On the stochastic Benjamin–Ono equation

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Abstract

We discuss the Cauchy problem for the stochastic Benjamin–Ono equation in the function class $H^s(R)$, $s > 3/2$. When there is a zero-order dissipation, we also establish the existence of an invariant measure with support in $H^2(R)$. Many authors have discussed the Cauchy problem for the deterministic Benjamin–Ono equation. But our results are new for the stochastic Benjamin–Ono equation. Our goal is to extend known results for the deterministic equation to the stochastic equation.

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0. Introduction

In this paper we will discuss the Cauchy problem for the stochastic Benjamin–Ono equation. Our main goal is to establish the existence of a solution to the Cauchy problem and to prove the existence of an invariant measure. The Cauchy problem is formulated as follows:

$$u_t + uu_x + H(u_{xx}) = \sum_{j=1}^{\infty} g_j \frac{dB_j}{dt}, \quad (t, x) \in (0, \infty) \times R,$$

(0.1)
where \( H(\cdots) \) is the Hilbert transform and \( g_j = g_j(t, x), j = 1, 2, \ldots \). The right-hand side corresponds to a random noise which is white in the time variable. When \( g_j \equiv 0 \) for all \( j \geq 1 \), (0.1) reduces to the well-known Benjamin–Ono equation, which describes unidirectional propagation of nonlinear dispersive waves [2,17], and has been extensively studied by many authors. See [1,8,9,12,15,16,18,20,23]. However, to the author’s knowledge, the Cauchy problem for the stochastic Benjamin–Ono equation (0.1) has not been investigated. Since the K-dV equation is the most well-known among all model equations which describe dispersive waves, other equations are often compared to the K-dV equation. The K-dV equation has stronger dispersion mechanism than the Benjamin–Ono equation, and the existence of a solution can be established through the variation of constants formula and the semigroup associated with the principal part of the equation. This approach has a stochastic version for stochastic evolution equations, and gives rise to a stochastic convolution when the forcing term is a white noise. Hence, for the stochastic K-dV equation, [6,19] proved the existence of a solution by careful analysis of the stochastic convolution. This approach covers a broad class of stochastic evolution equations. The monograph [4] presents a comprehensive study of the general stochastic evolution equations using this method. However [16] showed that Picard iteration scheme via the variation of constants formula fails the deterministic Benjamin–Ono equation. Thus we have to employ a different approach. By regularizing the equation and the data, we first obtain a pathwise solution which is sufficiently smooth in the space variable for each sample point. We need sufficient regularity of solutions to justify manipulations for the energy estimates. This first step is essentially the same as for the deterministic equation; see [1]. The second step is to obtain necessary stochastic a priori estimates, where integral invariants play a crucial role. This requires various new stochastic estimates. We will borrow some technical estimates for the deterministic equation from [1] which presents a comprehensive analysis of integral invariants. We also borrow some analytical tools from [18]. Then, by a measure-theoretic argument, we can obtain a desired solution for the original equation. Here we will establish an existence result in the function class \( H^s, s > 3/2 \). We also obtain estimates of the mean energy for \( s = 2 \). The details of proof will be presented in Section 2 below.

After the global Cauchy problem, we will prove the existence of an invariant measure when Eq. (0.1) includes an additional term of zero-order dissipation. Such a term can describe variable depth in the flow model; see [14]. An invariant measure is an important object in the study of stochastic dynamics. It corresponds to a stationary solution of a deterministic equation. If the initial datum has the probability distribution equal to an invariant measure, then the probability distribution of the evolving solution is invariant in time. There are some general results on the existence of invariant measures for stochastic evolution equations; see [4,5]. But the method of such results do not cover the stochastic Benjamin–Ono equation. Here we will use the recent result on a certain class of stochastic evolution equations [13, Theorem 1.1], where some sufficient conditions for the existence of an invariant measure are presented. We will verify those conditions. This involves various technical issues. One of the required conditions is that the time-average of the norm of a solution in the basic function class must be bounded uniformly in time. Hence, we need an extra term of zero-order dissipation, which dissipates the energy due to the ran-
dom noise. Even for a parabolic equation, such a term is necessary to make up for the lack of the Poincaré inequality in an unbounded space domain. In [5], it was shown that such a term is a necessary condition for a linear stochastic parabolic equation to have an invariant measure when the space domain is the whole space. Also, on account of our method of construction of a pathwise solution, we need extra work to show that the solution is a Markov process. All these technical issues are addressed in Section 3 below.

Finally, we state open questions. For the function class $H^s(R), s \leq 3/2$, the pathwise uniqueness of a solution for the Cauchy problem is open. For the deterministic equation, see [12,15,18,23]. In particular, [23] showed that the Cauchy problem is well-posed for $s = 1$. According to the known results for $1 \leq s \leq 3/2$, it seems that estimates of the Strichartz type and the gauge transformation are crucial. It is not known whether comparable estimates are possible for the stochastic Benjamin–Ono equation. Also, the uniqueness of an invariant measure is an open question. For nonlinear stochastic evolution equations, it is a difficult problem which has been resolved only for some special equations of parabolic type. See [5,7] and references therein.

1. Notation and preliminaries

Throughout this paper, $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a given stochastic basis, where $P$ is a probability measure, $\mathcal{F}$ is a $\sigma$-algebra and $\{\mathcal{F}_t\}_{t \geq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_0$ contains all $P$-negligible subsets. A point of $\Omega$ will be denoted by $\omega$. $\{B_j(t)\}_{j=1}^\infty$ is a sequence of mutually independent standard Brownian motions over $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. $E(\cdot)$ stands for expectation with respect to the probability measure $P$. In this paper, a stochastic integral is defined in the sense of Ito. When $O$ is a topological space, $B(O)$ denotes the Borel $\sigma$-algebra over $O$. When $X$ is a Banach space, an $X$-valued function $f$ is said to be $\mathcal{F}$-measurable if $f^{-1}(G) \in \mathcal{F}$ for every $G \in B(X)$. This coincides with strong measurability for Bochner integrals when the range of $f$ is separable. When $\mathcal{X}$ is a Banach space, we denote by $L^p(\Omega; \mathcal{X}), 1 \leq p < \infty$, the set of all $\mathcal{X}$-valued strongly measurable functions such that

$$\int_{\Omega} \|f\|_X^p \, dP < \infty.$$ 

An $\mathcal{X}$-valued stochastic process $Y(t)$ is said to be progressively measurable if $Y$ restricted to the interval $[0, t]$ is $B([0, t]) \otimes \mathcal{F}_t$-measurable for each $t \geq 0$. For general information on stochastic processes, see [4,10].

For $s \in R$, $H^s(R)$ denotes the usual Sobolev spaces. We will use the notation $J^s = (1 - \partial_{xx})^{s/2}$ so that $\|f\|_{H^s(R)} = \|J^s f\|_{L^2(R)}$. We will write

$$\|f\|_s = \|f\|_{H^s(R)}, \quad \langle f, g \rangle_0 = \langle f, g \rangle_{L^2(R)}.$$ 

$C_w(0, T; H^s(R))$ is the set of all $H^s(R)$-valued weakly continuous functions on the interval $[0, T]$. We present a list of basic analytical tools.
Lemma 1.1. Let $X \subset Y$ be two Hilbert spaces with continuous injection, and let $0 < T < \infty$. Then, it holds that

$$L^\infty(0, T; X) \cap C_w(0, T; Y) = C_w(0, T; X).$$

**Proof.** See [22].

Lemma 1.2. Let $s_1 < s_2 < s_3$, and $I$ be a bounded open interval on $\mathbb{R}$. If $\mathcal{G}$ is a bounded subset of $L^\infty(0, T; H^{s_3}(I))$ such that $\mathcal{G}_t = \{ f_t \mid f \in \mathcal{G} \}$ is bounded in $L^\infty(0, T; H^{s_1}(I))$, then $\mathcal{G}$ is relatively compact in $C([0, T]; H^{s_2}(I))$.

**Proof.** See [21].

Lemma 1.3. Let $\rho_{\epsilon}(\cdot)$ be the Friedrichs mollifier and $f \in C^1(\mathbb{R})$ with

$$\| f \|_{C^1} = \| f \|_{L^\infty(\mathbb{R})} + \| f_x \|_{L^\infty(\mathbb{R})} < \infty.$$

Then, it holds that

$$\| f(h_x * \rho_{\epsilon}) - \rho_{\epsilon} * (f h_x) \|_{L^2(\mathbb{R})} \leq C \| f \|_{C^1(\mathbb{R})} \| h \|_{L^2(\mathbb{R})}$$

for all $h \in L^2(\mathbb{R})$, for some constant $C > 0$, where $*$ denotes the convolution.

**Proof.** See [24].

Lemma 1.4. Let $s > 1/2$. It holds that

$$\| f \|_{L^\infty(\mathbb{R})} \leq C \left( 1 + \| f \|_{H^{1/2}(\mathbb{R})} \sqrt{\log(1 + \| f \|_{H^s(\mathbb{R})})} \right)$$

for all $f \in H^s(\mathbb{R})$, for some constant $C > 0$.

**Proof.** A slight technical modification of the proof of a similar inequality in [3, p. 678] yields the above inequality.

Lemma 1.5. Let $s \geq 0$ and $1 < p < \infty$. It holds that

$$\| J^s(f h) - f(J^s h) \|_{L^p(\mathbb{R})} \leq C \left( \| f_x \|_{L^\infty(\mathbb{R})} \| J^{s-1} h \|_{L^p(\mathbb{R})} + \| J^s(f) \|_{L^p(\mathbb{R})} \| h \|_{L^\infty(\mathbb{R})} \right)$$

for all $f, h \in C^\infty_0(\mathbb{R})$.

**Proof.** See [11].
**Definition 1.6.** Let \( u \) be a progressively measurable process such that \( u \in C([0, T]; H^s(\mathbb{R})) \), for almost all \( \omega \), for some \( s \geq 1 \) and \( T > 0 \). \( u \) is said to be a solution of (0.1), (0.2) if

\[
\partial_t \left( u - \sum_{j=1}^{\infty} \int_{0}^{t} g_j dB_j \right) + uu_x + \mathcal{H}(u_{xx}) = 0
\]

holds in the sense of distributions over \((0, T) \times \mathbb{R}, \) for almost all \( \omega \), and (0.2) holds for almost all \( \omega \).

2. The Cauchy problem

Throughout this section, we assume \( g_j = g_j(t, x) \), for all \( j \geq 1 \), and

\[
\sum_{j=1}^{\infty} \int_{0}^{T} \| g_j(t) \|_2^2 \, dt < \infty.
\]

(2.1)

for each \( T > 0 \), for some \( s > 3/2 \). We then have the following existence result.

**Theorem 2.1.** Let

\[
u_0 \in H^s(\mathbb{R}), \quad s > 3/2.
\]

(2.2)

Then, for each \( T > 0 \), there is a solution \( u \) of (0.1), (0.2) such that \( u \) is \( H^s(\mathbb{R}) \)-valued progressively measurable and

\[
u \in C([0, T]; H^s(\mathbb{R})), \quad \text{for almost all } \omega.
\]

(2.3)

The solution is pathwise unique in the function class \( C([0, T]; H^s(\mathbb{R})) \), \( s > 3/2 \).

We need more results when \( s = 2 \), for the existence of an invariant measure.

**Theorem 2.2.** Let (2.1) hold with \( s = 2 \), and suppose that \( u_0 \) is \( \mathcal{F}_0 \)-measurable and

\[
u_0 \in L^2(\Omega; H^2(\mathbb{R})) \cap L^{2\gamma}(\Omega; L^2(\mathbb{R})),
\]

(2.4)

for some \( \gamma \geq 9 \). Then, for each \( T > 0 \), there is a pathwise unique solution \( u \) of (0.1), (0.2) such that \( u \) is \( H^2(\mathbb{R}) \)-valued progressively measurable and

\[
u \in L^2(\Omega; C([0, T]; H^2(\mathbb{R}))) \cap L^{2\gamma}(\Omega; C([0, T]; L^2(\mathbb{R}))).
\]

(2.5)
The proof of these results consists of three steps. The first step is to regularize the equation and the data, and obtain solutions which are smooth in the space variable. The second step is to obtain a priori estimates. The last step is to construct a desired solution of the original equation.

2.1. Regularized equations

Following the regularization in [1], we consider the following problem:

\[ u_t + uu_x + \mathcal{H}(u_{xx}) + \epsilon (u_{xxx} + u) = \sum_{j=1}^{\infty} g_{j, \epsilon} \frac{dB_j}{dt}, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \]

\[ u(0, x) = u_{0, \epsilon}(x), \quad x \in \mathbb{R}. \]

Here we define

\[ g_{j, \epsilon} = g_j \ast \rho_\epsilon, \quad u_{0, \epsilon} = u_0 \ast \rho_\epsilon \]

where \( \rho_\epsilon \) is the Friedrichs mollifier in the space variable. Then, under the assumptions (2.1) and (2.2),

\[ \sum_{j=1}^{\infty} \int_0^T \left\| g_{j, \epsilon}(t) \right\|_v^2 dt < \infty \]

and

\[ u_{0, \epsilon} \in H^v(\mathbb{R}), \]

for every \( v \geq 1 \). For the Cauchy problem (2.6), (2.7), we define a continuous martingale

\[ W_\epsilon(t) = \sum_{j=1}^{\infty} \int_0^t g_{j, \epsilon}(\eta) dB_j(\eta) \]

and set

\[ v = u - W_\epsilon. \]

Then, (2.6) is equivalent to

\[ v_t + vv_x + W_\epsilon v_x + W_{\epsilon x} v + \mathcal{H}(v_{xx}) + \epsilon (v_{xxx} + v) = f_\epsilon \]

where \( f_\epsilon = -W_\epsilon W_{\epsilon x} - \mathcal{H}(W_{\epsilon xx}) - \epsilon (\partial_x^4 W_\epsilon + W_\epsilon). \)
Let us fix any $T > 0$. There is a subset $\hat{\Omega}$ such that $P(\Omega \setminus \hat{\Omega}) = 0$, and for each $\omega \in \hat{\Omega}$,

$$W_\epsilon \in C([0, T]; H^v(R))$$

for every $v \geq 1$. Thus, we can use the standard argument for the deterministic parabolic equation to obtain a unique solution $u_\epsilon$ such that

$$u_\epsilon \in C([0, T]; H^v(R))$$

for all $v \geq 1$, for each $\omega \in \hat{\Omega}$. It is easy to see that the mapping $W_\epsilon \mapsto u_\epsilon$ is continuous from $C([0, T]; H^4(R))$ into $C([0, T]; L^2(R))$. Thus, $u_\epsilon$ is $L^2(R)$-valued progressively measurable. Since every Borel subset of $H^v(R), v \geq 1$, is a Borel subset of $L^2(R)$, $u_\epsilon$ is $H^v(R)$-valued progressively measurable for each $v \geq 1$.

2.2. Basic estimates

Let $u_\epsilon$ be a solution of (2.6), (2.7). In this section, we fix any $\epsilon > 0$ and drop the subscript $\epsilon$ from $u_\epsilon, u_0, \epsilon, g_j, \epsilon$ and $W_\epsilon$. We also choose any large $T > 0$. Our first goal is to obtain $L^2(R)$-estimates. We define a stopping time

$$T_{\theta,N} = \begin{cases} 
\inf\{0 < t \leq T \mid \|u(t)\|_{\theta/2}(R) > N\}, \\
T & \text{if the set } \{\cdots\} \text{ is empty}
\end{cases}$$

where $N \geq 1$ and $\theta \geq 0$. Since $u \in C([0, T]; H^v(R))$, for all $v \geq 0$, for almost all $\omega$, there is some $0 < N(\theta, \omega) < \infty$, for almost all $\omega$, such that for all $N \geq N(\theta, \omega), T_{\theta,N}(\omega) = T$ holds.

By applying Ito’s rule to $\|u(t)\|_{0}^{2^v}, v \geq 1$, we obtain

$$\|u(t)\|_{0}^{2^v} = \|u_0\|_{0}^{2^v} - 2\gamma \epsilon \int_{0}^{t} \|u\|_{0}^{2^v-2} (\|u_{xx}\|_{0}^2 + \|u\|_{0}^2) \, d\eta$$

$$+ 2\gamma \sum_{j=1}^{\infty} \int_{0}^{t} \|u\|_{0}^{2^v-2} \langle u, g_j \rangle_{0} \, dB_j$$

$$+ 2\gamma (\gamma - 1) \sum_{j=1}^{\infty} \int_{0}^{t} \|u\|_{0}^{2(\gamma-2)} \|u, g_j\|_{0}^2 \, d\eta + \gamma \sum_{j=1}^{\infty} \int_{0}^{t} \|u\|_{0}^{2^v-2} \|g_j\|_{0}^2 \, d\eta$$

for all $t \in [0, T_{0,N}(\omega)]$, for almost all $\omega$. By the Burkholder–Davis–Gundy inequality, we can estimate the third term on the right-hand side:
\[ E \left( \sup_{0 \leq t \leq T_0, N} \left| \sum_{j=1}^{\infty} \int_0^t \|u\|_{0}^{2\gamma-2} \langle u, g_j \rangle_0 dB_j \right| \right) \]
\[ \leq CE \left( \sum_{j=1}^{\infty} \int_0^{T_0, N} \|u\|_0^{4\gamma-2} \|g_j\|_0^2 dt \right)^{1/2} \]
\[ \leq CE \left( \sup_{0 \leq t \leq T_0, N} \|u(t)\|_0^{2\gamma-2} \left( \sum_{j=1}^{\infty} \int_0^{T_0} \|g_j\|_0^2 dt \right)^{1/2} \right) \]
\[ \leq \delta E \left( \sup_{0 \leq t \leq T_0, N} \|u(t)\|_0^{2\gamma} \right) + \frac{C}{\delta^{2\gamma-1}} \left( \sum_{j=1}^{\infty} \int_0^{T_0} \|g_j\|_0^2 dt \right)^{\gamma} \] (2.15)

for every \( \delta > 0 \), where \( C \) denotes positive constants independent of \( T > 0, N \geq 1 \) and \( \epsilon > 0 \). We also have

\[ E \left( \sup_{0 \leq t \leq T_0, N} \text{the last two terms on the right-hand side of (2.14)} \right) \]
\[ \leq CE \left( \sup_{0 \leq t \leq T_0, N} \|u(t)\|_0^{2\gamma-2} \sum_{j=1}^{\infty} \int_0^{T_0} \|g_j\|_0^2 dt \right) \]
\[ \leq \delta E \left( \sup_{0 \leq t \leq T_0, N} \|u(t)\|_0^{2\gamma} \right) + \frac{C}{\delta^{2\gamma-1}} \left( \sum_{j=1}^{\infty} \int_0^{T_0} \|g_j\|_0^2 dt \right)^{\gamma} \] (2.16)

for every \( \delta > 0 \), where \( C \) denotes positive constants independent of \( T > 0, N \geq 1 \) and \( \epsilon > 0 \). Combining (2.13)–(2.16), we can derive by passing \( N \to \infty \),

\[ E \left( \sup_{0 \leq t \leq T} \|u(t)\|_0^{2\gamma} \right) \leq C_{\gamma, T} \] (2.17)

where \( C_{\gamma, T} \) is a positive constant independent of \( \epsilon > 0 \).

Next we will obtain estimates of integral invariants. There are infinitely many integral invariants \( \{I_n\}_{n=1}^{\infty} \) for the Benjamin–Ono equation. According to [1], each \( I_n(u) \) can be written as

\[ I_n(u) = \int_{-\infty}^{\infty} \frac{(-1)^n}{n+2} u^{n+2} dx \]
\[ + \sum_{m=1}^{n-1} \int_{-\infty}^{\infty} P_{n+2-m,m}(u) dx + d_n \int_{-\infty}^{\infty} |\xi|^n |\hat{u}(\xi)|^2 d\xi \] (2.18)
where $d_n$ is a positive number, and $\hat{u}$ is the Fourier transform of $u$. The polynomial $P_{j,k}(u)$ denotes the sum of all terms which are homogeneous of degree $j$ in $u$, and which involve exactly $k$ derivatives in $x$. It is easy to see that $I_n(\cdot)$ is an infinitely differentiable functional from $H^\frac{n}{2}(R)$ into $R$. Its first derivative is a continuous linear functional

$$ h \mapsto DI_n(u) \cdot h $$

from $H^\frac{n}{2}(R)$ into $R$. The second derivative is a continuous bilinear functional

$$(h_1, h_2) \mapsto D^2 I_n(u) \cdot (h_1, h_2)$$

from $H^\frac{n}{2}(R) \times H^\frac{n}{2}(R)$ into $R$. In the sequel, we will only need $I_3(u)$ and $I_4(u)$:

$$I_3(u) = \int_{-\infty}^{\infty} \left(-\frac{1}{5} u^5 - \left[\frac{4}{3} u^3 \mathcal{H}(u_x) + u^2 \mathcal{H}(uu_x)\right]\right.
-\left[2u \mathcal{H}(u_x)^2 + 6uu_x^2\right] + 4u \mathcal{H}(u_{xxx})\right) dx,$$

$$I_4(u) = \int_{-\infty}^{\infty} \left(\frac{1}{6} u^6 + \left[\frac{5}{4} u^4 \mathcal{H}(u_x) + \frac{5}{3} u^3 \mathcal{H}(uu_x)\right]\right.
+\frac{5}{2} \left[5u^2 u_x^2 + u^2 \mathcal{H}(u_x)^2 + 2u \mathcal{H}(u_x) \mathcal{H}(uu_x)\right]
-10 \left[u_x^2 \mathcal{H}(u_x) + 2uu_x \mathcal{H}(u_x)\right] + 8u_{xx}^2\left)\right. dx.$$

By Ito’s rule, it holds that

$$I_n(u(t)) - I_n(u(0)) = -\int_{0}^{t} DI_n(u) \cdot (uu_x + \mathcal{H}(u_{xx})) d\eta$$

$$-\epsilon \int_{0}^{t} DI_n(u) \cdot (u_{xxxx} + u) d\eta + \sum_{j=1}^{\infty} \int_{0}^{t} DI_n(u) \cdot g_j dB_j(\eta)$$

$$+\frac{1}{2} \sum_{j=1}^{\infty} \int_{0}^{t} D^2 I_n(u) \cdot (g_j, g_j) d\eta$$

(2.19)

for all $t \in [0, T_{n,N}(\omega)]$, for almost all $\omega$. We will estimate the stochastic integral on the right-hand side. To handle the term $P_{n+2-m,m}(u)$, we define a functional
\[ J_{n,m}(u) = \int_{-\infty}^{\infty} \left( \prod_{i=1}^{n+2-m} \partial_x^{\alpha_i} u \right) dx, \quad 1 \leq m \leq n - 1, \] (2.20)

where each \(0 \leq \alpha_i \leq \frac{m+1}{2}\) is a given integer such that

\[
\sum_{i=1}^{n+2-m} \alpha_i = m.
\]

Some terms of \(P_{n+2-m,m}(u)\) involve the Hilbert transform. But it is enough to consider the above functional because of the continuity of the Hilbert transform. It follows from the Burkholder–Davis–Gundy inequality that

\[
E\left( \sup_{0 \leq t \leq T_{n,N}} \left| \sum_{j=1}^{T_{n,N}} \int_0^t \left( \int_{-\infty}^{\infty} \left( \prod_{i=1 \atop i \neq k}^{n+2-m} \partial_x^{\alpha_i} g_j \right) \left( \prod_{i=1 \atop i \neq k}^{n+2-m} \partial_x^{\alpha_i} u \right) dx \right) dB_j \right| \right)
\]

\[
= E\left( \sup_{0 \leq t \leq T_{n,N}} \left| \sum_{j=1}^{T_{n,N}} \int_0^t \left( \int_{-\infty}^{\infty} \left( \prod_{i=1 \atop i \neq k}^{n+2-m} \partial_x^{\alpha_i} g_j \right) \left( \prod_{i=1 \atop i \neq k}^{n+2-m} \partial_x^{\alpha_i} u \right) dx \right) \right| dt \right)^{1/2}.
\] (2.21)

According to the estimates obtained in [1], we have

\[
\sum_{j=1}^{T_{n,N}} \int_0^T \left| \int_{-\infty}^{\infty} \left( \prod_{i=1 \atop i \neq k}^{n+2-m} \partial_x^{\alpha_i} g_j \right) \left( \prod_{i=1 \atop i \neq k}^{n+2-m} \partial_x^{\alpha_i} u \right) dx \right| \right|^2 dt
\]

\[
\leq C \sum_{j=1}^{T_{n,N}} \int_0^T \left( \sum_{k=1}^{n+2-m} \left( \|g_j\|_{L^2} \|g_j\|_0^{1-\theta_k} \prod_{i=1 \atop i \neq k}^{n+2-m} \|u\|_{L^2} \|u\|_0^{1-\theta_i} \right)^2 \right) dt
\]

\[
\leq C \sup_{0 \leq t \leq T_{n,N}} \left( \int_{-\infty}^{\infty} \left( \prod_{k=1}^{n+2-m} \left( \sum_{i=1 \atop i \neq k}^{n+2-m} \|u\|_{L^2} \|u\|_0^{1-\theta_i} \right)^2 \sum_{j=1}^{T_{n,N}} \|g_j\|_{L^2}^2 \right) dt \right)^{1/2}.
\] (2.22)

where \(0 \leq \theta_i \leq 1\), for each \(1 \leq i \leq n + 2 - m\), and

\[
\sum_{i=1}^{n+2-m} \theta_i = 1 + \frac{m}{n}.
\]
It follows that
\[
E \left( \sup_{0 \leq t \leq T_{n,N}} \left| \sum_{j=1}^{\infty} \int_{0}^{t} DJ_{n,m}(u) \cdot g_j \, dB_j \right| \right) \\
\leq \delta E \left( \sup_{0 \leq t \leq T_{n,N}} \left\| u(t) \right\|_{q}^{2} \right) \\
+ C(\delta, n, m) E \left( 1 + \sup_{0 \leq t \leq T_{n,N}} \left\| u(t) \right\|_{0}^{2n+2} \right) \left( 1 + \sum_{j=1}^{\infty} \int_{0}^{T} \left\| g_j \right\|_{q}^{2} \, dt \right)^{n} \tag{2.23}
\]
for all \( \delta > 0 \), where \( C(\delta, n, m) > 0 \) is a constant independent of \( \epsilon \) and \( N \).

Next we estimate \( D^2 J_{n,m}(u) \cdot (g_j, g_j) \)
\[
\left| D^2 J_{n,m}(u) \cdot (g_j, g_j) \right| \\
= \left| \sum_{k=1}^{n+2-m} \sum_{v=1}^{n+2-m} \int_{-\infty}^{\infty} \left( \frac{\partial^{\alpha_k} g_j}{\partial x} \frac{\partial^{\alpha_v} g_j}{\partial x} \right) \left( \prod_{i=1}^{n+2-m} \frac{\partial^{\alpha_i} u}{\partial x} \right) dx \right| \\
\leq C \sum_{k=1}^{n+2-m} \sum_{v=1}^{n+2-m} \left( \left\| g_j \right\|_{0}^{\theta_k} \left\| g_j \right\|_{0}^{\theta_v} \left\| g_j \right\|_{0}^{1-\theta_k} \left\| g_j \right\|_{0}^{1-\theta_v} \prod_{i=1}^{n+2-m} \left\| u \right\|_{0}^{\theta_i} \left\| u \right\|_{0}^{1-\theta_i} \right), \tag{2.24}
\]
which yields
\[
\sum_{j=1}^{\infty} \int_{0}^{T_{n,N}} \left| D^2 J_{n,m}(u) \cdot (g_j, g_j) \right| \, dt \\
\leq \delta \sup_{0 \leq t \leq T_{n,N}} \left\| u(t) \right\|_{q}^{2} \\
+ C(\delta, n, m) \left( 1 + \sup_{0 \leq t \leq T_{n,N}} \left\| u(t) \right\|_{0}^{2n+2} \right) \left( 1 + \sum_{j=1}^{\infty} \int_{0}^{T} \left\| g_j \right\|_{q}^{2} \, dt \right)^{2n} \tag{2.25}
\]
for all \( \delta > 0 \), where \( C(\delta, n, m) > 0 \) is a constant independent of \( \epsilon \) and \( N \).

The following estimates can be found in [1]. Let \( v \in H^\nu(R) \), for all \( \nu \geq 1 \). Then, it holds that
\[
I_n(v) \geq c_{n,1} \left\| v \right\|_{q}^{2} - c_{n,2} \left\| v \right\|_{0}^{2+2n} \tag{2.26}
\]
and
\[ DI_n(v) \cdot \partial^2_{xx} v \geq c_{n,3} \| v \|_{L^2}^2 + c_{n,4} \| v \|_0^{2+2n+4k}, \quad k = 0, 1, 2, \] (2.27)

where \( c_{n,j} \)'s are positive constants independent of \( v \).

It is also known that

\[ DI_n(v) \cdot (vv_x + \mathcal{H}(v_{xx})) = 0. \] (2.28)

The remaining terms on the right-hand side of (2.19) are easy to estimate. By combining (2.6), (2.17) with \( \gamma \geq n + 5 \), and (2.19)–(2.28), and by passing \( N \to \infty \), we can derive

\[ E\left( \sup_{0 \leq t \leq T} \left\| u(t) \right\|_{L^2}^2 \right) + E\left( \left\| \partial_t (u - W) \right\|_{L^\infty(0,T; H^{n/2-4}(R))} \right) \leq C_T \] (2.29)

for \( 0 < \epsilon \leq 1 \) in (2.6), where \( C_T \) is a positive constant which depends only on

\[ \| u_0 \|_{L^2}, \quad \| u_0 \|_{H^n}^2 \quad \text{and} \quad \sum_{j=1}^\infty \int_0^T \| g_j(t) \|_{L^2}^2 \, dt \]

provided they are finite.

2.3. Construction of solutions for \( s > 3/2 \)

Let us point out a major technical hurdle when we construct solutions to a stochastic nonlinear equation through approximate solutions. Typically we obtain uniform estimates of approximate solutions in \( L^p(\Omega; X), 1 \leq p < \infty \), where \( X \) is a certain Banach space. Suppose that \( Y \) is another Banach space such that the embedding \( X \to Y \) is compact. For deterministic equations, if we have uniform estimates of approximate solutions in \( X \), then we can extract a sequence which converges strongly in \( Y \). If \( Y \) is appropriately chosen, then this strong convergence can handle nonlinear terms. However, the embedding \( L^p(\Omega; X) \to L^p(\Omega; Y) \) is not compact. Thus, the well-known procedure for deterministic equations does not work for stochastic nonlinear equations. This justifies our measure-theoretic method to construct solutions. The main idea of our particular method is to find a subset \( \tilde{\Omega} \) such that \( P(\Omega \setminus \tilde{\Omega}) = 0 \), and for each fixed sample point \( \omega \in \tilde{\Omega} \), there is a sequence of approximate solutions which converges to a limit satisfying (1.1) in the sense of distributions. This involves various technical issues. We now present all the details.

Let us fix any \( s > 3/2 \), and assume the conditions (2.1) and (2.2). Let \( u = u_\epsilon \) be a solution of (2.6), (2.7). Since it holds that

\[ \| u_{0,\epsilon} \|_{\frac{3}{2}} \leq \| u_0 \|_{\frac{3}{2}} \quad \text{and} \quad \| g_{j,\epsilon} \|_{\frac{3}{2}} \leq \| g_j \|_{\frac{3}{2}} \] (2.30)

for all \( \epsilon > 0, \ j \geq 1, \ u = u_\epsilon \) satisfies (2.29) with \( n = 3 \), and a positive constant \( C_T \) independent of \( \epsilon \). With this \( u = u_\epsilon \), we apply Ito’s rule to the functional

\[ \log(1 + \left\| J^s u(t) \right\|_0^2), \]
and obtain

\[
\log\left(1 + \| J^s u(t) \|_0^2 \right) = \log\left(1 + \| J^s u_0 \|_0^2 \right) + \int_0^t \frac{2(J^s u_x, J^s(ux x) - J^s(uxx)))_0}{1 + \| J^s u \|_0^2} d\eta
\]
\[
+ \int_0^t \frac{2(J^s u_x - \epsilon J^s u_{xxxx} - \epsilon J^s u)_0}{1 + \| J^s u \|_0^2} d\eta + \sum_{j=1}^{\infty} \int_0^t \frac{2(J^s u, J^s g_j)_0}{1 + \| J^s u \|_0^2} dB_j
\]
\[
+ \sum_{j=1}^{\infty} \int_0^t \left( \frac{(J^s g_j, J^s g_j)_0}{1 + \| J^s u \|_0^2} - \frac{2((J^s u, J^s g_j)_0)^2}{(1 + \| J^s u \|_0^2)^2} \right) d\eta
\]  
(2.31)

for all \( t \in [0, T] \), for almost all \( \omega \).

Set

\[
h(t) = E\left( \sup_{0 \leq \eta \leq t \wedge T_3, N \wedge T_2, M} \log\left(1 + \| J^s u(\eta) \|_0^2 \right) \right),
\]  
(2.32)

where \( T_3, N \) and \( T_2, M \) are defined by (2.13) with \( N \geq 1 \), and \( M \geq 1 \), respectively.

By Lemmas 1.4 and 1.5, we have

\[
|\langle J^s(ux x), J^s u \rangle_0| \leq C \| u_x \|_{L^\infty(R)} \| J^s(u) \|_0^2
\]
\[
\leq C \left( 1 + \| u \|_{3/2} \sqrt{\log(1 + \| J^s u \|_0)} \right) \| J^s(u) \|_0^2.
\]  
(2.33)

It follows from (2.30)–(2.33) and the Burkholder–Davis–Gundy inequality that

\[
h(t) \leq \log\left(1 + \| J^s u_0 \|_0^2 \right) + C^* N \int_0^t h(\eta) d\eta + C \sum_{j=1}^{\infty} \int_0^T \| g_j \|^2_s \eta d\eta + C
\]  
(2.34)

for all \( 0 \leq t \leq T \), which yields

\[
h(T) = E\left( \sup_{0 \leq \eta \leq t \wedge T_3, N \wedge T_2, M} \log\left(1 + \| J^s u(\eta) \|_0^2 \right) \right) \leq C^*_T e^{C^* N T}
\]  
(2.35)

where \( C^*_T \) and \( C^* \) are positive constants independent of \( M, N \) and \( \epsilon \).
By passing $M \to \infty$, we derive from (2.35) that
\[ E \left( \sup_{0 \leq \eta \leq T_{3,N}} \log(1 + \| J^3 u(\eta) \|_0^2) \right) \leq C_T^* e^{C^* N T}. \] (2.36)

We now recall (2.10) and define
\[ W(t) = \sum_{j=1}^{\infty} \int_0^t g_j(\eta) dB_j(\eta). \] (2.37)

Then, there is a sequence \( \{m_k\}_{k=1}^{\infty} \) of increasing natural numbers such that
\[ W_{\epsilon_k} \to W \quad \text{in} \quad C([0, T]; H^s(R)) \] (2.38)

as $k \to \infty$, for almost all $\omega$, where
\[ \epsilon_k = 1/m_k. \] (2.39)

Next we denote by $u_k$ the solution of (2.6) and (2.7) for $\epsilon = \epsilon_k$, $k = 1, 2, \ldots$, and let $T_{3,N}^{(k)}$ stand for the stopping time defined by (2.13) with $u = u_k$ and $\theta = 3$. We then write
\[ G_{N,k} = \left\{ \sup_{0 \leq t \leq T} \| u_k(t) \|_{3/2} \leq N \right\} \cap \left\{ \| \partial_t (u_k - W_{\epsilon_k}) \|_{L^\infty(0,T;H^{-5/2}(R))} \leq N \right\} \cap \left\{ \sup_{0 \leq t \leq T_{3,N}^{(k)}} \log(1 + \| u_k(t) \|_s^2) \leq N e^{C^* N T} \right\}. \] (2.40)

Obviously, $T_{3,N}(\omega) = T$ for $\omega \in G_{N,k}$. By (2.29) and (2.36), it is apparent that
\[ P(G_{N,k}) \geq 1 - \frac{C_T}{N} - \frac{C_T}{N^2} - \frac{C_T^*}{N}, \] (2.41)

for all $N \geq 1$ and $k \geq 1$, and hence,
\[ P \left( \bigcup_{N=1}^{\infty} \bigcap_{\nu=1}^{\infty} \bigcup_{k=\nu}^{\infty} G_{N,k} \right) = 1. \] (2.42)

This implies that there is a subset
\[ \hat{\Omega} \subset \bigcup_{N=1}^{\infty} \bigcap_{\nu=1}^{\infty} \bigcup_{k=\nu}^{\infty} G_{N,k} \] (2.43)

with the following properties:
(i) \( P(\Omega \setminus \tilde{\Omega}) = 0; \)
(ii) for each \( \omega \in \tilde{\Omega}, \) \( u_k \in \bigcap_{\nu=1}^{\infty} C([0, T]; H^\nu(R)) \) satisfies (2.6) and (2.7) with \( \epsilon = \epsilon_k \) for all \( k \geq 1; \)
(iii) for each \( \omega \in \tilde{\Omega}, \)
\[
W_{\epsilon_k} \to W \quad \text{in} \quad C\left([0, T]; H^s(R)\right)
\]
as \( k \to \infty; \)
(iv) for each \( \omega \in \tilde{\Omega}, \) there is a subsequence \( \{k_j\}_{j=1}^{\infty} \) depending on \( \omega \) such that \( \{u_{k_j}\}_{j=1}^{\infty} \) satisfies
\[
\|u_{k_j}\|_{C([0,T];H^s(R))} + \|\partial_t(u_{k_j} - W_{\epsilon_{k_j}})\|_{L^\infty(0,T;H^{-5/2}(R))} \leq M(\omega) \quad (2.44)
\]
for all \( j \geq 1, \) for some constant \( M(\omega) \geq 1, \) and
\[
\begin{align*}
&u_{k_j} \to u \quad \text{weak star in} \quad L^\infty\left(0, T; H^s(R)\right), \quad (2.45) \\
&\partial_t(u_{k_j} - W_{\epsilon_{k_j}}) \to \partial_t(u - W) \quad \text{weak star in} \quad L^\infty\left(0, T; H^{-5/2}(R)\right) \quad (2.46)
\end{align*}
\]
for some \( u. \)

It follows from (2.44)–(2.46) and Lemma 1.1 that
\[
u \in C_w(0, T; H^s(R)) \quad (2.47)
\]
for each \( \omega \in \tilde{\Omega}. \) By virtue of (2.45), (2.46) and Lemma 1.2, we can use the diagonal process to extract a subsequence still denoted by \( \{u_{k_j}\}_{j=1}^{\infty} \) which converges to \( u \) strongly in \( C([0, T]; H^{1}(I)), \) for every bounded space interval \( I. \) Thus, \( u \) satisfies
\[
\partial_t(u - W) = -uu_x - \mathcal{H}(u_{xx}) \quad (2.48)
\]
in the sense of distributions over \( (0, T) \times R. \)

For each \( \omega \in \tilde{\Omega}, \) \( u = u(\omega) \) is uniquely determined by the following uniqueness result; see [20, p. 33].

**Lemma 2.3.** Let \( v_1 \) and \( v_2 \) belong to \( C_w(0, T; H^0(R)), \) \( \theta > 3/2. \) Suppose that \( v_1(0) = v_2(0) \) in \( H^0(R), \) and that
\[
\partial_t(v_1 - v_2) = v_2 v_{2x} - v_1 v_{1x} + \mathcal{H}(v_{2xx}) - \mathcal{H}(v_{1xx})
\]
holds in the sense of distributions over \( (0, T) \times R. \) Then, \( v_1 \equiv v_2. \)
Next we show that $u(t)$ is $H^s(R)$-valued $\mathcal{F}_t$-measurable for each $t$. Choose any closed ball $D$ in $H^1(R)$ and $\psi \in C_0^\infty (R)$. We claim that

$$\tilde{\Omega} \cap \{ \psi u(T) \in D \} = \tilde{\Omega} \cap \bigcap_{\sigma=1}^\infty \bigcap_{N=1}^\infty \bigcap_{v=1}^\infty \bigcap_{k=1}^\infty \{ \psi u_k(T) \in D_{\sigma} \} \cap G_{N,k}$$ (2.49)

where $D_{\sigma} = \{ y | \| y - z \|_{H^1(R)} \leq 1/\sigma, \text{for some } z \in D \}$. Suppose that $\omega$ belongs to the left-hand side. According to the construction of $u$ above, there is a subsequence $\{u_{k_j}\}_{j=1}^\infty$ such that $\omega \in G_{N,k_j}$ for all $j \geq 1$, for some $N \geq 1$, and $\{u_{k_j}\}_{j=1}^\infty$ satisfies (2.44)–(2.46), which implies

$$\psi u_{k_j}(T) \to \psi u(T) \text{ in } H^1(R).$$ (2.50)

Hence, $\omega$ belongs to the right-hand side. Next suppose $\omega$ belongs to the right-hand side. Then, there is a subsequence $\{u_{k_j}\}_{j=1}^\infty$ which satisfies (2.44)–(2.46). Hence, $\omega$ belongs to the left-hand side. Thus, $u(T)$ is $H^1(R)$-valued $\mathcal{F}_T$-measurable. If $0 < t^* < T$, then $u(t^*)$ is $H^1(R)$-valued $\mathcal{F}_{t^*}$-measurable by considering the uniqueness of a solution on the interval $[0, t^*]$ in the function class $C_w(0, t^*; H^s(R))$. Since every Borel subset of $H^s(R)$ is a Borel subset of $H^1(R)$ and $u$ is $H^s(R)$-valued weakly continuous in $t$, $u(t^*)$ is $H^s(R)$-valued $\mathcal{F}_{t^*}$-measurable.

Next we will show that for almost all $\omega$,

$$u \in C([0, T]; H^s(R)).$$ (2.51)

Since $u \in C_w(0, T; H^s(R))$, for almost all $\omega$, it is enough to show that

$$\| u(\eta) \|_s \to \| u(t) \|_s, \text{ as } \eta \to t$$

for each $t \in [0, T]$. For this, we apply Itô’s rule to the functional

$$\| (J^s u(t)) \ast \rho_\varepsilon \|_0^2$$

where $\rho_\varepsilon$ is the Friedrichs mollifier and the convolution is taken with respect to the space variable $x$:

$$\| (J^s u(t_2)) \ast \rho_\varepsilon \|_0^2 - \| (J^s u(t_1)) \ast \rho_\varepsilon \|_0^2 = -2 \int_{t_1}^{t_2} \left< (J^s(uu_x)) \ast \rho_\varepsilon, (J^s u) \ast \rho_\varepsilon \right>_0 \, d\eta$$

$$- 2 \sum_{j=1}^\infty \int_{t_1}^{t_2} \left< (J^s u) \ast \rho_\varepsilon, (J^s g_j) \ast \rho_\varepsilon \right>_0 \, dB_j(\eta)$$

$$+ \sum_{j=1}^\infty \int_{t_1}^{t_2} \| (J^s g_j) \ast \rho_\varepsilon \|_0^2 \, d\eta,$$ (2.52)
for all \( t_1 < t_2 \) on the interval \([0, T_{2s,N}(\omega)]\), for almost all \( \omega \). Here \( T_{2s,N} \) is a stopping time defined by (2.13) with \( u \) in (2.47). We can write

\[
(J^s(uu_x)) * \rho_\epsilon = u((J^5u_x) * \rho_\epsilon) + (uJ^5u_x) * \rho_\epsilon - u((J^5u_x) * \rho_\epsilon) + (J^5(uu_x) - uJ^5u_x) * \rho_\epsilon.
\]

(2.53)

By integration by parts, we have

\[
\left| \left< u((J^5u_x) * \rho_\epsilon), (J^5u) * \rho_\epsilon \right>_0 \right| \leq C \|u\|_{s}^3,
\]

(2.54)

and, by Lemmas 1.3 and 1.5, we also obtain

\[
\left\| (uJ^5u_x) * \rho_\epsilon - u((J^5u_x) * \rho_\epsilon) \right\|_0 \leq C \|u\|_{s}^2
\]

(2.55)

and

\[
\left\| (J^5(uu_x) - uJ^5u_x) * \rho_\epsilon \right\|_0 \leq C \|u\|_{s}^2,
\]

(2.56)

for some positive constants \( C \) independent of \( u \) and \( \epsilon > 0 \). It follows that

\[
\lim_{\epsilon \to 0} \left( \left\| \int_{t_1}^{t_2} (J^5(uu_x)) * \rho_\epsilon, (J^5u) * \rho_\epsilon \right\|_0 d\eta \right) \leq C |t_2 - t_1| \|u\|_{L_{s}^\infty(0,T;H^s(R))}^3,
\]

(2.57)

for all \( t_1 < t_2 \) on \([0, T]\), for almost all \( \omega \), where \( C > 0 \) is a constant independent of \( u \) and \( \omega \). Next we note that

\[
\sum_{j=1}^{\infty} \int_{0}^{T_{2s,N}} (J^s u) * \rho_\epsilon, (J^s g_j) * \rho_\epsilon \right\|_0 dB_j(\eta) \text{ and } \sum_{j=1}^{\infty} \int_{0}^{T_{2s,N}} (J^s u, J^s g_j)_0 dB_j(\eta)
\]

are continuous local martingales since (2.47) holds for almost all \( \omega \). By virtue of the Burkholder–Davis–Gundy inequality and the bounded convergence theorem, we see that

\[
E \left( \sup_{0 \leq t \leq T_{2s,N}} \left| \sum_{j=1}^{\infty} \int_{0}^{t} (J^s u, J^s g_j)_0 - (\langle J^s u \rangle * \rho_\epsilon, \langle J^s g_j \rangle * \rho_\epsilon \rangle_0) dB_j(\eta) \right| \right)
\]

\[
\leq CE \left( \sum_{j=1}^{\infty} \int_{0}^{T_{2s,N}} \left( \| J^s u - (J^s u) * \rho_\epsilon \|_0^2 \right) \| J^s g_j \|_0^2 dh_j \right)^{1/2} + \left( \| J^s u * \rho_\epsilon \|_0^2 \right) \| J^s g_j - (J^s g_j) * \rho_\epsilon \|_0^2 dh_j \}
\]

\[
\to 0, \quad \text{as } \epsilon \to 0.
\]

(2.58)
Hence, there is a sequence \( \{ \epsilon_k \}_{k=1}^\infty \) such that \( \epsilon_k \to 0 \), and

\[
\lim_{k \to \infty} \sum_{j=1}^{\infty} \int_{t_1}^{t_2} \langle \rho_{\epsilon_k} \ast (J^s g_j) \ast \rho_{\epsilon_k} \rangle_0 dB_j(\eta) = \sum_{j=1}^{\infty} \int_{t_1}^{t_2} \langle J^s u, J^s g_j \rangle_0 dB_j(\eta)
\]

(2.59)

for all \( 0 \leq t_1 < t_2 \leq T_{2s,N}(\omega) \), for almost all \( \omega \). Meanwhile, for almost all \( \omega \), there is \( 1 \leq N(\omega) < \infty \) such that \( T_{2s,N}(\omega) = T \), for all \( N \geq N(\omega) \). Hence, by the diagonal process, we can extract a subsequence still denoted by \( \{ \epsilon_k \}_{k=1}^\infty \) such that (2.59) holds for all \( 0 \leq t_1 < t_2 \leq T \), for almost all \( \omega \).

It follows that

\[
\| J^s u(t_2) \|_0^2 - \| J^s u(t_1) \|_0^2 + 2 \sum_{j=1}^{\infty} \int_{t_1}^{t_2} \langle J^s u, J^s g_j \rangle_0 dB_j(\eta) - \sum_{j=1}^{\infty} \int_{t_1}^{t_2} \| J^s g_j \|_0^2 d\eta \leq C(t_2 - t_1) \| u \|_{L^\infty(0,T; H^s(R))}^3
\]

(2.60)

for all \( t_1 < t_2 \) on the interval \([0, T]\), for almost all \( \omega \). Since

\[
\sum_{j=1}^{\infty} \int_{0}^{(1)} \langle J^s u, J^s g_j \rangle_0 dB_j(\eta) \in C([0, T])
\]

for almost all \( \omega \), it holds that

\[
\lim_{\eta \to t} \| J^s u(\eta) \|_0^2 = \| J^s u(t) \|_0^2
\]

(2.61)

for every \( t \in [0, T] \), for almost all \( \omega \). This implies that \( u \in C([0, T]; H^s(R)) \), for almost all \( \omega \). Now the proof of Theorem 2.1 is complete.

2.4. Construction of solutions for \( s = 2 \)

Here we assume (2.1) with \( s = 2 \) and (2.4). Let us fix any \( \gamma \geq 9 \), and \( T > 0 \). Let \( \hat{\Omega} \) be a subset such that \( P(\Omega \setminus \hat{\Omega}) = 0 \), and for each \( \omega \in \hat{\Omega} \),

\[
u_0, \epsilon \in H^\nu(R), \\
W_\epsilon \in C([0, T]; H^\nu(R))
\]

for all \( \nu \geq 1 \), and all \( \epsilon > 0 \). Hence, for each \( \omega \in \hat{\Omega} \), we can solve for \( v \) of (2.12), and there is a unique solution \( u_\epsilon \in C([0, T]; H^\nu(R)) \), for all \( \nu \geq 1 \). The mapping

\[
(u_0, \epsilon, W_\epsilon) \mapsto u_\epsilon
\]
is continuous from \( L^2(R) \times C([0, T]; H^4(R)) \) into \( C([0, T]; L^2(R)) \). It follows that \( u_\epsilon \) is \( H^v(R) \)-valued progressively measurable for all \( v \geq 1 \). Let \( u = u_\epsilon \) be a solution of (2.6), (2.7). By virtue of (2.17) and (2.29) with \( n = 4 \), we have

\[
E\left( \sup_{0 \leq t \leq T} \| u(t) \|_2^2 \right) + E\left( \sup_{0 \leq t \leq T} \| u(t) \|_{2^\gamma}^2 \right) + E\left( \| \partial_t (u - W) \|_{L^\infty(0,T;H^{-2}(R))} \right) \leq C_T \tag{2.62}
\]

for all \( 0 < \epsilon \leq 1 \), where \( C_T \) is a positive constant depending only on

\[
E(\| u_0 \|_2^2), \quad E(\| u_0 \|_{0}^{2^\gamma}) \quad \text{and} \quad \sum_{j=1}^{\infty} \int_0^T \| g_j \|_2^2 \, dt.
\]

We also recall (2.37)–(2.39) for \( s = 2 \). Let \( u_k \) be the solution of (2.6) and (2.7) for \( \epsilon = \epsilon_k \), \( k = 1, 2, \ldots \). Instead of \( G_{N,k} \) defined by (2.40), we will use

\[
G_{N,k} = \left\{ \sup_{0 \leq t \leq T} \| u_k(t) \|_2 \leq N \right\} \cap \left\{ \| \partial_t (u_k - W_{\epsilon_k}) \|_{L^\infty(0,T;H^{-2}(R))} \leq N \right\}. \tag{2.63}
\]

It follows from (2.62) that

\[
P\left( \bigcup_{N=1}^{\infty} \bigcap_{\nu=1}^{\infty} \bigcup_{k=\nu}^{\infty} G_{N,k} \right) = 1. \tag{2.64}
\]

Hence, there is a subset

\[
\Omega^* \subset \bigcup_{N=1}^{\infty} \bigcap_{\nu=1}^{\infty} \bigcup_{k=\nu}^{\infty} G_{N,k} \tag{2.65}
\]

with the following properties:

(i) \( P(\Omega \setminus \Omega^*) = 0 \);
(ii) for each \( \omega \in \Omega^* \), \( u_k \in \bigcap_{\nu=1}^{\infty} C([0, T]; H^v(R)) \) satisfies (2.6) and (2.7) with \( \epsilon = \epsilon_k \) for all \( k \geq 1 \);
(iii) for each \( \omega \in \Omega^* \),

\[
W_{\epsilon_k} \to W \quad \text{in} \quad C([0, T]; H^2(R))
\]

as \( k \to \infty \);
(iv) for each \( \omega \in \Omega^* \), there is a subsequence \( \{k_j\}_{j=1}^{\infty} \) depending on \( \omega \) such that \( \{u_{k_j}\}_{j=1}^{\infty} \) satisfies

\[
\| u_{k_j} \|_{C([0,T];H^2(R))} \leq N \tag{2.66}
\]
for all \( j \geq 1 \), for some \( N \geq 1 \), and
\[
uk_j \to u \quad \text{weak star in } L^{\infty}(0, T; H^2(R)),
\]
\[
\partial_t(uk_j - W_{\epsilon kj}) \to \partial_t(u - W) \quad \text{weak star in } L^{\infty}(0, T; H^{-2}(R))
\]
for some \( u \).

By repeating the same argument as above, \( u = u(\omega) \) is uniquely determined for each \( \omega \in \Omega^* \). This \( u \) is \( H^2(R) \)-valued progressively measurable, and is a solution of (0.1), (0.2). Furthermore,
\[
u \in C \left( [0, T]; H^2(R) \right), \quad \text{for almost all } \omega.
\]

It remains to show (2.5).

Choose any positive number \( K \). We will show the inequality
\[
\left( \| u(\omega) \|^2_{C([0,T];H^2(R))} + \| u(\omega) \|^2\gamma_{C([0,T];L^2(R))} \right) \wedge K \\
\leq \lim_{k \to \infty} \left( \| u_k(\omega) \|^2_{C([0,T];H^2(R))} + \| u_k(\omega) \|^2\gamma_{C([0,T];L^2(R))} \right) \wedge K
\]
for each \( \omega \in \Omega^* \).

If \( \lim_{k \to \infty} \left( \| u_k(\omega) \|^2_{C([0,T];H^2(R))} + \| u_k(\omega) \|^2\gamma_{C([0,T];L^2(R))} \right) \wedge K = K \), then the inequality is obvious.

If \( \lim_{k \to \infty} \left( \| u_k(\omega) \|^2_{C([0,T];H^2(R))} + \| u_k(\omega) \|^2\gamma_{C([0,T];L^2(R))} \right) \wedge K < K \), then there is a subsequence \( \{u_{kj}(\omega)\}_{j=1}^{\infty} \) such that
\[
\lim_{k \to \infty} \left( \| u_k(\omega) \|^2_{C([0,T];H^2(R))} + \| u_k(\omega) \|^2\gamma_{C([0,T];L^2(R))} \right) = \lim_{j \to \infty} \left( \| u_{kj}(\omega) \|^2_{C([0,T];H^2(R))} + \| u_{kj}(\omega) \|^2\gamma_{C([0,T];L^2(R))} \right) < K,
\]
and
\[
u_{kj}(\omega) \to u^* \quad \text{weak star in } L^{\infty}(0, T; H^2(R))
\]
for some function \( u^* \). By (2.72) and the equation
\[
\partial_t(u_{kj} - W_{\epsilon kj}) = -u_{kj}u_{kj,x} - H(u_{kj,xx}) - \epsilon_{kj} \left( \partial_x^4 u_{kj} + u_{kj} \right),
\]
we find that \( \partial_t(u_{kj} - W_{\epsilon kj}) \) is uniformly bounded in \( C([0, T]; H^{-2}(R)) \). As above, we use Lemma 1.2 to find that \( u^* \) must satisfy (0.1) and (0.2). By the pathwise uniqueness of a solution at this \( \omega \), \( u^* = u(\omega) \) holds. Thus,
\[ \|u(\omega)\|_{C([0,T];H^2(R))}^2 + \|u(\omega)\|_{C([0,T];L^2(R))}^{2\gamma} \leq \lim_{j \to \infty} \|u_{kj}(\omega)\|_{C([0,T];H^2(R))}^2 + \lim_{j \to \infty} \|u_{kj}(\omega)\|_{C([0,T];L^2(R))}^{2\gamma} \]
\[ \leq \lim_{j \to \infty} \left( \|u_{kj}(\omega)\|_{C([0,T];H^2(R))}^2 + \|u_{kj}(\omega)\|_{C([0,T];L^2(R))}^{2\gamma} \right). \quad (2.73) \]

Hence, (2.70) is true. By Fatou’s lemma, it holds that
\[ \int_{\Omega} \left( \|u\|_{C([0,T];H^2(R))}^2 + \|u\|_{C([0,T];L^2(R))}^{2\gamma} \right) dP \leq \lim_{k \to \infty} \int_{\Omega} \left( \|u_k(\omega)\|_{C([0,T];H^2(R))}^2 + \|u_k(\omega)\|_{C([0,T];L^2(R))}^{2\gamma} \right) dP \]
\[ \leq \lim_{k \to \infty} \int_{\Omega} \left( \|u_k(\omega)\|_{C([0,T];H^2(R))}^2 + \|u_k(\omega)\|_{C([0,T];L^2(R))}^{2\gamma} \right) dP \leq C_T \quad (2.74) \]

where \( C_T \) is the same positive constant as in (2.62). By passing \( K \uparrow \infty \), we arrive at
\[ \int_{\Omega} \left( \|u\|_{C([0,T];H^2(R))}^2 + \|u\|_{C([0,T];L^2(R))}^{2\gamma} \right) dP \leq C_T \quad (2.75) \]

where \( C_T \) is the same as above, and depends only on
\[ E\left(\|u_0\|_{L^2}^2\right), \quad E\left(\|u_0\|_{L^2}^{2\gamma}\right) \quad \text{and} \quad \sum_{j=1}^{\infty} \int_0^T \|g_j\|_{L^2}^2 dt. \]

This yields (2.5), and the proof of Theorem 2.2 is complete.

3. Invariant measures

Here we consider Eq. (0.1) with an additional term.
\[ u_t + uu_x + \mathcal{H}(u_{xx}) + \alpha u = \sum_{j=1}^{\infty} g_j \frac{dB_j}{dt} \quad (3.1) \]

where \( \alpha > 0 \) is a constant. The additional term \( \alpha u \) does not change any result in the above sections. Throughout this section, we assume \( g_j = g_j(x) \) for all \( j \geq 1 \), and
\[ \sum_{j=1}^{\infty} \|g_j\|_{H^2(R)}^2 < \infty. \quad (3.2) \]
Theorem 3.1. Under the assumption (3.2), there is an invariant measure for (3.1) with support in $H^2(R)$.

We first establish the necessary energy estimates under assumption (3.2).

Lemma 3.2. Let $u$ be the solution in Theorem 2.2. Then, it holds that

$$
\frac{1}{T} \int_0^T E(\|u(t)\|_2^2) \, dt \leq M, \tag{3.3}
$$

for all $T \geq 1$, for some constant $M > 0$.

Proof. Choose any $T > 0$. $\gamma \geq 9$ is the same as in Theorem 2.2. Since (2.5) holds, it follows from Ito’s rule that

$$
\begin{align*}
\|u(t)\|_0^{2\gamma} &= \|u_0\|_0^{2\gamma} - 2\gamma \alpha \int_0^t \|u(\eta)\|_0^{2\gamma} \, d\eta + 2\gamma \sum_{j=1}^\infty \int_0^t \|u\|_0^{2\gamma - 2} \langle u, g_j \rangle_0 \, dB_j(\eta) \\
&\quad + 2\gamma (\gamma - 1) \sum_{j=1}^\infty \int_0^t \|u\|_0^{2(\gamma - 2)} \|g_j\|_0^2 \, d\eta + \gamma \sum_{j=1}^\infty \int_0^t \|u\|_0^{2\gamma - 2} \|g_j\|_0^2 \, d\eta.
\end{align*}
$$

(3.4)

for all $t \in [0, T]$, for almost all $\omega$. By taking the expectation,

$$
\begin{align*}
E(\|u(t)\|_0^{2\gamma}) &= E(\|u_0\|_0^{2\gamma}) - 2\gamma \alpha \int_0^t E(\|u(\eta)\|_0^{2\gamma}) \, d\eta \\
&\quad + 2\gamma (\gamma - 1) E \left( \sum_{j=1}^\infty \int_0^t \|u\|_0^{2(\gamma - 2)} \|g_j\|_0^2 \, d\eta \right) \\
&\quad + \gamma E \left( \sum_{j=1}^\infty \int_0^t \|u\|_0^{2\gamma - 2} \|g_j\|_0^2 \, d\eta \right), \tag{3.5}
\end{align*}
$$

for all $t \in [0, T]$. We also have

$$
E \left( \sum_{j=1}^\infty \int_0^t \|u(\eta)\|_0^{2\gamma - 2} \|g_j\|_0^2 \, d\eta \right) \leq \delta \int_0^t E(\|u(\eta)\|_0^{2\gamma}) \, d\eta + C_\delta \left( \sum_{j=1}^\infty \|g_j\|_0^\gamma \right)^\gamma. \tag{3.6}
$$
for all \( \delta > 0 \), and all \( t \in [0, T] \), for some constant \( C_\delta > 0 \) independent of \( T > 0 \). It follows from (3.5) and (3.6) that

\[
E(\|u(t)\|_0^{2\gamma}) + \gamma \alpha \int_0^t E(\|u(\eta)\|_0^{2\gamma}) \, d\eta \leq Mt + M, \tag{3.7}
\]

for all \( t \in [0, T] \), where \( M \) denotes positive constants independent of \( T > 0 \).

Again by Ito’s rule and (2.28), we have

\[
I_4(u(t)) - I_4(u(0)) = -\alpha \int_0^t DI_4(u(\eta)) \cdot u(\eta) \, d\eta + \sum_{j=1}^\infty \int_0^t DI_4(u(\eta)) \cdot g_j \, dB_j(\eta)
\]

\[
+ \frac{1}{2} \sum_{j=1}^\infty \int_0^t D^2I_4(u(\eta)) \cdot (g_j, g_j) \, d\eta, \tag{3.8}
\]

for all \( t \in [0, T] \), for almost all \( \omega \). Here we need a different version of (2.25). It follows from (2.24) that

\[
\sum_{j=1}^\infty \int_0^t \left| D^2J_{4,m}(u) \cdot (g_j, g_j) \right| \, d\eta 
\]

\[
\leq \delta \int_0^t \|u(\eta)\|_2^2 \, d\eta + C(\delta, m) \left( \int_0^t (1 + \|u(\eta)\|_0^{10}) \, d\eta \right) \left( 1 + \sum_{j=1}^\infty \|g_j\|_2^2 \right)^8, \tag{3.9}
\]

for all \( \delta > 0, t \in [0, T] \), for some constant \( C(\delta, m) > 0 \) independent of \( T > 0 \). Other terms in the last integral of (3.8) are easy to estimate. By combining (2.26), (2.27) and (3.7)–(3.9), we arrive at

\[
\int_0^t E(\|u(\eta)\|_2^2) \, d\eta \leq Mt + M, \tag{3.10}
\]

for all \( t \in [0, T] \), where \( M \) denotes positive constants independent of \( T > 0 \). \( \square \)

Next we will show that the solution process is a Markov process. We need to introduce a new function class:

\[
\mathcal{Y} = \{ f \in L^2_{\text{loc}}(R) \mid (1 + x^2)^{-1} f \in L^2(R) \} \tag{3.11}
\]

equipped with the inner product

\[
\langle f, g \rangle_{\mathcal{Y}} \overset{\text{def}}{=} \langle (1 + x^2)^{-1} f, (1 + x^2)^{-1} g \rangle_0. \tag{3.12}
\]
Lemma 3.3. $H^2(R)$ is dense in $\mathcal{Y}$, and the embedding $H^2(R) \to \mathcal{Y}$ is compact.

Proof. Choose any $f \in \mathcal{Y}$ and $\epsilon > 0$. There is some $g \in C_0^\infty(R)$ such that

$$\| (1 + x^2)^{-1} f - g \|_{L^2(R)} < \epsilon.$$ 

Let $h = (1 + x^2)g \in C_0^\infty(R)$. Then,

$$\| f - h \|_{\mathcal{Y}} = \| (1 + x^2)^{-1} f - g \|_{L^2(R)} < \epsilon.$$ 

Thus, $C_0^\infty(R)$ is dense in $\mathcal{Y}$, and $H^2(R)$ is also dense in $\mathcal{Y}$. Next, let $\{f_m\}_{m=1}^\infty$ be a bounded sequence in $H^2(R)$. Then,

$$\| f_m \|_{H^2(R)} \leq M, \quad \text{for all } m \geq 1,$$

for some positive constant $M$. It follows that

$$\int_{|x| \geq K} (1 + x^2)^{-2} |f_m(x)|^2 \, dx \leq \frac{M}{K^4},$$

for all $m \geq 1$ and $K > 0$. Hence,

$$\| f_n - f_m \|_{\mathcal{Y}}^2 \leq \frac{4M}{K^4} + \int_{I_K} |f_n - f_m|^2 \, dx$$

where $I_K = \{x \in R \mid |x| < K\}$. For each $K > 0$, the embedding $H^2(I_K) \to L^2(I_K)$ is compact. Thus, by the diagonal process along expanding $I_K$ to $R$, we can extract a subsequence $\{f_{m_k}\}_{k=1}^\infty$ such that it is convergent in $L^2(I_K)$ for every $K > 0$. Thus, the subsequence $\{f_{m_k}\}_{k=1}^\infty$ is convergent in $\mathcal{Y}$. \[\square\]

Lemma 3.4. For any bounded continuous function $\psi$ on $H^2(R)$, there is a sequence of functions $\{\psi_k\}_{k=1}^\infty$ which are bounded uniformly in $k$, and continuous on $\mathcal{Y}$ such that

$$\psi_k(y) \to \psi(y), \quad \text{as } k \to \infty,$$

for each $y \in H^2(R)$.

Proof. Let $\psi$ be a bounded continuous function on $H^2(R)$. For each $y \in \mathcal{Y}$, we set

$$\Lambda_k(y) = (\chi_k y) \ast \rho_{1/k}$$

(3.13)

where $\ast$ denotes the convolution. Here $\rho_{1/k}$ is the Friedrichs mollifier, and

$$\chi_k(x) = \chi(x/k), \quad \text{for } k \geq 1,$$

(3.14)
where $\chi \in C_0^\infty (R)$ such that

$$
\chi(x) = 1, \quad \text{for } |x| \leq 1, \quad \text{and} \quad \chi(x) = 0, \quad \text{for } |x| \geq 2.
$$

Fix any $k \geq 1$. Let $y_m \to y$ in $\mathcal{Y}$, as $m \to \infty$.

It follows from the inequality

$$
\|f * g\|_0 \leq \|f\|_0 \|g\|_{L^1(R)}
$$

that

$$
\Lambda_k(y_m) \to \Lambda_k(y) \quad \text{in} \quad H^2(R), \quad \text{as} \quad m \to \infty. \quad (3.15)
$$

Next fix any $y \in H^2(R)$. It now holds that

$$
\|y - \Lambda_k(y)\|_2 \leq \|y - y * \rho_{1/k}\|_2 + \|(1 - \chi_k)y) * \rho_{1/k}\|_2
$$

$$
\leq \|y - y * \rho_{1/k}\|_2 + \|(1 - \chi_k)y\|_2 \|\rho_{1/k}\|_{L^1(R)} \to 0, \quad \text{as} \quad k \to \infty. \quad (3.16)
$$

Finally, we set

$$
\psi_k(y) = \psi(\Lambda_k(y)) \quad (3.17)
$$

so that the required properties be satisfied. □

Choose any $\eta \geq 0$. Let $X = X(t; \eta, \xi)$ be a solution of (0.1) for $t > \eta$ satisfying

$$
X(\eta; \eta, \xi) = \xi
$$

(3.18)

where $\xi$ is $H^2(R)$-valued $\mathcal{F}_\eta$-measurable such that

$$
\xi \in L^2(\Omega; H^2(R)) \cap L^{2\gamma}(\Omega; L^2(R)), \quad \gamma \geq 9.
$$

**Lemma 3.5.** Let $z \in H^2(R)$. For any $0 \leq \tau \leq \eta \leq t < \infty$, and any bounded continuous function $\psi$ on $H^2(R)$, it holds that

$$
E(\psi(X(t; \tau, z)) | \mathcal{F}_\eta) = P_{\eta,t}(\psi)(X(\eta; \tau, z)), \quad (3.19)
$$

for almost all $\omega$, where the operator $P_{\eta,t}$ is defined by

$$
P_{\eta,t}(\psi)(y) = E(\psi(X(t; \eta, y))), \quad \text{for} \quad y \in H^2(R).
$$
Proof. By the pathwise uniqueness of a solution, we have

\[ X(t; \tau, z) = X(t; \eta, X(\eta; \tau, z)), \quad \text{for almost all } \omega. \]

By setting \( \xi = X(\eta; \tau, z) \), (3.19) can be written as

\[ E\left( \psi(X(t; \eta, \xi)) \mid \mathcal{F}_\eta \right) = P_{\eta,t}(\psi)(\xi), \quad (3.20) \]

for almost all \( \omega \), where \( \xi = X(\eta; \tau, z) \). Let \( \{\xi_k\}_{k=1}^\infty \) be a sequence of \( H^2(R) \)-valued \( \mathcal{F}_\eta \)-measurable step maps such that

\[ \xi_k \to \xi \quad \text{in } L^2(\Omega; H^2(R)) \cap L^{2\gamma}(\Omega; L^2(R)), \quad (3.21) \]

and, for almost all \( \omega \),

\[ \xi_k \to \xi \quad \text{in } H^2(R), \quad \text{as } k \to \infty. \quad (3.22) \]

Let \( \Omega^\dagger \) be a subset such that \( P(\Omega \setminus \Omega^\dagger) = 0 \), and for each \( \omega \in \Omega^\dagger \),

(i) \( X(\cdot; \eta, \xi) \) and \( X(\cdot; \eta, \xi_k) \) belong to \( C([\eta, \infty); H^2(R)) \) for all \( k \geq 1 \);
(ii) \( W(\omega) \in C([0, \infty); H^2(R)) \), where \( W \) was defined by (2.37);
(iii) \( u = X(\cdot; \eta, \xi) \) and \( u = X(\cdot; \eta, \xi_k), k \geq 1 \), satisfy

\[ \partial_t(u - W) + uu_x + \mathcal{H}(u_{xx}) + au = 0 \quad (3.23) \]

in the sense of distributions over \((\eta, \infty) \times R\);
(iv) \( \xi_k(\omega) \to \xi(\omega) \) in \( H^2(R), \) as \( k \to \infty \).

Let \( \phi \) be a bounded continuous function on \( Y \). We will show that

\[ \int_A \phi(X(T; \eta, \xi)) \, dP = \lim_{k \to \infty} \int_A \phi(X(T; \eta, \xi_k)) \, dP \quad (3.24) \]

for every \( A \in \mathcal{F}_\eta \), and \( T > \eta \). Here we note that for any fixed \( \omega \in \Omega^\dagger \), we do not have an estimate of \( X(T; \eta, \xi_k) \) uniform in \( k \). Thus, we need some extra work to show (3.24).

Define

\[ \mathcal{M}_{N,k} = \{ \omega \mid \| X(\cdot; \eta, \xi_k) \|_{C([\eta, T]; H^2(R))} \geq N \} \]

or \( \| \partial_t(X(\cdot; \eta, \xi_k) - W) \|_{C([\eta, T]; L^2(R))} \geq N \}. \quad (3.25) \]
We assume that \( \phi \) is nonnegative and define
\[
C = \sup_{z \in \mathcal{Y}} \phi(z). \tag{3.26}
\]

Let \( \chi_{N,k} \) be the characteristic function of the set \( \mathcal{M}_{N,k} \).

For given \( \epsilon > 0 \), we use (2.75) and (3.21) to choose \( N \) independent of \( k \) such that
\[
P(\mathcal{M}_{N,k}) < \epsilon. \tag{3.27}
\]

We will show that for each \( \omega \in \Omega^\dagger \),
\[
\phi(\{X(T; \eta, \xi)\}) \geq \lim_{k \to \infty} \phi(\{X(T; \eta, \xi_k)\})(1 - \chi_{N,k}). \tag{3.28}
\]

If \( \lim_{k \to \infty} \phi(\{X(T; \eta, \xi_k)\})(1 - \chi_{N,k}) = 0 \), for \( \omega = \omega^* \in \Omega^\dagger \), then it is true at \( \omega^* \).

If \( \lim_{k \to \infty} \phi(\{X(T; \eta, \xi_k)\})(1 - \chi_{N,k}) > 0 \), for \( \omega = \omega^* \in \Omega^\dagger \), then there is a subsequence \( \{k_m\}_{m=1}^\infty \) such that
\[
\lim_{k \to \infty} \phi(\{X(T; \eta, \xi_k)\})(1 - \chi_{N,k}) = \lim_{m \to \infty} \phi(\{X(T; \eta, \xi_{k_m})\}), \tag{3.29}
\]
\[
\|X(\cdot; \eta, \xi_{k_m})\|_{C([\eta,T]; H^2(R))} < N \tag{3.30}
\]
and
\[
\|\partial_t (X(\cdot; \eta, \xi_{k_m}) - W)\|_{C([\eta,T]; L^2(R))} < N, \tag{3.31}
\]
for all \( k_m \), at \( \omega = \omega^* \). By virtue of Lemmas 1.2, 3.3, and the pathwise uniqueness of a solution, it follows that
\[
X(T; \eta, \xi_{k_m}) \to X(T; \eta, \xi) \quad \text{in} \quad \mathcal{Y}, \quad \text{as} \quad m \to \infty, \tag{3.32}
\]
at \( \omega = \omega^* \), and
\[
\phi(\{X(T; \eta, \xi_{k_m})\}) \to \phi(\{X(T; \eta, \xi)\}).
\]

Thus, (3.28) holds. We then have
\[
\int_A \phi(\{X(T; \eta, \xi)\}) dP \geq \int_A \lim_{k \to \infty} \phi(\{X(T; \eta, \xi_k)\})(1 - \chi_{N,k}) dP
\]
\[
\geq \lim_{k \to \infty} \int_A \phi(\{X(T; \eta, \xi_k)\}) dP - C\epsilon, \tag{3.33}
\]
for all \( A \in \mathcal{F}_\eta \). Next we will show that for each \( \omega \in \Omega^\dagger \),
\[
\phi(X(T; \eta, \xi)) \leq \lim_{k \to \infty} \phi(X(T; \eta, \xi_k))(1 - \chi_{N,k}) \vee C \chi_{N,k}. \tag{3.34}
\]

If \(\lim_{k \to \infty} \phi(X(T; \eta, \xi_k))(1 - \chi_{N,k}) \vee C \chi_{N,k} = C\), for \(\omega = \omega^* \in \Omega^\dagger\), then (3.34) is obvious by (3.26).

If \(\lim_{k \to \infty} \phi(X(T; \eta, \xi_k))(1 - \chi_{N,k}) \vee C \chi_{N,k} < C\), for \(\omega = \omega^* \in \Omega^\dagger\), then there is a subsequence \(\{k_m\}_{m=1}^\infty\) such that

\[
\lim_{k \to \infty} \phi(X(T; \eta, \xi_k))(1 - \chi_{N,k}) \vee C \chi_{N,k} = \lim_{m \to \infty} \phi(X(T; \eta, \xi_{k_m})), \tag{3.35}
\]

and

\[
\|X(\cdot; \eta, \xi_{k_m})\|_{C([\eta, T]; H^2(R))} < N \tag{3.36}
\]

and

\[
\|\partial_t(X(\cdot; \eta, \xi_{k_m}) - W)\|_{C([\eta, T]; L^2(R))} < N, \tag{3.37}
\]

for all \(k_m\), at \(\omega = \omega^*\). It follows that

\[
\phi(X(T; \eta, \xi_{k_m})) \to \phi(X(T; \eta, \xi)) \tag{3.38}
\]

at \(\omega = \omega^*\). Thus, (3.34) holds. It follows that

\[
\int_A \phi(X(T; \eta, \xi)) \, dP \leq \int_A \lim_{k \to \infty} \phi(X(T; \eta, \xi_k))(1 - \chi_{N,k}) \vee C \chi_{N,k} \, dP
\]

\[
\leq \lim_{k \to \infty} \int_A \phi(X(T; \eta, \xi_k)) \, dP + C \epsilon, \tag{3.39}
\]

for all \(A \in \mathcal{F}_\eta\). Hence, we have (3.24) for every bounded continuous nonnegative function \(\phi\) on \(\mathcal{Y}\), and all \(A \in \mathcal{F}_\eta\). For a general bounded continuous function \(\phi\) on \(\mathcal{Y}\), (3.24) follows by writing

\[
\phi = \phi^+ - \phi^-.
\]

In the meantime, by (3.22), for each bounded continuous function \(\phi\) on \(\mathcal{Y}\),

\[
P_{\eta,t}(\phi)(\xi_k) \to P_{\eta,t}(\phi)(\xi), \quad \text{as} \ k \to \infty, \quad \text{for almost all} \ \omega. \tag{3.40}
\]

Since \(\xi_k\) is an \(\mathcal{F}_\eta\)-measurable step map, it is easy to see that for each \(k \geq 1\),

\[
E(\phi(X(t; \eta, \xi_k)) \mid \mathcal{F}_\eta) = P_{\eta,t}(\phi)(\xi_k) \tag{3.41}
\]

for almost all \(\omega\). Hence, it follows from (3.24) and (3.40) that
\[
\int_A E(\phi(X(t; \eta, \xi)) \mid \mathcal{F}_\eta) \, dP = \int_A P_{\eta,t}(\phi)(\xi) \, dP, 
\]
(3.42)
for every \(A \in \mathcal{F}_\eta\) and each bounded continuous function \(\phi\) on \(\mathcal{Y}\). If \(\psi\) is a bounded continuous function on \(H^2(R)\), let \(\{\psi_k\}_{k=1}^\infty\) be a sequence defined in Lemma 3.4. For each \(\psi_k\), (3.42) holds. Thus, it holds for \(\psi\), which yields (3.19) \(\square\)

**Lemma 3.6.** Let \(\{z_k\}_{k=1}^\infty\) be a bounded sequence in \(H^2(R)\) such that \(z_k \to z\) in \(\mathcal{Y}\), as \(k \to \infty\). For any \(T > 0\), and any bounded continuous function \(\phi\) on \(\mathcal{Y}\),

\[
E(\phi(X(T; 0, z_k))) \to E(\phi(X(T; 0, z))), \quad as \ k \to \infty.
\]

**Proof.** Here \(z_k \to z\) weakly in \(H^2(R)\). But we can repeat the same arguments as for (3.24), because (3.32) and (3.38) are still valid by virtue of the pathwise uniqueness of a solution. \(\square\)

Next we define a transition function

\[
\mathcal{P}(\eta, z; t, \Gamma) = P(X(t; \eta, z) \in \Gamma),
\]
for \(0 \leq \eta \leq t < \infty\), and \(\Gamma \in \mathcal{B}(H^2(R))\).

**Lemma 3.7.** For all \(0 \leq \eta \leq t < \infty\), and \(\Gamma \in \mathcal{B}(H^2(R))\), \(\mathcal{P}(\eta, \cdot; t, \Gamma)\) is \(\mathcal{B}(H^2(R))\)-measurable.

**Proof.** Fix any \(0 \leq \eta \leq t < \infty\). It follows from Lemma 3.6 that \(E(\phi(X(t; \eta, z)))\) is continuous in \(z \in H^2(R)\), for each bounded continuous function \(\phi\) on \(\mathcal{Y}\). Let \(\psi\) be a bounded continuous function on \(H^2(R)\). Let \(\{\psi_k\}_{k=1}^\infty\) be a sequence of bounded continuous functions on \(\mathcal{Y}\) constructed in Lemma 3.4. Then,

\[
E(\psi(X(t; \eta, z))) = \lim_{k \to \infty} E(\psi_k(X(t; \eta, z))), \quad for \ each \ z \in H^2(R).
\]

Hence, \(E(\psi(X(t; \eta, \cdot)))\) is \(\mathcal{B}(H^2(R))\)-measurable. It follows that \(\mathcal{P}(\eta, \cdot; t, \Gamma)\) is \(\mathcal{B}(H^2(R))\)-measurable for every \(\Gamma \in \mathcal{B}(H^2(R))\). \(\square\)

**Lemma 3.8.** For each \(t \geq 0\), \(\tau \geq 0\), \(z \in H^2(R)\) and \(\Gamma \in \mathcal{B}(H^2(R))\), it holds that

\[
\mathcal{P}(0, z; t, \Gamma) = \mathcal{P}(\tau, z; t + \tau, \Gamma).
\]
(3.43)

**Proof.** Fix any \(T > 0\), \(\tau > 0\) and \(z \in H^2(R)\). Let us write

\[
B_j^\tau(t) = B_j(t + \tau) - B_j(\tau),
\]
(3.44)
\[
W^\tau(t) = W(t + \tau) - W(\tau) = \sum_{j=1}^\infty g_j B_j^\tau(t)
\]
(3.45)
and, recalling (2.10),

$$W^*_\epsilon(t) = \sum_{j=1}^{\infty} g_{j,\epsilon} B^*_j(t).$$

Let $u_\epsilon$ and $u^*_\epsilon$ be solutions of (2.6) satisfying $u_\epsilon(0) = u^*_\epsilon(0) = z \ast \rho_\epsilon$, corresponding to $B_j$’s and $B^*_j$’s, respectively. Since the mapping $W_\epsilon \mapsto u_\epsilon$ is continuous from $C([0, T]; H^4(R))$ into $C([0, T]; L^2(R))$, for almost all $\omega$, and $W_\epsilon$ and $W^*_\epsilon$ have the same probability distribution, it holds that

$$P\left(u_\epsilon(t) \in \Gamma \right) = P\left(u^*_\epsilon(t) \in \Gamma \right),$$

for all $t \in [0, T]$, and for all $\Gamma \in \mathcal{B}(L^2(R))$. Next let $u$ be a solution of (0.1) satisfying $u(0) = z$. Replace $B_j$’s by $B^*_j$’s in (0.1), and let $u^*$ be the corresponding solution satisfying $u^*(0) = z$. By the same argument as in the proof of Lemma 3.5, we find that

$$\int_{\Omega} \psi(u(t)) \, dP = \lim_{k \to \infty} \int_{\Omega} \psi(u_{\epsilon_k}(t)) \, dP$$

and

$$\int_{\Omega} \psi(u^*(t)) \, dP = \lim_{k \to \infty} \int_{\Omega} \psi(u^*_\epsilon(t)) \, dP$$

for every bounded continuous function $\psi$ on $\mathcal{Y}$, and all $t \in [0, T]$. Here $\epsilon_k$ is the same as in (2.39) to accommodate the convergence of $W_{\epsilon_k}$ to $W$, and the convergence of $W^*_\epsilon$ to $W^*$ in (2.38) with $s = 2$. Since $L^2(R)$ is embedded into $\mathcal{Y}$, (3.47) implies that

$$\int_{\Omega} \psi(u_{\epsilon_k}(t)) \, dP = \int_{\Omega} \psi(u^*_\epsilon(t)) \, dP$$

for all $k \geq 1$. Hence it follows from (3.48) and (3.49) that

$$\int_{\Omega} \psi(u(t)) \, dP = \int_{\Omega} \psi(u^*(t)) \, dP.$$ 

This implies that (3.43) holds for all $\Gamma \in \mathcal{B}(\mathcal{Y})$, and hence, for all $\Gamma \in \mathcal{B}(H^2(R))$. \qed

We borrow the following result from [13].

Suppose that $X(t, s; z), 0 \leq s \leq t < \infty$, is a pathwise unique solution of a certain stochastic evolution equation such that $X(s, s; z) = z$. We define a function

$$\mathcal{P}(s, z; t, \Gamma) = P\left(X(t, s; z) \in \Gamma \right), \quad \text{for each } \Gamma \in \mathcal{B}(\mathcal{Y}), \ 0 \leq s \leq t < \infty, \ z \in \mathcal{Y}.$$
We make the following assumptions.

[I] $X(\cdot, s; z)$ is a $\Xi$-valued continuous process adapted to $\{F_t\}_{t \geq s}$ for each $z \in \Xi$ and $s \geq 0$, where $\Xi$ is a separable Banach space.

[II] $\mathcal{P}(\cdot; \cdot; \cdot, \cdot)$ is a time-homogeneous transition probability function. In other words, it satisfies the following conditions:

(i) $\mathcal{P}(s, z; t, \cdot)$ is a probability measure over $\{\Xi, B(\Xi)\}$ for all $z \in \Xi$, and $0 \leq s < t < \infty$;

(ii) $\mathcal{P}(s, \cdot; t, \Gamma)$ is $B(\Xi)$-measurable for all $0 \leq s < t < \infty$ and $\Gamma \in B(\Xi)$;

(iii) for all $0 \leq s < t < \xi < \infty$ and $\Gamma \in B(\Xi)$,

$$\mathcal{P}(s, z; \xi, \Gamma) = \int_\Xi \mathcal{P}(s, z; t, dy) \mathcal{P}(t, y; \xi, \Gamma);$$

(iv) $\mathcal{P}(s, \cdot; t, \cdot) = \mathcal{P}(s + h, \cdot; t + h, \cdot)$ for all $0 \leq s < t < \infty$ and $h > 0$.

[III] There is some $z \in \Xi$ such that

$$E\left(\|X(t, 0; z)\|_\Xi\right) \leq M, \quad \text{for all } t \geq 0,$$

for some positive constant $M$.

[IV] There is a Banach space $\Upsilon$ such that $\Xi \subset \Upsilon$, the embedding $\Xi \rightarrow \Upsilon$ is continuous, and each closed ball of finite radius in $\Xi$ is a compact subset of $\Upsilon$. Furthermore, for each bounded continuous function $\psi$ on $\Xi$, there is a sequence of continuous functions $\{\psi_k\}_{k=1}^\infty$ on $\Upsilon$ such that $\psi_k$ is bounded uniformly in $k$ and

$$\lim_{k \rightarrow \infty} \psi_k(y) = \psi(y), \quad \text{for each } y \in \Xi.$$

[V] For each fixed $0 \leq t < \infty$, and each fixed closed ball $S$ of finite radius in $\Xi$, if $\{z_n\}_{n=1}^\infty$ is a sequence in $S$ such that

$$z_n \rightarrow z \quad \text{in } \Upsilon,$$

then

$$E\left(\psi\left(X(t, 0; z_n)\right)\right) \rightarrow E\left(\psi\left(X(t, 0; z)\right)\right),$$

for every bounded continuous function $\psi$ on $\Upsilon$.

**Theorem.** [13] Under assumptions [I]–[V], there is an invariant measure for the above process $X(\cdot)$. In other words, there is a probability measure $\mu$ on $\Xi$ such that

$$\int_\Xi E\left(\psi\left(X(t, 0; y)\right)\right) \mu(dy) = \int_\Xi \psi(y) \mu(dy)$$

for all $t \geq 0$, and every bounded continuous function $\psi$ on $\Xi$. 
Here we take $\Xi = H^2(R)$ and $\Upsilon = \Upsilon$ defined by (3.11). Condition [II] is satisfied by Lemmas 3.5, 3.7 and 3.8. Condition [III] may be replaced by (3.3). Lemma 3.4 yields condition [IV]. Condition [V] follows from Lemma 3.6. Now we can apply the above Theorem [13] to complete the proof of Theorem 3.1.

References