Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 898 (2015) 605–626
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Topologically stratified energy minimizers in a product Abelian field theory

Xiaosen Han [a], Yisong Yang [b,c,*]

[a] *Institute of Contemporary Mathematics, School of Mathematics and Statistics, Henan University, Kaifeng, Henan 475004, PR China*
[b] *Department of Mathematics, Polytechnic School of Engineering, New York University, Brooklyn, NY 11201, USA*
[c] *NYU-ECNU Institute of Mathematical Sciences, New York University–Shanghai, 3663 North Zhongshan Road, Shanghai 200062, PR China*

Received 11 April 2015; received in revised form 15 July 2015; accepted 20 July 2015

Available online 23 July 2015

Editor: Hubert Saleur

Abstract

We study a recently developed product Abelian gauge field theory by Tong and Wong hosting magnetic impurities. We first obtain a necessary and sufficient condition for the existence of a unique solution realizing such impurities in the form of multiple vortices. We next reformulate the theory into an extended model that allows the coexistence of vortices and anti-vortices. The two Abelian gauge fields in the model induce two species of magnetic vortex-lines resulting from N_s vortices and P_s anti-vortices ($s = 1, 2$) realized as the zeros and poles of two complex-valued Higgs fields, respectively. An existence theorem is established for the governing equations over a compact Riemann surface S which states that a solution with prescribed N_1, N_2 vortices and P_1, P_2 anti-vortices of two designated species exists if and only if the inequalities

$$|N_1 + N_2 - (P_1 + P_2)| < \frac{|S|}{\pi}, \quad |N_1 + 2N_2 - (P_1 + 2P_2)| < \frac{|S|}{\pi},$$

hold simultaneously, which give bounds for the 'differences' of the vortex and anti-vortex numbers in terms of the total surface area of S. The minimum energy of these solutions is shown to assume the explicit value

$$E = 4\pi(N_1 + N_2 + P_1 + P_2),$$

given in terms of several topological invariants, measuring the total tension of the vortex-lines.

* Corresponding author.
E-mail addresses: hanxiaosen@henu.edu.cn (X. Han), yisongyang@nyu.edu (Y. Yang).

http://dx.doi.org/10.1016/j.nuclphysb.2015.07.022

1. Introduction

It is well known that the simplest and most important quantum field theory model is the Abelian Higgs model [18,28] which embodies an electromagnetic gauge field and spontaneously broken symmetry and allows mass generation through the Higgs mechanism. In the temporal gauge, its static limit gives rise to the classical Ginzburg–Landau theory for superconductivity [17] so that in its two dimensional setting mixed-state configurations known as the Abrikosov vortices [1] can be rigorously constructed [7,18,33,34,38]. Inspired by the gauged sigma model of Schroers [29,30], the classical Abelian Higgs model is extended in [42,43] to allow the co-existence of vortices and anti-vortices. This extended model is also shown to generate cosmic strings and anti-strings when gravitation is switched on by the Einstein equations which give rise to curvature and mass concentrations essential for matter accretion in the early universe [19,20,36,37,40,41]. In order to understand the topological contents of such an extended Abelian Higgs model, a reformulation of it is carried in [31] in the context of a complex line bundle over a compact Riemann surface S as in [7,15,25,26]. In a sharp and interesting contrast with the Abelian Higgs model where vortices are topologically characterized by the first Chern class, the vortices and anti-vortices in the extended Abelian Higgs model [42,43] are characterized jointly and elegantly [31] by the first Chern class of the line bundle and the Thom class [32] of the associated dual bundle. In the former case, there are only finitely many minimum energy values which can be attained due to the fact that the total number of vortices is confined by the total area $|S|$ of the two-surface S where vortices reside. In the latter case, however, the confinement is made instead to the difference of the numbers of vortices and anti-vortices, but the minimum energy is proportional to the sum of these numbers. Hence the possible minimum energy values becomes an explicitly determined infinite sequence as in the situation of vortices over a non-compact surface in the classical Abelian Higgs theory [18,33,34].

In a recent interesting work of Tong and Wong [35], a product Abelian gauge field theory is formulated to include magnetic impurities in the form of an extra gauge-matter sector. This gauge-matter sector is not treated as a background source but as a fully coupled sector. In other words, this is a product Abelian gauge field theory with two complex Higgs fields. It is shown in [35] that, like in the classical Abelian Higgs model, the new product model allows a BPS (after Bogomol'nyi [6] and Prasad–Sommerfield [27]) reduction, hence a construction of magnetic vortices as in [18]. The present paper aims to enrich our understanding of Abelian (magnetic) vortices by achieving two goals. The first is to extend the product Abelian gauge field theory of Tong and Wong [35] using the ideas in [29,30,42,43] into a new product field theory that allows the coexistence of two species of vortices and anti-vortices. The second is to establish an existence theorem for such vortices of beautiful topological characteristics. For clarity and simplicity, the underlying domain for the vortices to live is assumed to be a compact Riemann surface, as in [31].

In order to put our study in an appropriate perspective, we shall first present a reformulation of the Tong–Wong model [35] in terms of a complex line bundle over a compact Riemann surface. In such a context, we show that a Bradlow type bound or limit appears as in the Abelian Higgs theory for the existence of multiple vortices [7,15,25,26,38], which is a preparation for our work regarding the extended model.

An outline of the rest of the paper is as follows. In Section 2, we present the Tong–Wong theory and our extended product Abelian gauge field theory, in their static limits. We describe in detail the field-theoretical properties of the extended theory and derive its BPS equations. We then state our main existence theorems for the existence of multiple vortices in the Tong–Wong theory and for the coexistence of multiple vortices and anti-vortices, of two species. In Section 3, we convert the BPS equations into systems of nonlinear elliptic equations, state the main existence theorems in terms of these equations, and carry out some preliminary discussion. In Section 4, we establish the existence and uniqueness theorem for the Tong–Wong multiple vortex solutions by calculus of variations. In Section 5, we prove the existence theorem for the vortex and anti-vortex solutions of our extended model by using a Leray–Schauder fixed-point theorem argument [16] under a necessary and sufficient condition. In Section 6, we explicitly compute the (minimum) energy of a vortex and anti-vortex solution and show that such energy arises topologically and is proportional to the sum of vortex and anti-vortex numbers of two species. In Section 7, we make some concluding remarks regarding coexisting vortices and anti-vortices in the extended model.

2. Energy functionals, BPS reductions, and existence theorems

Let L be complex Hermitian line bundle over a Riemann surface S. Use q, p to denote two sections $L \to S$ and Dq, Dp the connections induced from the real-valued connection 1-forms $\hat{A}$, $\tilde{A}$, respectively, so that

$$Dq = \mathrm{d}q - \mathrm{i}(\hat{A}–\tilde{A})q, \quad Dp = \mathrm{d}p - \mathrm{i}\tilde{A}p. \tag{2.1}$$

Using $*$ to denote the usual Hodge dual operating on differential forms, the energy density of the Tong–Wong model [35] for a product Abelian Higgs theory implementing magnetic impurities may be rewritten as

$$\begin{aligned}\mathcal{E} = {} & \frac{1}{2} * (\hat{F} \wedge *\hat{F}) + \frac{1}{2} * (\tilde{F} \wedge *\tilde{F}) + *(Dq \wedge *\overline{Dq}) + *(Dp \wedge *\overline{Dp}) \\ & + \frac{1}{2}(1 - |q|^2)^2 + \frac{1}{2}([1 - |q|^2] + [|p|^2 - 1])^2,\end{aligned} \tag{2.2}$$

where $\hat{F} = \mathrm{d}\hat{A}$, $\tilde{F} = \mathrm{d}\tilde{A}$ are curvature 2-forms, which recovers the classical Ginzburg–Landau model [17] when impurities are switched off by setting

$$\tilde{A} = 0, \quad p = 1. \tag{2.3}$$

Note also that there holds the identity

$$\begin{aligned}Dq \wedge *\overline{Dq} + (*Dq) \wedge \overline{Dq} = {} & (Dq \pm \mathrm{i} * Dq) \wedge *\overline{(Dq \pm \mathrm{i} * Dq)} \\ & \pm \mathrm{i}(Dq \wedge \overline{Dq} - (*Dq) \wedge (*\overline{Dq})).\end{aligned} \tag{2.4}$$

Thus we get

$$\begin{aligned}\mathcal{E} = {} & \frac{1}{2} \left|\hat{F} \mp *(1 - |q|^2)\right|^2 + \frac{1}{2} \left|\tilde{F} \pm (*[1 - |q|^2] + *[|p|^2 - 1])\right|^2 \\ & \pm *\hat{F}(1 - |q|^2) \mp *\tilde{F}([1 - |q|^2] + [|p|^2 - 1]) \\ & + |Dq \pm \mathrm{i} * Dq|^2 + |Dp \pm \mathrm{i} * Dp|^2 \\ & \pm \frac{\mathrm{i}}{2}(Dq \wedge \overline{Dq} - (*Dq) \wedge (*\overline{Dq})) \pm \frac{\mathrm{i}}{2}(Dp \wedge \overline{Dp} - (*Dp) \wedge (*\overline{Dp})).\end{aligned} \tag{2.5}$$

On the other hand, with the current densities

$$J(q)=\frac{\mathrm{i}}{2}(q\overline{Dq}-\overline{q}Dq),\quad J(p)=\frac{\mathrm{i}}{2}(p\overline{Dp}-\overline{p}Dp), \tag{2.6}$$

we have

$$\mathrm{d}J(q)=-(\hat{F}-\tilde{F})|q|^2+\frac{\mathrm{i}}{2}(Dq\wedge\overline{Dq}-*Dq\wedge*\overline{Dq}), \tag{2.7}$$

$$\mathrm{d}J(p)=-\tilde{F}|p|^2+\frac{\mathrm{i}}{2}(Dp\wedge\overline{Dp}-*Dp\wedge*\overline{Dp}). \tag{2.8}$$

Inserting (2.7) and (2.8) into (2.5), we have

$$\begin{aligned}\mathcal{E}=&\frac{1}{2}\left|\hat{F}\mp*(1-|q|^2)\right|^2+\frac{1}{2}\left|\tilde{F}\pm(*[1-|q|^2]+*[|p|^2-1])\right|^2\\&+|Dq\pm\mathrm{i}*Dq|^2+|Dp\pm\mathrm{i}*Dp|^2\\&\pm*(\hat{F}-\tilde{F})\pm*\tilde{F}\pm*\mathrm{d}J(q)\pm*\mathrm{d}J(p),\end{aligned} \tag{2.9}$$

which leads to the topological energy lower bound

$$E=\int_S\mathcal{E}*1\geq\left|\int_S\hat{F}\right|. \tag{2.10}$$

From the form of (2.9) it is clear that the lower bound in (2.10) is attained by the solutions of the BPS equations

$$\hat{F}=\pm*(1-|q|^2), \tag{2.11}$$

$$\tilde{F}=\mp*([1-|q|^2]+[|p|^2-1]), \tag{2.12}$$

$$Dq\pm\mathrm{i}*Dq=0, \tag{2.13}$$

$$Dp\pm\mathrm{i}*Dp=0, \tag{2.14}$$

as derived by Tong and Wong in [35].

The equations of motion of (2.2) are

$$D*Dq=-(1-|q|^2)q-([1-|q|^2]+[|p|^2-1])q, \tag{2.15}$$

$$\mathrm{d}*\hat{F}=\mathrm{i}(\overline{q}Dq-q\overline{Dq}), \tag{2.16}$$

$$D*Dp=([1-|q|^2]+[|p|^2-1])p, \tag{2.17}$$

$$\mathrm{d}*\tilde{F}=-\mathrm{i}(\overline{q}Dq-q\overline{Dq})+\mathrm{i}(\overline{p}Dp-p\overline{Dp}), \tag{2.18}$$

which contain (2.11)–(2.14) as its first integral and may be viewed as a reduced form of (2.15)–(2.18). These reduced first-order equations are often referred to as the BPS equations after Bogomol'nyi [6] and Prasad–Sommerfield [27] who pioneered the idea of such reduction for the classical Yang–Mills–Higgs equations. When the upper sign is taken, the system is said to be self-dual; the lower, anti-self-dual. It may also be checked that the self-dual and anti-self-dual cases are related to each other through the transformation

$$\hat{A}\rightleftharpoons-\hat{A},\quad\tilde{A}\rightleftharpoons-\tilde{A},\quad q\rightleftharpoons\overline{q},\quad p\rightleftharpoons\overline{p}. \tag{2.19}$$

Hence, in the sequel, we will only consider the self-dual situation.

The structure of (2.13) and (2.14) indicates that the zeros of q, p are isolated and of integer multiplicities which may be assumed to be

$$\mathcal{Z}(q)=\{z_{1,1},\dots,z_{1,N_1}\},\quad \mathcal{Z}(p)=\{z_{2,1},\dots,z_{2,N_2}\},\tag{2.20}$$

where for convenience a zero of multiplicity m is counted as m zeros in the zero set. The quantities $\frac{1}{2\pi}\int(\hat{F}-\tilde{F})$ and $\frac{1}{2\pi}\int\tilde{F}$ are the first Chern numbers induced from the connections $\hat{A}$–$\tilde{A}$ and $\tilde{A}$ over $L\to S$ which are determined by the numbers of zeros, N_1 and N_2, by the formulas

$$\frac{1}{2\pi}\int_S(\hat{F}-\tilde{F})=N_1,\tag{2.21}$$

$$\frac{1}{2\pi}\int_S\tilde{F}=N_2,\tag{2.22}$$

respectively.

Regarding the Tong–Wong BPS equations (2.11)–(2.14), here is our existence theorem.

Theorem 2.1. *For the BPS system consisting of equations (2.11)–(2.14) over a compact Riemann surface S with canonical total area $|S|$ governing two connection 1-forms $\hat{A}$, $\tilde{A}$ and two cross sections q, p with the prescribed sets of zeros given in (2.20), there exists a solution to realize these sets of zeros if and only if N_1 and N_2 satisfy the bound*

$$N_1+2N_2<\frac{|S|}{2\pi}.\tag{2.23}$$

Such a solution carries a minimum energy of the form

$$E=2\pi(N_1+N_2),\tag{2.24}$$

and is unique up to gauge transformations.

The condition stated in (2.23) is analogous to the so-called Bradlow bound [5,9,24] in the classical Abelian Higgs model [7,15,38] which was actually deduced earlier by Noguchi [25,26].

Next, following [42,43] based on the idea of gauged sigma model, we show that we may extend the Tong–Wong model [35] to accommodate vortices and anti-vortices by considering the modified energy density

$$\begin{aligned}\mathcal{E}=&\frac{1}{2}*(\hat{F}\wedge*\hat{F})+\frac{1}{2}*(\tilde{F}\wedge*\tilde{F})+\frac{4}{(1+|q|^2)^2}*(Dq\wedge*\overline{Dq})\\&+\frac{4}{(1+|p|^2)^2}*(Dp\wedge*\overline{Dp})\\&+2\left(\frac{1-|q|^2}{1+|q|^2}\right)^2+2\left(\frac{1-|q|^2}{1+|q|^2}+\frac{|p|^2-1}{1+|p|^2}\right)^2.\end{aligned}\tag{2.25}$$

In fact, let ϕ and ψ be two S^2-valued scalar fields. Fix $\mathbf{n}=(0,0,1)\in S^2$ and define the vacuum manifold of the model to be

$$\mathbf{n}\cdot\phi=0,\quad \mathbf{n}\cdot\psi=0.\tag{2.26}$$

Then we modify (2.2) into the form

$$\mathcal{E}=\frac{1}{2}*(\hat{F}\wedge*\hat{F})+\frac{1}{2}*(\tilde{F}\wedge*\tilde{F})+|D\phi|^2+|D\psi|^2+2(\mathbf{n}\cdot\phi)^2+2(\mathbf{n}\cdot[\phi-\psi])^2, \quad (2.27)$$

where

$$D\phi=\mathrm{d}\phi-(\mathbf{n}\times\phi)(\hat{A}–\tilde{A}),\quad D\psi=\mathrm{d}\psi-(\mathbf{n}\times\psi)\tilde{A}. \quad (2.28)$$

Use the stereographic projection from the south pole $-\mathbf{n}$ of S^2 to represent ϕ and ψ by complex-valued functions q and p, respectively, so that

$$\phi=\left(\frac{2\Re\{q\}}{1+|q|^2},\frac{2\Im\{q\}}{1+|q|^2},\frac{1-|q|^2}{1+|q|^2}\right),\quad \psi=\left(\frac{2\Re\{p\}}{1+|p|^2},\frac{2\Im\{p\}}{1+|p|^2},\frac{1-|p|^2}{1+|p|^2}\right), \quad (2.29)$$

where $\Re\{c\}$ and $\Im\{c\}$ denote the real and imaginary parts of a complex number c. Inserting (2.29) into (2.27), we arrive at (2.25).

It is interesting to observe that (2.2) is recovered from (2.25) when taking the limit $|q|\to 1$, $|p|\to 1$ in the denominators $1+|q|^2$ and $1+|p|^2$ of (2.25). The Euler–Lagrange equations of the energy density are found to be

$$\begin{aligned}D*\left(\frac{Dq}{(1+|q|^2)^2}\right)&=\frac{1}{(1+|q|^2)^3}(Dq\wedge*\overline{Dq})+2*\left(\frac{1-|q|^2}{(1+|q|^2)^3}\right)q\\&\quad+2*\left(\frac{1-|q|^2}{1+|q|^2}+\frac{|p|^2-1}{1+|p|^2}\right)\left(\frac{1-|q|^2}{(1+|q|^2)^2}\right)q, &(2.30)\\ \mathrm{d}*\hat{F}&=4\mathrm{i}\frac{(\overline{q}Dq-q\overline{Dq})}{(1+|q|^2)^2}, &(2.31)\\ D*\left(\frac{Dp}{(1+|p|^2)^2}\right)&=\frac{1}{(1+|p|^2)^3}(Dp\wedge*\overline{Dp})\\&\quad+2*\left(\frac{1-|q|^2}{1+|q|^2}+\frac{|p|^2-1}{1+|p|^2}\right)\left(\frac{|p|^2-1}{(1+|p|^2)^2}\right)p, &(2.32)\\ \mathrm{d}*\tilde{F}&=-4\mathrm{i}\frac{(\overline{q}Dq-q\overline{Dq})}{(1+|q|^2)^2}+4\mathrm{i}\frac{(\overline{p}Dp-p\overline{Dp})}{(1+|p|^2)^2}, &(2.33)\end{aligned}$$

which appear rather complicated and intractable. In order to obtain interesting solutions of these equations, we follow [35,42] to pursue a BPS reduction.

Introduce the current densities

$$J(q)=\frac{\mathrm{i}}{1+|q|^2}(q\overline{Dq}-\overline{q}Dq),\quad J(p)=\frac{\mathrm{i}}{1+|p|^2}(p\overline{Dp}-\overline{p}Dp). \quad (2.34)$$

Then we have

$$\begin{aligned}K(q)&=\mathrm{d}J(q)\\&=-\frac{2|q|^2}{1+|q|^2}(\hat{F}-\tilde{F})+\mathrm{i}\left(\frac{Dq\wedge\overline{Dq}-*Dq\wedge*\overline{Dq}}{(1+|q|^2)^2}\right), &(2.35)\\ K(p)&=\mathrm{d}J(p)\\&=-\frac{2|q|^2}{1+|q|^2}\tilde{F}+\mathrm{i}\left(\frac{Dp\wedge\overline{Dp}-*Dp\wedge*\overline{Dp}}{(1+|p|^2)^2}\right). &(2.36)\end{aligned}$$

So, with $|Dq|^2 = *(Dq \wedge *\overline{Dq})$, etc., we arrive at the decomposition

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\left|\hat{F} \mp 2*\left(\frac{1-|q|^2}{1+|q|^2}\right)\right|^2 + \frac{1}{2}\left|\tilde{F} \pm \left(2*\left(\frac{1-|q|^2}{1+|q|^2}\right) + 2*\left(\frac{|p|^2-1}{1+|p|^2}\right)\right)\right|^2 \\ &\quad + \frac{2}{(1+|q|^2)^2}|Dq \pm \mathrm{i} * Dq|^2 + \frac{2}{(1+|p|^2)^2}|Dp \pm \mathrm{i} * Dp|^2 \\ &\quad \pm 2*(\hat{F} - \tilde{F}) \pm 2*K(q) \pm 2*\tilde{F} \pm 2*K(p). \end{aligned} \tag{2.37}$$

The quantities $\frac{1}{4\pi}\int K(q)$ and $\frac{1}{4\pi}\int K(p)$ are the Thom classes over $L^* \to S$, respectively [31]. Thus, the sum

$$\tau = 2\hat{F} + 2K(q) + 2K(p) \tag{2.38}$$

is a topological density which leads to the topological energy lower bound

$$E = \int_M \mathcal{E} * 1 \geq \left|\int_M \tau\right|, \tag{2.39}$$

measuring the tension [8,13,11,12,10] of the vortex-lines, so that the lower bound is saturated when the quartet $(q, p, \hat{A}, \tilde{A})$ satisfies the equations

$$Dq \pm \mathrm{i} * Dq = 0, \tag{2.40}$$

$$Dp \pm \mathrm{i} * Dp = 0, \tag{2.41}$$

$$\hat{F} = \pm 2*\left(\frac{1-|q|^2}{1+|q|^2}\right), \tag{2.42}$$

$$\tilde{F} = \mp\left(2*\left(\frac{1-|q|^2}{1+|q|^2}\right) + 2*\left(\frac{|p|^2-1}{1+|p|^2}\right)\right). \tag{2.43}$$

It may directly be checked that (2.40)–(2.43) imply (2.30)–(2.33). In other words, (2.40)–(2.43) may be regarded as a reduction of the system of equations (2.30)–(2.33).

From (2.40) and (2.41), we know [18,42,43] that the zeros and poles of the sections q, p are isolated and possess integer multiplicities. For simplicity, we may denote the sets of zeros and poles of q, p by

$$\mathcal{Z}(q) = \{z'_{1,1}, \ldots, z'_{1,N_1}\}, \quad \mathcal{P}(q) = \{z''_{1,1}, \ldots, z''_{1,P_1}\}, \tag{2.44}$$

$$\mathcal{Z}(p) = \{z'_{2,1}, \ldots, z'_{2,N_2}\}, \quad \mathcal{P}(p) = \{z''_{2,1}, \ldots, z''_{2,P_2}\}, \tag{2.45}$$

respectively, so that the associated multiplicities of the zeros and poles are naturally counted by their repeated appearances in the above collections of points.

If we interpret $*\hat{F}$ as a magnetic or vorticity field, (2.42) indicates that it attains its maximum $*\hat{F} = 2$ at the zeros and minimum $*\hat{F} = -2$ at the poles of q. Thus, the zeros and poles of q may be viewed as centers of vortices and anti-vortices. In other words, we may identify the zeros and poles of q as the locations of vortices and anti-vortices generated from the connection 1-form $\hat{A}$. Similarly, the zeros and poles of p may be interpreted as vortices and anti-vortices generated from the connection 1-form $\hat{A} + \tilde{A}$. Therefore, in what follows, the zeros and poles of q, p are interchangeably and generically referred to as the vortices and anti-vortices of a solution configuration $(\hat{A}, \tilde{A}, q, p)$.

Here is our existence theorem for the BPS equations (2.40)–(2.43).

Theorem 2.2. *Consider the BPS system consisting of equations (2.40)–(2.43) of the energy density (2.25) formulated over a complex Hermitian line bundle L over a compact Riemann surface S with canonical total area $|S|$ governing two connection 1-forms $\hat{A}$, $\tilde{A}$ and two cross sections q, p and comprising a reduction of the Euler–Lagrange equations (2.30)–(2.33). For the prescribed sets of zeros and poles for the fields q and p given respectively in (2.44) and (2.45), the coupled equations (2.40)–(2.43) have a solution to realize these sets of zeros and poles, if and only if the inequalities*

$$|N_1 + N_2 - (P_1 + P_2)| < \frac{|S|}{\pi}, \tag{2.46}$$

$$|N_1 + 2N_2 - (P_1 + 2P_2)| < \frac{|S|}{\pi}, \tag{2.47}$$

regarding the total numbers of zeros and poles are fulfilled simultaneously. Moreover, such a solution carries a minimum energy of the form

$$E = 4\pi(N_1 + N_2 + P_1 + P_2), \tag{2.48}$$

which is seen to be stratified topologically by the Chern and Thom classes of the line bundle L and its dual respectively. In particular, in terms of energy, zeros (vortices) and poles (anti-vortices) of q, p contribute equally.

It is interesting to note that the inequalities (2.46) and (2.47) imply that the differences of vortices and anti-vortices must stay within suitable ranges to ensure the existence of a solution:

$$\Big|N_1 - P_1\Big| < \frac{3|S|}{\pi}, \tag{2.49}$$

$$\Big|N_2 - P_2\Big| < \frac{2|S|}{\pi}. \tag{2.50}$$

However, it may be checked that the conditions (2.49) and (2.50) do not lead to (2.46) and (2.47). The latter may be called the difference of total numbers of vortices and anti-vortices and the difference of 'weighted total numbers' of vortices and anti-vortices. We note that (2.49) and (2.50) give the upper bounds of the total 'magnetic fluxes'

$$\int_S (\hat{F} - \tilde{F}) = 2\pi(N_1 - P_1), \tag{2.51}$$

$$\int_S \tilde{F} = 2\pi(N_2 - P_2), \tag{2.52}$$

generated by the 'magnetic fields' $\hat{F} - \tilde{F}$ and $\tilde{F}$, respectively, which may be compared with (2.21) and (2.22) for the fluxes of the Tong–Wong model [35]. See Section 6 for details of calculation.

3. Governing elliptic equations and basic properties

To proceed, we set

$$u = \ln|q|^2, \quad v = \ln|p|^2, \tag{3.1}$$

in (2.11)–(2.14). Thus by [18,42,43] we are led to the following equivalent governing elliptic equations

$$\Delta u = 4(\mathrm{e}^u - 1) - 2(\mathrm{e}^v - 1) + 4\pi \sum_{z\in\mathcal{Z}(q)} \delta_z, \tag{3.2}$$

$$\Delta v = -2(\mathrm{e}^u - 1) + 2(\mathrm{e}^v - 1) + 4\pi \sum_{z\in\mathcal{Z}(p)} \delta_z, \tag{3.3}$$

where Δ is the Laplace–Beltrami operator on (S, g) defined by

$$\Delta u = \frac{1}{\sqrt{g}}\partial_j(g^{jk}\sqrt{g}\partial_k u), \tag{3.4}$$

and δ_z denotes the Dirac measure concentrated at the point $z \in S$ with respect to the Riemannian metric g over S.

In what follows, we use $\mathrm{d}\Omega_g$ to denote the canonical surface element and $|S|$ the associated total area of the Riemann surface (S, g).

The results stated in Theorem 2.1 are contained in the following theorem concerning the coupled elliptic equations (3.2) and (3.3).

Theorem 3.1. *The system of equations consisting of (3.2) and (3.3) has a solution if and only if*

$$N_1 + 2N_2 < \frac{|S|}{2\pi}. \tag{3.5}$$

Moreover, if a solution exists, it must be unique and satisfies the quantization conditions

$$\int_S (1 - \mathrm{e}^u)\, \mathrm{d}\Omega_g = 2\pi(N_1 + N_2), \tag{3.6}$$

$$\int_S (1 - \mathrm{e}^v)\, \mathrm{d}\Omega_g = 2\pi(N_1 + 2N_2). \tag{3.7}$$

Similarly, setting (3.1) in (2.40)–(2.43), we obtain

$$\Delta u = \frac{8(\mathrm{e}^u - 1)}{\mathrm{e}^u + 1} - \frac{4(\mathrm{e}^v - 1)}{\mathrm{e}^v + 1} + 4\pi \sum_{z\in\mathcal{Z}(q)} \delta_z - 4\pi \sum_{z\in\mathcal{P}(q)} \delta_z, \tag{3.8}$$

$$\Delta v = -\frac{4(\mathrm{e}^u - 1)}{\mathrm{e}^u + 1} + \frac{4(\mathrm{e}^v - 1)}{\mathrm{e}^v + 1} + 4\pi \sum_{z\in\mathcal{Z}(p)} \delta_z - 4\pi \sum_{z\in\mathcal{P}(p)} \delta_z. \tag{3.9}$$

Regarding the equivalently reduced equations (3.8) and (3.9) from (2.40)–(2.43), we have

Theorem 3.2. *The coupled equations (3.8) and (3.9) admit a solution (u, v) with the prescribed sets $\mathcal{Z}(q), \mathcal{P}(q), \mathcal{Z}(p), \mathcal{P}(p)$ in S specified in (2.44) and (2.45) if and only if the inequalities (2.46) and (2.47) are satisfied simultaneously. Moreover, for the solution to equations (3.8) and (3.9) obtained above, there hold the quantized integrals*

$$\int_S \frac{1 - \mathrm{e}^u}{1 + \mathrm{e}^u}\, \mathrm{d}\Omega_g = \pi\,(N_1 - P_1 + N_2 - P_2)\,, \tag{3.10}$$

$$\int_S \frac{1-e^{v}}{1+e^{v}}\,d\Omega_g = \pi\left(N_1 - P_1 + 2(N_2 - P_2)\right). \tag{3.11}$$

For convenience, we first need to take care of the Dirac distributions by subtracting suitable background functions. To do so, we let $u_0^1, u_0^2, v_0^1, v_0^2$ be the normalized solutions of the equations that determine the source functions arising from the sets $\mathcal{Z}(q), \mathcal{P}(q), \mathcal{Z}(p), \mathcal{P}(p)$, respectively. For instance, u_0^1 is the unique solution [4] to

$$\Delta u_0^1 = -\frac{4\pi N_1}{|S|} + 4\pi \sum_{z\in\mathcal{Z}(q)} \delta_z, \qquad \int_S u_0^1\,d\Omega_g = 0. \tag{3.12}$$

Set $u = u_0^1 + U$, $v = v_0^1 + V$. Then we can rewrite (3.2) and (3.3) as

$$\Delta U = 4(e^{u_0^1+U} - 1) - 2(e^{v_0^1+V} - 1) + \frac{4\pi N_1}{|S|}, \tag{3.13}$$

$$\Delta V = -2(e^{u_0^1+U} - 1) + 2(e^{v_0^1+V} - 1) + \frac{4\pi N_2}{|S|}. \tag{3.14}$$

We first show the necessity of the condition (3.5). If there is a solution of (3.13)–(3.14), integration of which over S gives

$$\int_S e^{u_0^1+U}\,d\Omega_g = |S| - 2\pi(N_1 + N_2) \equiv a_1 > 0, \tag{3.15}$$

$$\int_S e^{v_0^1+V}\,d\Omega_g = |S| - 2\pi(N_1 + 2N_2) \equiv a_2 > 0. \tag{3.16}$$

Then we see that the condition (3.5) is necessary to ensure the existence of a solution to the system (3.13)–(3.14). The quantized integrals (3.6)–(3.7) follow from (3.15)–(3.16).

Let $u = u_0^1 - u_0^2 + U$, $v = v_0^1 - v_0^2 + V$. Then equations (3.8) and (3.9) can be rewritten as

$$\Delta U = 8f(u_0^1, u_0^2, U) - 4f(v_0^1, v_0^2, V) + \frac{4\pi(N_1 - P_1)}{|S|}, \tag{3.17}$$

$$\Delta V = -4f(u_0^1, u_0^2, U) + 4f(v_0^1, v_0^2, V) + \frac{4\pi(N_2 - P_2)}{|S|}, \tag{3.18}$$

where and in what follows we use the notation

$$f(s^1, s^2, t) \equiv \frac{e^{s^1-s^2+t} - 1}{e^{s^1-s^2+t} + 1} = \frac{e^{s^1+t} - e^{s^2}}{e^{s^1+t} + e^{s^2}}, \qquad s^1, s^2, t \in \mathbb{R}. \tag{3.19}$$

For fixed $s^1, s^2 \in \mathbb{R}$, we have

$$0 < \frac{d}{dt} f(s^1, s^2, t) = \frac{2e^{s^1+t}e^{s^2}}{(e^{s^1+t} + e^{s^2})^2} \le \frac{1}{2}, \qquad \forall\, t \in \mathbb{R}. \tag{3.20}$$

We now show that the condition consisting of (2.46) and (2.47) is necessary for the existence of solutions for (3.17)–(3.18). In fact, integrating (3.17)–(3.18) over S, we find

$$\int_S f(u_0^1, u_0^2, U)\,\mathrm{d}\Omega_g = a|S|, \tag{3.21}$$

$$\int_S f(v_0^1, v_0^2, V)\,\mathrm{d}\Omega_g = b|S|, \tag{3.22}$$

where a, b are constants defined by

$$a \equiv -\frac{\pi}{|S|}(N_1 - P_1 + N_2 - P_2), \tag{3.23}$$

$$b \equiv -\frac{\pi}{|S|}(N_1 - P_1 + 2(N_2 - P_2)). \tag{3.24}$$

From (3.21)–(3.22) we see that the quantized integrals (3.10)–(3.11) hold.

On the other hand, noting

$$-1 < f(s^1, s^2, t) < 1 \quad \text{for any} \quad s^1, s^2, t \in \mathbb{R}, \tag{3.25}$$

we arrive at

$$|a| < 1 \quad \text{and} \quad |b| < 1, \tag{3.26}$$

which is equivalent to (2.46) and (2.47). Thus the inequalities (2.46) and (2.47) are necessary for a solution to exist.

4. Proof of existence for the Tong–Wong system

In this section we establish Theorem 2.1 or Theorem 3.1 through a thorough study of the coupled vortex equations (3.2) and (3.3). To this end, we recast the problem into a variational problem and apply a direct minimization approach recently developed in [22].

We have shown that the condition (3.5) is necessary to the existence of a solution to (3.2)–(3.3) on S, and in what follows we prove that it is also sufficient.

To formulate the problem into a variational structure, we set

$$f = U, \quad h = U + 2V, \quad \text{or} \quad U = f, \quad V = \frac{h - f}{2}. \tag{4.1}$$

Hence we rewrite equations (3.13)–(3.14) equivalently as

$$\Delta f = 4\left(\mathrm{e}^{u_0^1 + f} - 1\right) - 2\left(\mathrm{e}^{v_0^1 + \frac{h-f}{2}} - 1\right) + \frac{4\pi N_1}{|S|}, \tag{4.2}$$

$$\Delta h = 2\left(\mathrm{e}^{v_0^1 + \frac{h-f}{2}} - 1\right) + \frac{4\pi(N_1 + 2N_2)}{|S|}. \tag{4.3}$$

Then we directly check that equations (4.2)–(4.3) are the Euler–Lagrange equations of the following functional

$$I(f, h) = \frac{1}{2}\left(\|\nabla f\|_2^2 + \|\nabla h\|_2^2\right) + 4\int_S \left(\mathrm{e}^{u_0^1 + f} - f + \mathrm{e}^{v_0^1 + \frac{h-f}{2}} - \frac{h - f}{2}\right)\mathrm{d}\Omega_g$$
$$+ \frac{4\pi N_1}{|S|}\int_S f\,\mathrm{d}\Omega_g + \frac{4\pi(N_1 + 2N_2)}{|S|}\int_S h\,\mathrm{d}\Omega_g. \tag{4.4}$$

Here and in what follows we use the following notation

$$\|\nabla w\|_2^2 = \int_S |\nabla w|^2 \mathrm{d}\Omega_g \equiv \int_S g^{jk}\partial_j w \partial_k w \mathrm{d}\Omega_g. \tag{4.5}$$

We know that the Sobolev space $W^{1,2}(S)$ (cf. [4]) can be decomposed as $W^{1,2}(S) = \dot{W}^{1,2}(S) \oplus \mathbb{R}$, where

$$\dot{W}^{1,2}(S) \equiv \left\{ w \in W^{1,2}(S) \Big| \quad \int_S w \,\mathrm{d}\Omega_g = 0 \right\} \tag{4.6}$$

is a closed subspace of $W^{1,2}(S)$.

To save notation, in the following of this paper we also use $W^{1,2}(S)$, $\dot{W}^{1,2}(S)$ and $L^p(S)$ to denote the spaces of vector-valued functions.

For $f, h \in W^{1,2}(S)$ we decompose them as

$$f = f' + \overline{f},\ h = h' + \overline{h}, \quad f', h' \in \dot{W}^{1,2}(S), \quad \overline{f}, \overline{h} \in \mathbb{R}. \tag{4.7}$$

On the subspace $\dot{W}^{1,2}(S)$ there hold the Poincaré inequality

$$\int_S w^2 \mathrm{d}\Omega_g \equiv \|w\|_2^2 \le C\|\nabla w\|_2^2, \quad w \in \dot{W}^{1,2}(S) \tag{4.8}$$

and the Moser–Trudinger inequality [4,14]

$$\int_S \mathrm{e}^w \mathrm{d}\Omega_g \le C \exp\left(\frac{1}{16\pi} \int_S |\nabla w|^2 \mathrm{d}x \right), \quad w \in \dot{W}^{1,2}(S), \tag{4.9}$$

where C is a generic positive constant. We see from (4.9) that the functional I defined by (4.4) is a C^1-functional.

By the definition of I and the decomposition (4.7) we have

$$\begin{aligned} I(f,h) = {} & \frac{1}{2}(\|\nabla f'\|_2^2 + \|\nabla h'\|_2^2) + 4\left(\mathrm{e}^{\overline{f}} \int_S \mathrm{e}^{u_0^1 + f'} \mathrm{d}\Omega_g - a_1 \overline{f} \right) \\ & + 4\left(\mathrm{e}^{\frac{\overline{h}-\overline{f}}{2}} \int_S \mathrm{e}^{v_0^1 + \frac{h'-f'}{2}} \mathrm{d}\Omega_g - a_2 \frac{[\overline{h}-\overline{f}]}{2} \right), \end{aligned} \tag{4.10}$$

where a_1, a_2, defined by (3.15)–(3.16), are positive, as ensured by (3.5).

Hence, by (4.10) and the Jensen inequality, we obtain

$$I(f,h) - \frac{1}{2}(\|\nabla f'\|_2^2 + \|\nabla h'\|_2^2) \ge 4\left(|S|\mathrm{e}^{\overline{f}} - a_1\overline{f} + |S|\mathrm{e}^{\frac{\overline{h}-\overline{f}}{2}} - a_2\frac{[\overline{h}-\overline{f}]}{2} \right). \tag{4.11}$$

From (4.10)–(4.11) we also see that

$$I(f,h) \ge 4\left(\ln\frac{|S|}{a_1} + \ln\frac{|S|}{a_2} \right), \tag{4.12}$$

which implies the functional I is bounded from below and the minimization problem

$$a_0 \equiv \min\left\{ I(f,h) \middle| (f,h) \in W^{1,2}(S) \right\} \tag{4.13}$$

is well-defined.

Let $\{(f_k, h_k)\}$ be a minimizing sequence. Noting that the function $m(t) = \alpha e^t - \beta$, $\alpha, \beta > 0$ satisfies $m(t) \to +\infty$ as $t \to \pm\infty$, we see from (4.11) that $\overline{f}_k, \frac{\overline{h}_k - \overline{f}_k}{2}$ must be bounded for all k, which implies $\{(\overline{f}_k, \overline{h}_k)\}$ are bounded for all k. And (4.11) also implies $\{(\nabla f'_k, \nabla h'_k)\}$ are bounded in $L^2(S)$ for all k. Then by the Poincaré inequality (4.8), we conclude that $\{(f'_k, h'_k)\}$ are bounded in $\dot{W}^{1,2}(S)$, which with the boundedness of $\{(\overline{f}_k, \overline{h}_k)\}$ imply that $\{(f_k, h_k)\}$ are bounded in $W^{1,2}(S)$ for all k. Hence, there exits a subsequence of $\{(f_k, h_k)\}$, still denoted by $\{(f_k, h_k)\}$, such that $\{(f_k, h_k)\}$ converges weakly to some $(f_\infty, h_\infty) \in W^{1,2}(S)$.

It is easy to see that the functional I is also weakly lower semi-continuous. Then the limit $(f_\infty, h_\infty) \in W^{1,2}(S)$ is a critical point of I. Of course, it gives a solution for the system (4.2)–(4.3), and hence for (3.13)–(3.14). So the sufficiency of (3.5) follows.

We directly see that the functional I is strictly convex. Therefore the functional I admits at most one critical point. That is to say, a solution of (3.13)–(3.14) must be unique.

Therefore we have completed the proof of Theorem 3.1.

5. Proof of existence for the vortex and anti-vortex system

In this section, we prove that the condition comprised of (2.46) and (2.47) is also sufficient for the existence of a solution of the coupled equations (3.8) and (3.9). We will extend a fixed-point theorem argument used in [44] when treating a single equation.

To do so, it is convenient to rewrite equations (3.17) and (3.18) equivalently as

$$\Delta U = 8\left(f(u_0^1, u_0^2, U) - a\right) - 4\left(f(v_0^1, v_0^2, V) - b\right), \tag{5.1}$$

$$\Delta V = -4\left(f(u_0^1, u_0^2, U) - a\right) + 4\left(f(v_0^1, v_0^2, V) - b\right), \tag{5.2}$$

where a, b are defined by (3.23)–(3.24).

We begin with the following lemma.

Lemma 5.1. *For any $(U', V') \in \dot{W}^{1,2}(S)$, there exists a unique pair $(c_1(U'), c_2(V')) \in \mathbb{R}^2$ such that*

$$\int_S f(u_0^1, u_0^2, U' + c_1(U'))\, d\Omega_g = a|S|, \tag{5.3}$$

$$\int_S f(v_0^1, v_0^2, V' + c_2(V'))\, d\Omega_g = b|S|, \tag{5.4}$$

where a, b are defined by (3.23)–(3.24).

Proof. Under the condition consisting of (2.46) and (2.47), we easily see that

$$-1 < a, b < 1. \tag{5.5}$$

Noting the expression (3.19), for any $(U', V') \in \dot{W}^{1,2}(S)$, we have

$$\int\limits_S f(u_0^1, u_0^2, U' + t)\mathrm{d}\Omega_g, \int\limits_S f(v_0^1, v_0^2, V' + t)\mathrm{d}\Omega_g \to |S| \quad \text{as} \quad t \to \infty \tag{5.6}$$

and

$$\int\limits_S f(u_0^1, u_0^2, U' + t)\mathrm{d}\Omega_g, \int\limits_S f(v_0^1, v_0^2, V' + t)\mathrm{d}\Omega_g \to -|S| \quad \text{as} \quad t \to -\infty. \tag{5.7}$$

Then, for any $(U', V') \in \dot{W}^{1,2}(S)$, we conclude from (5.5), (5.6) and (5.7) that there exists a point $(c_1(U'), c_2(V')) \in \mathbb{R}^2$ such that (5.3) and (5.4) hold.

The uniqueness of $(c_1(U'), c_2(V'))$ follows from the strict monotonicity of $f(s^1, s^2, t)$ with respect to t (see (3.20)). $\square$

Lemma 5.2. *For any $(U', V') \in \dot{W}^{1,2}(S)$, let $(c_1(U'), c_2(V'))$ be defined in Lemma 5.1. Then, the mapping $(c_1(\cdot), c_2(\cdot)) : \dot{W}^{1,2}(S) \to \mathbb{R}^2$, is continuous with respect to the weak topology of $\dot{W}^{1,2}(S)$.*

Proof. Take a weakly convergent sequence $\{(U_k', V_k')\}$ in $\dot{W}^{1,2}(S)$ such that $(U_k', V_k') \to (U_0', V_0')$ weakly in $\dot{W}^{1,2}(S)$. Then we see that

$$(U_k', V_k') \to (U_0', V_0') \quad \text{strongly in} \quad L^p(S) \quad \text{for any} \quad p \geq 1, \tag{5.8}$$

by the compact embedding $W^{1,2}(S) \hookrightarrow L^p(S)$ $(p \geq 1)$. We aim to prove that $(c_1(U_k'), c_2(V_k')) \to (c_1(U_0'), c_2(V_0'))$ as $k \to \infty$.

Claim: The sequence $\{(c_1(U_k'), c_2(V_k'))\}$ is bounded.

To show this claim we first prove that $\{(c_1(U_k'), c_2(V_k'))\}$ is bounded from above. We argue by contradiction. Without loss of generality, assume $c_1(U_k') \to \infty$ as $k \to \infty$. Noting (5.8) and using the Egorov theorem, we see that for any $\varepsilon > 0$, there is a large constant $K_\varepsilon > 0$ and a subset $S_\varepsilon \subset S$ such that

$$|U_k'| \leq K_\varepsilon, \quad x \in S \setminus S_\varepsilon, \quad |S_\varepsilon| < \varepsilon, \quad \forall k. \tag{5.9}$$

Then by (5.9) and (3.25) we have

$$\begin{aligned} |a||S| &= \left| \int\limits_{S\setminus S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))\mathrm{d}\Omega_g + \int\limits_{S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))\mathrm{d}\Omega_g \right| \\ &\geq \left| \int\limits_{S\setminus S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))\mathrm{d}\Omega_g \right| - \left| \int\limits_{S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))\mathrm{d}\Omega_g \right| \\ &\geq \int\limits_{S\setminus S_\varepsilon} f(u_0^1, u_0^2, c_1(U_k') - K_\varepsilon)\mathrm{d}\Omega_g - \varepsilon. \end{aligned} \tag{5.10}$$

Hence taking $k \to \infty$ in (5.10) we get

$$|a||S| \geq |S \setminus S_\varepsilon| - \varepsilon \geq |S| - 2\varepsilon.$$

Noting that ε is arbitrary, we obtain

$$|a| \geq 1,$$

which contradicts the condition (2.46) ($|a| < 1$). Hence the sequence $\{(c_1(U_k'), c_2(V_k'))\}$ is bounded from above.

Now we show that $\{(c_1(U_k'), c_2(V_k'))\}$ is also bounded from below. In fact, we may suppose $c_1(U_k') \to -\infty$ as $k \to \infty$. Using (5.9) and (3.25), we have

$$\begin{aligned} a|S| &= \int\limits_{S\setminus S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))\mathrm{d}\Omega_g + \int\limits_{S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))\mathrm{d}\Omega_g \\ &\le \int\limits_{S\setminus S_\varepsilon} f(u_0^1, u_0^2, c_1(U_k') + K_\varepsilon)\mathrm{d}\Omega_g + \varepsilon. \end{aligned} \tag{5.11}$$

Then letting $k \to \infty$ in (5.11), we obtain

$$a|S| \le -|S \setminus S_\varepsilon| + \varepsilon \le -|S| + 2\varepsilon,$$

which implies $a \le -1$ since $\varepsilon > 0$ is arbitrary. Hence we get a contradiction with the condition (2.46) again. So the sequence $\{(c_1(U_k'), c_2(V_k'))\}$ is bounded from below. Therefore the claim follows.

By the claim above, up to a subsequence, we may assume that

$$(c_1(U_k'), c_2(V_k')) \to (c_1', c_2') \quad \text{as} \quad k \to \infty \quad \text{for some} \quad (c_1', c_2') \in \mathbb{R}^2. \tag{5.12}$$

Then, using (3.20), the Schwartz inequality, (5.8) and (5.12) we have

$$\begin{aligned} &\left| \int\limits_S f(u_0^1, u_0^2, U_k' + c_1(U_k'))\,\mathrm{d}\Omega_g - \int\limits_S f(u_0^1, u_0^2, U_0' + c_1')\,\mathrm{d}\Omega_g \right| \\ &= \left| \int\limits_S f_t(u_0^1, u_0^2, \theta[U_k' + c_1(U_k')] + [1-\theta][U_0' + c_1'])(U_k' - U_0' + c_1(U_k') - c_1')\,\mathrm{d}\Omega_g \right| \\ &\le \frac{1}{2} \int\limits_S \left|U_k' - U_0' + c_1(U_k') - c_1'\right| \mathrm{d}\Omega_g \\ &\le \frac{1}{2}\left(|S|^{\frac{1}{2}}\|U_k' - U_0'\|_2 + |S||c_1(U_k') - c_1'|\right) \to 0 \quad \text{as} \quad k \to \infty, \end{aligned} \tag{5.13}$$

where $\theta \in (0, 1)$. Noting (5.13) and

$$\int\limits_S f(u_0^1, u_0^2, U_k' + c_1(U_k'))\,\mathrm{d}\Omega_g = a|S|, \tag{5.14}$$

we have

$$\int\limits_S f(u_0^1, u_0^2, U_0' + c_1')\,\mathrm{d}\Omega_g = a|S|. \tag{5.15}$$

Similarly, we get

$$\int\limits_S f(v_0^1, v_0^2, V_0' + c_2')\,\mathrm{d}\Omega_g = b|S|. \tag{5.16}$$

Hence from Lemma 5.1 we see that $(c_1', c_2') = (c_1(U_0'), c_2(V_0'))$. Then Lemma 5.2 follows.
At this point we can define an operator

$$T : \dot{W}^{1,2}(S) \to \dot{W}^{1,2}(S)$$

as follows. For $(U', V') \in \dot{W}^{1,2}(S)$, let $(c_1(U'), c_2(V'))$ be defined by Lemma 5.1. Define $(\tilde{U}', \tilde{V}') = T(U', V')$ where $\tilde{U}'$ and $\tilde{V}'$ are the unique solutions of

$$\Delta \tilde{U}' = 8\left(f(u_0^1, u_0^2, U' + c_1(U')) - a\right) - 4\left(f(v_0^1, v_0^2, V' + c_2(V')) - b\right), \tag{5.17}$$

$$\Delta \tilde{V}' = -4\left(f(u_0^1, u_0^2, U' + c_1(U')) - a\right) + 4\left(f(v_0^1, v_0^2, V' + c_2(V')) - b\right), \tag{5.18}$$

respectively. In fact, for any $(U', V') \in \dot{W}^{1,2}(S)$, since the right-hand sides of (5.17) and (5.18) have zero averages, the solutions $\tilde{U}'$ and $\tilde{V}'$ of (5.17) and (5.18), respectively, are unique (cf. [4]). □

Next we show that the operator T admits a fixed point in $\dot{W}^{1,2}(S)$. To this end, we first establish the following lemma.

Lemma 5.3. *The above operator $T : \dot{W}^{1,2}(S) \to \dot{W}^{1,2}(S)$ is completely continuous.*

Proof. Assume $(U_k', V_k') \to (U_0', V_0')$ weakly in $\dot{W}^{1,2}(S)$. Hence by the compact embedding theorem we see that (5.8) holds.
Denote

$$(\tilde{U}_k', \tilde{V}_k') = T(U_k', V_k') \quad \text{and} \quad (\tilde{U}_0', \tilde{V}_0') = T(U_0', V_0'). \tag{5.19}$$

Therefore we have

$$\begin{aligned}
\Delta(\tilde{U}_k' - \tilde{U}_0') &= 8\big(f(u_0^1, u_0^2, U_k' + c_1(U_k')) - f(u_0^1, u_0^2, U_0' + c_1(U_0'))\big) \\
&\quad - 4\big(f(v_0^1, v_0^2, V_k' + c_2(V_k')) - f(v_0^1, v_0^2, V_0' + c_2(V_0'))\big) \\
&= 8 f_t(u_0^1, u_0^2, \hat{U}' + \hat{c}_1)\big(U_k' - U_0' + c_1(U_k') - c_1(U_0')\big) \\
&\quad - 4 f_t(v_0^1, v_0^2, \hat{V}' + \hat{c}_2)\big(V_k' - V_0' + c_2(V_k') - c_2(V_0')\big),
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
\Delta(\tilde{V}_k' - \tilde{V}_0') &= -4\big(f(u_0^1, u_0^2, U_k' + c_1(U_k')) - f(u_0^1, u_0^2, U_0' + c_1(U_0'))\big) \\
&\quad + 4\big(f(v_0^1, v_0^2, V_k' + c_2(V_k')) - f(v_0^1, v_0^2, V_0' + c_2(V_0'))\big) \\
&= -4 f_t(u_0^1, u_0^2, \hat{U}' + \hat{c}_1)\big(U_k' - U_0' + c_1(U_k') - c_1(U_0')\big) \\
&\quad + 4 f_t(v_0^1, v_0^2, \hat{V}' + \hat{c}_2)\big(V_k' - V_0' + c_2(V_k') - c_2(V_0')\big),
\end{aligned} \tag{5.21}$$

where $\hat{U}_k'$ lies between U_k' and U_0', $\hat{V}_k'$ between V_k' and V_0', $\hat{c}_1$ between $c_1(U_k')$ and $c_1(U_0')$, and $\hat{c}_2$ between $c_2(V_k')$ and $c_2(V_0')$.

Multiplying both sides of (5.20) and (5.21) by $\tilde{U}_k' - \tilde{U}_0'$ and $\tilde{V}_k' - \tilde{V}_0'$, respectively, and integrating by parts, we obtain

$$\begin{aligned}
\|\nabla(\tilde{U}_k' - \tilde{U}_0')\|_2^2 \le \int_S \Big\{ & 4\big(|U_k' - U_0'| + |c_1(U_k') - c_1(U_0')|\big) \\
& + 2\big(|V_k' - V_0'| + |c_2(V_k') - c_2(V_0')|\big)\Big\} |\tilde{U}_k' - \tilde{U}_0'| \, \mathrm{d}\Omega_g,
\end{aligned} \tag{5.22}$$

$$\|\nabla(\tilde{V}_k' - \tilde{V}_0')\|_2^2 \le 2\int_S \Big(|U_k' - U_0'| + |c_1(U_k') - c_1(U_0')| + |V_k' - V_0'| + |c_2(V_k') - c_2(V_0')|\Big)|\tilde{V}_k' - \tilde{V}_0'|\,\mathrm{d}\Omega_g, \tag{5.23}$$

where the property (3.20) is used.

Combining (5.22) with (5.23), and using the Poincaré inequality, we arrive at

$$\|\nabla(\tilde{U}_k' - \tilde{U}_0')\|_2^2 + \|\nabla(\tilde{V}_k' - \tilde{V}_0')\|_2^2 \le C\Big(\|U_k' - U_0'\|_2^2 + \|V_k' - V_0'\|_2^2 + |c_1(U_k') - c_1(U_0')|^2 + |c_2(V_k') - c_2(V_0')|^2\Big) \tag{5.24}$$

for some $C > 0$. Then, from (5.8), Lemma 5.2, and (5.24), we see that

$$(\nabla\tilde{U}_k', \nabla\tilde{V}_k') \to (\nabla\tilde{U}_0', \nabla\tilde{V}_0') \quad \text{strongly in} \quad L^2(S) \quad \text{as} \quad k \to \infty,$$

which, with (5.8), yields

$$(\tilde{U}_k', \tilde{V}_k') \to (\tilde{U}_0', \tilde{V}_0') \quad \text{strongly in} \quad \dot{W}^{1,2}(S) \quad \text{as} \quad k \to \infty.$$

Then the proof of Lemma 5.3 is complete. □

Before applying the Leray–Schauder fixed-point theory, we need to estimate the solution of the fixed-point equation,

$$(U_t', V_t') = tT(U_t', V_t'), \quad 0 \le t \le 1. \tag{5.25}$$

Lemma 5.4. *For any (U_t', V_t') satisfying (5.25), there exists a constant $C > 0$ independent of $t \in [0, 1]$ such that*

$$\|U_t'\|_{\dot{W}^{1,2}(S)} + \|V_t'\|_{\dot{W}^{1,2}(S)} \le C. \tag{5.26}$$

Proof. From (5.25) we have

$$\Delta U_t' = 8t\left(f(u_0^1, u_0^2, U_t' + c_1(U_t')) - a\right) - 4t\left(f(v_0^1, v_0^2, V_t' + c_2(V_t')) - b\right), \tag{5.27}$$

$$\Delta V_t' = -4t\left(f(u_0^1, u_0^2, U_t' + c_1(U_t')) - a\right) + 4t\left(f(v_0^1, v_0^2, V_t' + c_2(V_t')) - b\right). \tag{5.28}$$

Multiplying both sides of (5.27) and (5.28) by U_t' and V_t', respectively, and integrating by parts, we see that

$$\begin{aligned}\|\nabla U_t'\|_2^2 &\le \int_S \Big(8|f(u_0^1, u_0^2, U_t' + c_1(U_t'))| + 4|f(v_0^1, v_0^2, V_t' + c_2(V_t'))|\Big)|U_t'|\mathrm{d}\Omega_g\\ &\le 12\int_S |U_t'|\mathrm{d}\Omega_g,\\ \|\nabla V_t'\|_2^2 &\le \int_S \Big(4|f(u_0^1, u_0^2, U_t' + c_1(U_t'))| + 4|f(v_0^1, v_0^2, V_t' + c_2(V_t'))|\Big)|V_t'|\mathrm{d}\Omega_g\\ &\le 8\int_S |V_t'|\mathrm{d}\Omega_g,\end{aligned}$$

where we have used (3.25). Then by the Poincaré inequality, we get the desired estimate (5.26).

Now using Lemmas 5.3, 5.4, and the Leray–Schauder fixed-point theorem (cf. [17]), we see that the operator T admits a fixed point, say (U', V'), in $\dot{W}^{1,2}(S)$. Thus $(U' + c_1(U'), V' + c_2(V'))$ is a solution of (5.1) and (5.2), i.e. a solution of (3.17) and (3.18).

Hence we have completed the proof of Theorem 3.2. □

6. Explicit calculation of minimum energy

In this section we establish the minimum energy formula (2.48) and show how it is stratified topologically.

By equations (2.40)–(2.43), the fact $*1 = \mathrm{d}\Omega_g$, and (3.10)–(3.11), we see that

$$\int_S (\hat{F} - \tilde{F}) = 4\int_S *\frac{1-\mathrm{e}^u}{\mathrm{e}^u+1} - 2\int_S *\frac{1-\mathrm{e}^v}{\mathrm{e}^v+1} = 2\pi(N_1 - P_1), \tag{6.1}$$

$$\int_S \tilde{F} = -2\int_S *\frac{1-\mathrm{e}^u}{\mathrm{e}^u+1} + 2\int_S *\frac{1-\mathrm{e}^v}{\mathrm{e}^v+1} = 2\pi(N_2 - P_2), \tag{6.2}$$

are valid, which give us

$$\int_S \hat{F} = 2\pi(N_1 - P_1 + N_2 - P_2). \tag{6.3}$$

To calculate the lower bound of the energy, we need to compute the fluxes contributed by the current densities $K(q)$ and $K(p)$.

Take a coordinate chart $\{\mathcal{U}_j\}$ of S. Assume $z''_{1,j} \in \mathcal{U}_j$, $j = 1, \dots, P_1$. In local coordinates, we have $D_i q = \partial_i q - \mathrm{i}(\hat{A}_i - \tilde{A}_i)q$, $i = 1, 2$ and the density $K(q)$ in $\mathcal{U}_j$ can be written as

$$K(q) = -\frac{2|q|^2}{1+|q|^2}(\hat{F} - \tilde{F}) + \mathrm{i}\frac{D_i q\overline{D_j q} - \overline{D_i q} D_j q}{(1+|q|^2)^2}\mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{6.4}$$

Besides, in $K(q) = \mathrm{d}J(q)$, we have

$$J(q) = \frac{\mathrm{i}}{1+|q|^2}(q\overline{D_i q} - \overline{q} D_i q)\mathrm{d}x^i. \tag{6.5}$$

Then it follows from the Stokes formula that

$$\int_S K(q) = \int_S \mathrm{d}J(q) = \sum_{j=1}^{P_1} \lim_{r\to 0} \oint_{\partial B(z''_{1,j}, r)} J(q), \tag{6.6}$$

where $B(z, r)$ denotes a disc centered at z with radius $r > 0$ and all the line integrals are taken counterclockwise.

Note that near $z''_{1,j} \in \mathcal{P}(q)$, the section q has the representation

$$q(z) = z^{-1}h_j(z, \overline{z}), \quad z = x^1 + \mathrm{i}x^2, \quad x^1(z''_{1,j}) = x^2(z''_{1,j}) = 0, \tag{6.7}$$

where h_j is a non-vanishing function defined near $z''_{1,j}$.

From equation (2.40) we see that

$$\hat{A}_1 - \tilde{A}_1 = -2\mathrm{Re}(\mathrm{i}\overline{\partial}\ln u), \quad \hat{A}_2 - \tilde{A}_2 = -2\mathrm{Im}(\mathrm{i}\overline{\partial}\ln u), \tag{6.8}$$

which, with $u = \ln |q|^2$, implies

$$D_1 q = (\partial + \overline{\partial})q + \left(\frac{\partial \overline{q}}{\overline{q}} - \frac{\overline{\partial} q}{q}\right) q = q\partial u, \tag{6.9}$$

$$D_2 q = \mathrm{i}(\partial - \overline{\partial})q + \mathrm{i}\left(\frac{\overline{\partial} q}{q} + \frac{\partial \overline{q}}{\overline{q}}\right) q = \mathrm{i}q\partial u. \tag{6.10}$$

Then, by (6.6), (6.9), and (6.10), we have

$$\begin{aligned}\oint\limits_{\partial B(z''_{1,j},r)} J(q) &= \mathrm{i} \oint\limits_{\partial B(z''_{1,j},r)} \frac{|q|^2}{1+|q|^2}(\overline{[\overline{\partial} - \partial]u\mathrm{d}x^1} - \mathrm{i}[\overline{\partial} + \partial]u\mathrm{d}x^2) \\ &= \oint\limits_{\partial B(z''_{1,j},r)} \frac{\mathrm{e}^u}{1+\mathrm{e}^u}(\partial_2 u\mathrm{d}x^1 - \partial_1 u\mathrm{d}x^2).\end{aligned} \tag{6.11}$$

Noting (6.7), near $z''_{1,j} \in \mathcal{P}(q)$, we see that

$$u = -2\ln|z| + w_j, \tag{6.12}$$

where w_j is a smooth function. Thus we obtain

$$\lim_{r\to 0} \oint\limits_{\partial B(z''_{1,j},r)} J(q) = 4\pi, \tag{6.13}$$

which, with (6.6), gives

$$\int\limits_S K(q) = 4\pi P_1. \tag{6.14}$$

Following a similar procedure, we have

$$\int\limits_S K(p) = 4\pi P_2. \tag{6.15}$$

As described in [31], the normalized integrals $\frac{1}{4\pi}\int K(q)$ and $\frac{1}{4\pi}\int K(p)$, counting the numbers P_1, P_2 of anti-vortices of the two species, are the Thom classes of the dual bundle $L^* \to S$, of two respective classification (Chern) classes, $\frac{1}{2\pi}\int(\hat{F} - \tilde{F})$ and $\frac{1}{2\pi}\int \tilde{F}$.

Hence, by (2.37)–(2.39), (6.3), (6.14), and (6.15), we obtain the following topologically stratified minimum energy

$$E = \int\limits_S 2([\hat{F} - \tilde{F}] + \tilde{F} + K(q) + K(p)) = 4\pi(N_1 + P_1 + N_2 + P_2), \tag{6.16}$$

as stated in Theorem 2.2.

7. Conclusions and remarks

In this work we have extended the formalism of Tong and Wong [35] of a product Abelian Higgs theory describing a coupled vortex system with magnetic impurities to accommodate co-existing vortices and anti-vortices of two species realized as topological solitons governed by a BPS system of equations. In additional to the usual first Chern classes suited over a complex Hermitian line bundle, the presence of anti-vortices switches on the Thom classes over the dual bundle, as in [31]. When the underlying Riemann surface S where vortices and anti-vortices reside is compact, we have established a theorem which spells out a necessary and sufficient condition, consisting of two inequalities, (2.46) and (2.47), for prescribed N_1, N_2 vortices and P_1, P_2 anti-vortices, of two respective species, to exist.

This necessary and sufficient condition contains a few special situations worthy of mentioning.

(i) When $N_2 = P_2 = 0$ (only vortices and anti-vortices of the first species are present), the condition becomes

$$|N_1 - P_1| < \frac{|S|}{\pi}. \tag{7.1}$$

(ii) When $N_1 = P_1 = 0$ (only vortices and anti-vortices of the second species are present), the condition reads

$$|N_2 - P_2| < \frac{|S|}{2\pi}. \tag{7.2}$$

(iii) When $N_1 = N_2 = N$ and $P_1 = P_2 = P$ (there are equal numbers of vortices and anti-vortices, respectively, of two species), the condition is

$$|N - P| < \frac{|S|}{3\pi}. \tag{7.3}$$

In all these situations, the numbers of vortices and anti-vortices may be arbitrarily large, provided that the differences of these numbers are kept in suitable ranges as given.

Although the vortices and anti-vortices of the two species do not appear in the model in a symmetric manner as seen in the field-theoretical Lagrangian density and the governing equations, they make equal contributions to the total topologically stratified minimum energy as stated in (2.48) of an elegant form.

Let $\mathcal{M}(N_1, P_1, N_2, P_2)$ denote the moduli space of solutions of the BPS equations (2.40)–(2.43) with $N_1 + N_2$ and $P_1 + P_2$ prescribed vortices and anti-vortices, of two respective species. Since these solutions depend on at least $2(N_1 + N_2 + P_1 + P_2)$ continuous parameters which specify the locations of zeros and poles of the two sections q, p, respectively, we obtain the following lower bound for the dimensionality of $\mathcal{M}(N_1, P_1, N_2, P_2)$:

$$\dim(\mathcal{M}(N_1, P_1, N_2, P_2)) \geq 2(N_1 + N_2 + P_1 + P_2). \tag{7.4}$$

Since we have not established the uniqueness of a solution with $N_1 + N_2$ and $P_1 + P_2$ prescribed vortices and anti-vortices of the two species yet, we do not know whether the inequality (7.4) is actually an equality. In this regard, it will be interesting to carry out an investigation along the (well-known classical) index theory work of Atiyah, Hitchin, and Singer [2,3] on the Yang–Mills instantons, of Weinberg [39] on the BPS system of the Abelian Higgs model, and of Lee [21] on supersymmetric domain walls, for our new system of equations (2.40)–(2.43).

In a sharp contrast, if we use $\mathcal{M}(N_1, N_2)$ to denote the moduli space of the solutions of the Tong–Wong equations (2.11)–(2.14) with N_1 and N_2 prescribed vortices, of two respective species, the established uniqueness of the solutions indicates the result

$$\dim(\mathcal{M}(N_1, N_2)) = 2(N_1 + N_2). \tag{7.5}$$

See [23] for some recent related work.

Acknowledgements

Han was partially supported by National Natural Science Foundation of China under Grant 11201118 and the Key Foundation for Henan colleges under Grant 15A110013. Both authors were partially supported by National Natural Science Foundation of China under Grants 11471100 and 11471099.

References

[1] A.A. Abrikosov, On the magnetic properties of superconductors of the second group, Sov. Phys. JETP 5 (1957) 1174–1182.
[2] M.F. Atiyah, N.J. Hitchin, I.M. Singer, Deformation of instantons, Proc. Natl. Acad. Sci. USA 74 (1977) 2662–2663.
[3] M.F. Atiyah, N.J. Hitchin, I.M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. A 362 (1978) 425–461.
[4] T. Aubin, Nonlinear Analysis on Manifolds: Monge–Ampére Equations, Springer, Berlin, New York, 1982.
[5] J.M. Baptista, N.S. Manton, The dynamics of vortices on S^2 near the Bradlow limit, J. Math. Phys. 44 (2003) 3495–3508.
[6] E.B. Bogomol'nyi, The stability of classical solutions, Sov. J. Nucl. Phys. 24 (1976) 449–454.
[7] S. Bradlow, Vortices in holomorphic line bundles over closed Kähler manifolds, Commun. Math. Phys. 135 (1990) 1–17.
[8] S. Chen, Y. Yang, Existence of multiple vortices in supersymmetric gauge field theory, Proc. R. Soc. A 468 (2012) 3923–3946.
[9] S. Demoulini, Convergence of the self-dual Ginzburg–Landau gradient flow, Math. Phys. Anal. Geom. 16 (2013) 195–212.
[10] M. Eto, M. Nitta, N. Sakai, Effective theory on non-Abelian vortices in six dimensions, Nucl. Phys. B 701 (2004) 247–272.
[11] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Solitons in the Higgs phase – the moduli matrix approach, J. Phys. A 39 (2006) R315–R392.
[12] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, N. Sakai, Moduli space of non-Abelian vortices, Phys. Rev. Lett. 96 (2006) 161601.
[13] M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, N. Sakai, Multiple layer structure of non-Abelian vortex, Phys. Lett. B 678 (2009) 254–258.
[14] L. Fontana, Sharp borderline Sobolev inequalities on compact Riemannian manifolds, Comment. Math. Helv. 68 (1993) 415–454.
[15] O. Garcia-Prada, A direct existence proof for the vortex equations over a compact Riemann surface, Bull. Lond. Math. Soc. 26 (1994) 88–96.
[16] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, New York, 1977.
[17] V.L. Ginzburg, L.D. Landau, On the theory of superconductivity, in: D. Ter Haar (Ed.), Collected Papers of L.D. Landau, Pergamon, New York, 1965, pp. 546–568.
[18] A. Jaffe, C.H. Taubes, Vortices and Monopoles, Birkhäuser, Boston, 1980.
[19] T.W.B. Kibble, Some implications of a cosmological phase transition, Phys. Rep. 69 (1980) 183–199.
[20] T.W.B. Kibble, Cosmic strings – an overview, in: G. Gibbons, S. Hawking, T. Vachaspati (Eds.), The Formation and Evolution of Cosmic Strings, Cambridge University Press, 1990, pp. 3–34.
[21] K.S.M. Lee, An index theorem for domain walls in supersymmetric gauge theories, Phys. Rev. D 67 (2003) 045009.
[22] E.H. Lieb, Y. Yang, Non-Abelian vortices in supersymmetric gauge field theory via direct methods, Commun. Math. Phys. 313 (2012) 445–478.

[23] G.W. Moore, A.B. Royston, D. Van den Bleeken, Parameter counting for singular monopoles on $\mathbb{R}^3$, J. High Energy Phys. 1410 (2014) 142.
[24] S.M. Nasir, Vortices and flat connections, Phys. Lett. B 419 (1998) 253–257.
[25] M. Noguchi, Abelian Higgs Theory on Riemann Surfaces, Thesis, Duke University, 1985.
[26] M. Noguchi, Yang–Mills–Higgs theory on a compact Riemann surface, J. Math. Phys. 28 (1987) 2343–2346.
[27] M.K. Prasad, C.M. Sommerfield, Exact classical solutions for the 't Hooft monopole and the Julia–Zee dyon, Phys. Rev. Lett. 35 (1975) 760–762.
[28] L.H. Ryder, Quantum Field Theory, 2nd edn., Cambridge University Press, Cambridge, UK, 1997.
[29] B.J. Schroers, Bogomol'nyi solitons in a gauged $O(3)$ sigma model, Phys. Lett. B 356 (1995) 291–296.
[30] B.J. Schroers, The spectrum of Bogomol'nyi solitons in gauged linear sigma models, Nucl. Phys. B 475 (1996) 440–468.
[31] L.M. Sibner, R.J. Sibner, Y. Yang, Abelian gauge theory on Riemann surfaces and new topological invariants, Proc. R. Soc. A 456 (2000) 593–613.
[32] D. Sullivan, René Thom's work on geometric homology and bordism, Bull. Am. Math. Soc. 41 (2004) 341–350.
[33] C.H. Taubes, Arbitrary N-vortex solutions to the first order Ginzburg–Landau equations, Commun. Math. Phys. 72 (1980) 277–292.
[34] C.H. Taubes, On the equivalence of the first and second order equations for gauge theories, Commun. Math. Phys. 75 (1980) 207–227.
[35] D. Tong, K. Wong, Vortices and impurities, J. High Energy Phys. 1401 (2014) 090.
[36] A. Vilenkin, Cosmic strings and domain walls, Phys. Rep. 121 (1985) 263–315.
[37] A. Vilenkin, E.P.S. Shellard, Cosmic Strings and Other Topological Defects, Cambridge University Press, 1994.
[38] S. Wang, Y. Yang, Abrikosov's vortices in the critical coupling, SIAM J. Math. Anal. 23 (1992) 1125–1140.
[39] E.J. Weinberg, Multivortex solutions of the Ginzburg–Landau equations, Phys. Rev. D 19 (1979) 3008–3012.
[40] Y. Yang, Obstructions to the existence of static cosmic strings in an Abelian Higgs model, Phys. Rev. Lett. 73 (1994) 10–13.
[41] Y. Yang, Prescribing topological defects for the coupled Einstein and Abelian Higgs equations, Commun. Math. Phys. 170 (1995) 541–582.
[42] Y. Yang, Coexistence of vortices and antivortices in an Abelian gauge theory, Phys. Rev. Lett. 80 (1998) 26–29.
[43] Y. Yang, Strings of opposite magnetic charges in a gauge field theory, Proc. R. Soc. Lond. A 455 (1999) 601–629.
[44] Y. Yang, Solitons in Field Theory Nonlinear Analysis, Springer, New York, 2001.