*

Proof. Under the hypothesis below and from Lemma 7 H^* is a graph. If H is not linear there are two hyperedges E_i and E_j which contains two x_i, x_j at least. These two vertices become two hyperedges in H^*. These two hyperedges contain both e_i and e_j as vertices. Because H^* is a graph $\{e_j; e_i\}$ give rise to a multi-edge. Hence H^* is not simple.

In the same way if H^* is not a simple graph one can deduce that H is not linear.

It is easy to prove that the graph H^* has no loop, it is equivalent to say that any vertex of H belong to exactly two hyperedges. □

A graph $\Gamma = (V; E)$ is *k-connected*, ($k \in \mathbb{N}$) if $|V| > k$ and $V \setminus X$ is connected for every $X \subseteq V$ with $|X| < k$.

One define the *k-edge connectivity* in the same way.

In Godsil and Gordon (2001), p. 38 we have the following result.

Lemma 9. *If Γ is a connected vertex transitive graph, then its edge connectivity is equal to its valence.*

The following result can be found in Chartrand and Stewart (1969).

Theorem 10. *Let Γ be a k-edge connected graph, then the line graph $L(\Gamma)$ is k-connected and $2k-2$-edge connected.*

We will also need the following.

Theorem 11. *Every line graph of a 4-edge connected graph is Hamiltonian.*

The proof of this theorem can be found in Zhan (1986).

Now from these considerations we are able to prove the following theorem:

Theorem 12. *Let G be a group and S be set of generators of G such that $|S| \neq 3$ and for all $s \in S$, $o(s) = 2$. The $\mathbb{G}$-graph $\tilde{\Phi}((G; S)) = (V; E)$ is Hamiltonian.*

Proof. Because for all $s \in S$, $o(s) = 2$, any vertex of the principal clique hypergraph-i.e. the hypergraph having as vertex set the vertex set of $\tilde{\Phi}((G; S))$ and the hyperedge set the set of principal clique – has a degree equal to 2, moreover this hypergraph is linear; so from Lemma 7 and Proposition 8 the dual of H is a simple graph.

This graph is the Cayley graph $Cay((G; S)$: the line graph of the principal clique hypergraph is the graph $\mathcal{K}((G, S))$ and from Theorem 4 it is isomorphic to $Cay((G, S^*)) = Cay((G; S)$, (because for all $s \in S$, $o(s) = 2$).

From Remark 6 we have $L(Cay((G; S))) = \tilde{\Phi}((G; S))$.

If $|S| = 1, 2$, $Cay((G; S))$ is a cycle so it is Hamiltonian, hence $L(Cay((G; S))) = \tilde{\Phi}((G; S))$ is Hamiltonian.

Suppose now that $|S| = k > 3$. Hence the valence of $Cay((G; S))$ is $k \geq 4$ and this graph is vertex transitive, from Lemma 9 it is k-edge connected, $k \geq 4$ consequently from Theorem 10, $L(Cay((G; S))) = \tilde{\Phi}((G; S))$ is k-connected, $k \geq 4$ and $2k - 2$-edge connected. From Theorem 11 $\tilde{\Phi}((G; S))$ is hamiltonian. □

7. Application

We are now using the results above for the hamiltonicity of Cayley graphs.

7.1. The group $(\frac{\mathbb{Z}}{2\mathbb{Z}})^t$

Let $G = (\frac{\mathbb{Z}}{2\mathbb{Z}})^t$, $t \geq 2$ be a group with the generator set $S = \{e_i; 1 \leq i \leq t\}$ with $e_i = (\underbrace{0, 0 \cdots, 0, 0}_{i-1}, 1, 0, 0, \cdots, 0, 0)$, $|S| = t \geq 4$, $o(e_i) = 2$.

Define now $\alpha(e_i) = e_{i+1}\ i \mod t$ and $x = \sum_{1\leq i\leq t} \lambda_i e_i$.

The map $\alpha(x) = \sum_{1\leq i\leq t} \lambda_i e_{i+1},\ i \mod t$ is a bijection, moreover we have $\alpha(x + y) = \alpha(\sum_{1\leq i\leq t} \lambda_i e_i + \mu_i e_i) = \sum_{1\leq i\leq t} \lambda_i e_{i+1} + \sum_{1\leq i\leq t} \mu_i e_{i+1},\ i \mod t = \alpha(x) + \alpha(y)$. So α is an automorphism.

The group $\langle\alpha\rangle$, $|\langle\alpha\rangle| = t$ acts transitively on S: for $e_i, e_{i+1} \in S, i < j$ and $j = i + k, \mod t$ we have $\alpha^k(e_i) = e_j$.

By the orbit stabilizer theorem we have $|\langle\alpha\rangle x| = |S| = \frac{|\langle\alpha\rangle|}{|\langle\alpha\rangle_x|} = t$, consequently $|\langle\alpha\rangle_x| = 1$. Hence $\langle\alpha\rangle$ acts regularly on S.

Let K be the set $\{x \in G, x = \sum_{1\leq i\leq t} \lambda_i e_i,$ such that $\sum_{1\leq i\leq t} \lambda_i = 0\}$. It is easy to show that K is a subgroup of G.

We have $K \cap \langle e_i\rangle = \{0\}$, $1 \leq i \leq t$, because if $x = \sum_{1\leq i\leq t} \lambda_i e_i = e_j$, $\sum_{1\leq i\leq t} \lambda_i = 1$; contradiction.

$\alpha(K) \subset K$ because for $x \in K$ $\alpha(x) = \sum_{1\leq i\leq t} \lambda_i \alpha(e_i) = \sum_{1\leq i\leq t} \lambda_i e_{i+1}$ and $\sum_{1\leq i\leq t} \lambda_i = 0$.

We can now conclude the following.

Theorem 13. *The G-graph $\widetilde{\Phi}((\frac{\mathbb{Z}}{2\mathbb{Z}})^t; S)$ is a Cayley graph : $Cay(\delta(K) \rtimes \langle\alpha\rangle; T)$ and this one is hamiltonian.*

7.2. The group S_n

Let G be the group S_n with the generator set $S = \{(i, i+1), 1 \leq i \leq n\}$, $n \geq 4$. Let ρ be the cycle $(1, 2, 3, \ldots, n)$. Settle $\alpha = \phi_\rho : \phi_\rho((i, i+1)) = \rho(i, i+1)\rho^{-1}$. We have $\rho(i, i+1)\rho^{-1}(\rho(i)) = \rho(i, i+1)(i) = \rho(i+1) = i+2$. Now $\rho(i, i+1)\rho^{-1}(\rho(i+1)) = \rho(i, i+1)(i+1) = \rho(i) = i+1$. Hence $\rho(i, i+1)\rho^{-1} = (i+1, i+2)$.

The group $\langle\alpha\rangle$ acts transitively on S: $\alpha^2 = \phi_\rho^2 = \phi_\rho \circ \phi_\rho = \phi_\rho(\rho(i, i+1)\rho^{-1}) = \phi_\rho((i+1, i+2)) = \rho(i+1, i+2)\rho^{-1} = (i+2, i+3)$, by induction $\phi_\rho((i+k, i+k+1)) = \rho(i+k, i+k+1)\rho^{-1} = (i+k+1, i+k+2)$. Consequently for $(u, v)\ (w, z) \in S$ there is t such that $\phi_\rho^t((u, v)) = (w, z)$ and $\langle\alpha\rangle$ acts transitively on S.

The $\langle\alpha\rangle$ acts regularly on S: like above by applying the orbit stabilizer theorem.

Let K be the alternating group, i.e. $K = A_n$. For any transposition τ the signature of it is $\epsilon(\tau) = -1$; the group A_n is the kernel of the epimorphism $\epsilon : S_n \longrightarrow \{-1, 1\}$, so for all $\rho \in A_n$ we have $\epsilon(\rho) = 1$; from this: $A_n \cap \langle\alpha\rangle = \{1\}$.

$\alpha(K) \subset K$ because for $x \in K$, $\epsilon(x) = 1$ and $\alpha(x) = \rho x \rho^{-1}$ has the same parity of x, i.e. $\epsilon(\rho x \rho^{-1}) = 1$ We can now conclude the following.

Theorem 14. *The G-graph $\widetilde{\Phi}(S_n; S)$ is a Cayley graph : $Cay(\delta(K) \rtimes \langle\alpha\rangle; T)$ and this one is hamiltonian.*

Remark. The results above give us a new method to produce some new classes of hamiltonian Cayley graphs.

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