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Sequences of Binomial Type with Persistent Roots

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We find all sequences of polynomials $(p_n)_{n \geq 0}$ with persistent roots (i.e., $p_n(x) = c_n(x - r_1)(x - r_2) \cdots (x - r_n)$) that are of binomial type in Viskov's generalization of Rota's umbral calculus to generalized Appell polynomials. We show that such sequences only exist in the classical umbral calculus, the divided difference umbral calculus, and the new "hyperbolic" umbral calculus generated by $(d/d\sqrt{x})^2$. In each of these three umbral calculi, we also find all Sheffer sequences with persistent roots. © 1996 Academic Press, Inc.

1. INTRODUCTION

Most sequences of polynomials that interest mathematicians fall into one of the following three categories or their generalizations:

- sequences of binomial type [13–15],
- sequences with persistent roots [4, 5] and
- sequences of orthogonal polynomials [3].

These theories are very rich; powerful formulas allow the computation of one such sequence of polynomials in terms of another of the same type.

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However, to go from, for example, an arbitrary (generalized) convolution sequence to an arbitrary sequence with persistent roots, it is necessary to pass through a sequence that is both, for example, $x^n/n!$.

This gives rise to the following three questions:

What are the convolution sequences of orthogonal polynomials? This first question is treated by [1, 3, 10, 16].

What are the persistent sequences of orthogonal polynomials? Since persistent sequences obey a two-term recurrence and orthogonal sequences obey a three-term recurrence (Favard's theorem), there are no persistent sequences of orthogonal polynomials.

What are the persistent sequences of binomial type?

The objective of this paper is the complete resolution of this last question. Up to rescaling, all persistent convolution sequences can be found in Table I.

A sequence of polynomials $(p_n)_{n\geq 0}$ is said to be of *binomial type* if it obeys the identity

$$E^y p_n(x) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(x) p_k(y), \tag{1}$$

where $E^y p(x) = p(x+y)$. Well-known examples include the powers of x (in which case Eq. (1) is called the binomial theorem), the Abel polynomials $x(x-na)^{n-1}$, the Laguerre polynomials, and so on [13, 14].

TABLE I
Complete List of Persistent Convolution Sequences

Umbral calculus			Roots	Delta operator	Polynomials
Name	b_n	Φ	r_n	$f(D_b)p(x)$	$q_n(x)$
any	any	any	$0, 0, 0, 0, \ldots$	$D_b p(x)$	$\dfrac{x^n}{b_n}$
divided difference	1	$\dfrac{1}{1-t}$	$0, 1, 1, 1, \ldots$	$\dfrac{xp(x) - p(1)}{x-1}$	$x(x-1)^{n-1}$
classical	$n!$	exp	$0, 1, 2, 3, \ldots$	$p(x+1) - p(x)$	$\dbinom{x}{n}$
hyperbolic	$(2n)!$	cosh	$0, 1, 4, 9, \ldots$	$\frac{1}{2}\Big[p\big([\sqrt{x}+1]^2\big) + p\big([\sqrt{x}-1]^2\big) - 2p(x)\Big]$	$\dfrac{x(x-1)\cdots\left(x-(n-1)^2\right)}{(2n)!}$

If $(p_n)_{n\geq 0}$ is of binomial type, then the sister sequence $(q_n)_{n\geq 0} = (p_n/n!)_{n\geq 0}$ obeys the identity

$$E^y q_n(x) = \sum_{k=0}^{n} q_{n-k}(x) q_k(y).$$

It is often more convenient to consider the sister *convolution sequence* rather than the sequence of binomial type itself.

An important generalization of the umbral calculus involves replacing the derivative D with a homogeneous linear operator of the form $D_b x^n = b_n x^{n-1}/b_{n-1}$ (where $b_n \neq 0$ for all n) provided that the shift E^y is replaced with $E_b^y = \Phi(yD_b)$ where $\Phi(t) = \sum_{n=0}^{\infty} t^n/b_n$ [11, 12, 20, 21]. The resulting polynomials are sometimes called generalized Appell polynomials.

However, umbral calculus does not include all important sequences of polynomials even as generalized above. For that reason, we now turn to sequences with persistent roots.

A sequence $(p_n)_{n\geq 0}$ is said to have *persistent roots* or be *persistent* if $p_n(x)$ divides $p_{n+1}(x)$ for all n. That is to say, all of the roots in p_n persist in p_{n+1}.

In Section 3, we consider the classical umbral calculus ($b_n = n!$). All persistent convolution sequences have their roots in an arithmetic progression

$$p_n(x) = x(x-a)\cdots(x-(n-1)a).$$

Moreover, if we allow Sheffer sequences, then we are no longer limited to progressions which start at zero.

We then consider two other umbral calculi and characterize all of their convolution and Sheffer sequences with persistent roots. In Section 4, we consider the divided difference umbral calculus ($b_n = 1$) [8, 17−19] which is important in interpolation theory. In Section 5, we consider a new umbral calculus ($b_n = (2n)!$) which we call the hyperbolic umbral calculus since $\Phi(t) = \cosh(\sqrt{t})$.

Surprisingly, these three examples give essentially all persistent convolution sequences. Obviously, any sequence $(q_n(x))_{n\geq 0} = (c^n x^n/b_n)_{n\geq 0}$ has persistent roots (all zero), and is the basic convolution sequence for the delta operator D_b/c. However, in Section 6, we show that, other than these trivial examples, all persistent convolution sequences belong to the classical, divided difference, or hyperbolic umbral calculus. We have thus explicitly characterized all persistent divided difference polynomial sequences.

Our work was assisted by extensive use of the computer algebra system **Maple** and our umbral calculus package [2].

2. GENERATING FUNCTIONS

We show in this section that in any umbral calculus, there is, up to vertical and horizontal scaling, at most one nontrivial persistent convolution sequence. These results are superseded by the explicit examples in Sections 3–5, and the completeness results in Section 6 which follow as consequences of these uniqueness results.

2.1. *Umbral Calculus*

Modern umbral calculus employs powerful operator methods to derive results on polynomials [13–15].

Different commutation classes of operators give yield to analogous results concerning different sequences of polynomials [9, 21].

The classical umbral calculus concerns the family of operators that commute with the derivative D. These operators can all be expressed as formal power series in the derivative. Thus, they all commute with each other. In particular, they commute with the shift operator $E^y = \exp(yD)$. Hence, these operators are usually called shift-invariant operators.

The situation in other umbral calculi is analogous. A *generalized derivative* is defined by a non-zero generalized factorial function b_n as

$$D_b x^n = \begin{cases} \dfrac{b_n}{b_{n-1}} x^{n-1} & \text{if } n > 0, \text{ and} \\ 0 & \text{if } n = 0. \end{cases}$$

The *generalized shift operator* is then $E_b^y = \Phi(yD_b)$ where $\Phi(t) = \sum_{n=0}^{\infty} t^n / b_n$ so that we have the following generalized binomial theorem [13, 20]

$$E_b^y x^n = \sum_{k=0}^{n} \frac{b_n}{b_k b_{n-k}} x^k y^{n-k}.$$

2.2. *Convolution Sequences*

The sequences of polynomials $(p_n)_{n \geq 0}$ under consideration in the umbral calculus are required to satisfy $\deg(p_n) = n$. These sequences are studied in terms of the operators which define them.

For example, let f be a *delta series*, i.e., a formal power series with zero constant term. Then any sequence $(q_n)_{n \geq 0}$ satisfying $f(D_b) q_n(x) = q_{n-1}(x)$ will be said to be Sheffer. If further we have $q_n(0) = 0$ for $n > 0$, and $q_0 \equiv 1$, then $(q_n)_{n \geq 0}$ is said to be a convolution sequence.

In the classical umbral calculus, convolution sequences q_n have generating functions of the form

$$\sum_{n=0}^{\infty} q_n(x)t^n = \exp\big(xf^{(-1)}(t)\big),$$

where $f^{(-1)}$ denotes the compositional inverse of f.

In general, convolution sequences q_n are characterized by generating functions of the form

$$\sum_{n=0}^{\infty} q_n(x)t^n = \Phi\big(xf^{(-1)}(t)\big). \tag{2}$$

The delta operator of $q_n(x)$ is then $f(D_b)$.

Thus, each convolution sequence $(q_n)_{n\geq 0}$ is determined uniquely by an umbral calculus b and a delta series f. Conversely, every convolution sequence belongs to essentially only one umbral calculus b (Lemma 6.1).

However, if we require that $(q_n)_{n\geq 0}$ have persistent roots, then we can say much more. In this case, $(q_n)_{n\geq 0}$ is determined by the umbral calculus b and the polynomial q_2 alone.

PROPOSITION 2.1. *Let $(q_n)_{n\geq 0}$ be a persistent convolution sequence in the umbral calculus b with respect to the delta operator $f(D_b)$ with roots r_n. Then either*:

- *The sequence is trivial, that is, $q_n(x) = c^n x^n/b_n$, $f(D_b) = D_b/c$, and $r_n = 0$ for all $n \geq 1$, or else*
- $r_1 = 0$, $r_2 \neq 0$, *and* $f(D_b) = c(E_b^{r_2} - 1)/r_2$. (*For* $c = r_2 = 1$, $f(D_b)$ *is called the forward difference operator* Δ_b.)

We will give two proofs of this important result. The first one uses operator methods, and the second one uses generating functions.

Proof 1. *Case* $r_2 \neq 0$. By the generalized binomial theorem, $E_b^{r_2} q_n(x) = \sum_{k=0}^{n} q_{n-k}(x) q_k(r_2)$. However, $q_k(r_2) = 0$ for $k \geq 2$, and $q_1(x) = x/c$. Thus, $E_b^{r_2} q_n(x) = q_n(x) + (r_2/c) q_{n-1}(x)$. Hence, $q_{n-1}(x) = c(E_b^{r_2} - 1)q_n/r_2$, and $Q = c(E_b^{r_2} - 1)/r_2$.

Case $r_2 = 0$. Assume $Q = D_b/c + R$ where R is an operator of order $n \geq 2$. The constant term of $p_{n-1} = Qp_n = D_b p_n/c + Rp_n$ is zero, so the constant term of Rp_n is zero. This contradicts the fact that R is an operator of order n in which case Rp_n should be a non-zero constant. ∎

Proof 2. *Case* $r_2 \neq 0$. Evaluating (2) at $x = r_2$, annihilates all but the first two terms on the left hand side leaving only

$$1 + ct = \Phi\big(r_2 f^{(-1)}(t)\big).$$

Substituting $t = f(D_b)$ yields

$$1 + r_2 f(D_b) = \Phi(r_2 D_b)$$

$$f(D_b) = \big(\Phi(r_2 D_b) - 1\big)/r_2.$$

Case $r_2 = 0$. Suppose $f^{(-1)}(t) = ct + R(t)$ where R is a formal power series of order at least two. The coefficient of x on the left hand side of (2) is ct and on the right hand side $ct + R(t)/b_1$. Thus, $R(t) = 0$. ∎

Proposition 2.1 gives a necessary condition for persistent convolution sequences. As we will see, this condition is not sufficient. In fact, this condition only guarantees that the first two roots persist. That is, $x(x - r_2)$ divides $q_n(x)$ for $n \geq 2$.

2.3. *Sheffer Sequences*

Given a convolution sequence $(q_n)_{n \geq 0}$, we say that a polynomial sequence $(s_n)_{n \geq 0}$ is Sheffer relative to $(q_n)_{n \geq 0}$ if

$$E_b^y s_n(x) = \sum_{k=0}^{n} q_{n-k}(x) s_k(y),$$

or equivalently if there is an invertible operator A such that $As_k = q_k$ for all k. The generating function for s_n is of the form

$$\sum_{n=0}^{\infty} s_n(x) t^n = \Phi\big(xf^{(-1)}(t)\big)/a\big(f^{(-1)}(t)\big), \tag{3}$$

where f and Φ are as above and $A = a(D_b)$ (see [20]). Examples here include the Bernoulli polynomials and the generalized Laguerre polynomials.

Proposition 2.1 has a succinct Sheffer generalization. A Sheffer sequence $(s_n)_{n \geq 0}$ is determined uniquely by an umbral calculus b and delta series f and a. However, if $(s_n)_{n \geq 0}$ is known to have persistent roots, then $(s_n)_{n \geq 0}$ is determined by the umbral calculus b, $q_1(x)$, and $q_2(x)$ alone.

PROPOSITION 2.2. *Let* $(s_n)_{n \geq 0}$ *be a persistent Sheffer sequence in the umbral calculus* b *with respect to the delta operator* $f(D_b)$ *and invertible operator* $a(D_b)$ *with roots* r_n.

- *If* $r_1 = r_2$, *then*

$$a(D_b) = E_b^{r_1}/k$$

and

$$f(D_b) = \frac{kD_b \Phi'(r_1 D_b)}{c\Phi(r_1 D_b)}.$$

- *If* $r_1 \neq r_2$, *then*

$$a(D_b) = kE_b^{r_1}/k$$

and

$$f(D_b) = k\left(\frac{E_b^{r_2}}{E_b^{r_1}} - 1\right)\Big/ c(r_2 - r_1).$$

Note that if $r_1 = 0$, then $(s_n)_{n \geq 0}$ is actually a convolution sequence. Thus, $a \equiv 1$, and we may apply Proposition 2.1.

Proof. *Case* $r_1 \neq r_2$. Substituting $x = r_1$ and $t = f(u)$ into (3), we obtain

$$a(u) = \Phi(r_1 u)/k, \tag{4}$$

where $k = s_0(0)$. Now, substitute $x = r_2$ and $t = f(u)$ into Eq. (3) which yields

$$k + cf(u)(r_2 - r_1) = \frac{k}{\Phi(r_1 u)}\Phi(r_2 u).$$

Solving for $f(u)$, we conclude

$$f(u) = k\left(\frac{\Phi(r_2 u)}{\Phi(r_1 u)} - 1\right)\Big/ c(r_2 - r_1).$$

Case $r_1 = r_2$. Shift the generating function (3) by r_1,

$$\sum_{n=0}^{\infty} s_n(x + r_1)t^n = \Phi\big((x + r_1)g(t)\big)/a(g(t)),$$

where $g(t) = f^{(-1)}(t)$. Compare the coefficients of x on both sides,

$$\begin{aligned} ct &= a(g(t))\left(\frac{g(t)}{b_1} + \frac{2r_1 g(t)^2}{b_2} + \frac{3r_1^2 g(t)^3}{b_3} + \cdots\right) \\ &= \Phi'(r_1 g(t))g(t)a(g(t)). \end{aligned}$$

Equation (4) now implies that

$$\begin{aligned} \Phi(r_1 g(t))ct/k &= \Phi'(r_1 g(t))f(t) \\ \Phi(r_1 u)cf(u) &= k\Phi'(r_1 u)u \\ f(u) &= ku\Phi'(r_1 u)/c\Phi(r_1 u). \end{aligned}$$

∎

Note that Proposition 2.2 gives a necessary condition for persistent Sheffer sequences. As we will see, this condition is not sufficient. In fact, this condition only guarantees that the first two roots persist. That is, $(x - r_1)(x - r_2)$ divides $s_n(x)$ for $n \geq 2$.

3. CLASSICAL UMBRAL CALCULUS

3.1. *Umbral Calculus*

What are the persistent convolution sequences in the classical umbral calculus?

In the classical umbral calculus, $b_n = n!$ is the usual factorial, $D_b = D$ is the usual derivative, $E_b^y p(x) = p(x+y)$ is the usual shift operator E^y, and $\Phi(t) = \exp(t)$.

3.2. *Convolution Sequences*

Up to scaling (vertical and horizontal) the only persistent convolution sequences in the classical umbral calculus are $x^n/n!$ and $\binom{x}{n}$.

PROPOSITION 3.1. *Let $(q_n)_{n \geq 0}$ be a classical persistent convolution sequence. Then either*:

- $q_n(x) = c^n x^n / n!$ *with*
  —*delta operator* $f(D) = D/c$, *and*
  —*roots* $r_n = 0$ *for all* $n \geq 1$, *or else*

- $q_n(x) = c^n x(x - r_2) \cdots (x - (n-1)r_2)/n! = (cr_2)^n \binom{x/r_2}{n}$ *with*
  —*delta operator* $f(D) = (E^{r_2} - 1)/cr_2$, *and*
  —*roots* $r_n = (n-1)r_2$ *for all* $n \geq 1$.

The classical forward difference operator is thus given by $\Delta_b p(x) = p(x+1) - p(x)$.

Proof. $f(D)$ must be as given by Proposition 2.1. Note that $(E^1 - 1)\binom{x}{n} = \binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1}$. ∎

3.3. *Sheffer Sequences*

The result for Sheffer sequences is surprisingly similar. Up to scaling (vertical and horizontal) the only persistent Sheffer sequences in the classical umbral calculus are of the form $x^n/n!$, $(x-1)^n/n!$, or $\binom{x-a}{n}$.

PROPOSITION 3.2. *Let* $(s_n)_{n \geq 0}$ *be a classical persistent Sheffer sequence. Then either*:

- $s_n(x) = c^n(x - r_1)^n / kn!$, *which is Sheffer*
  —*relative to* $q_n(x) = c^n x^n / kn!$,
  —*with delta operator* $f(D) = D/c$,
  —*invertible operator* $a(D) = E^{r_1}/k$, *and*
  —*with roots* $r_n = r_1$ *for all* n, *or else*

- $s_n(x) = c^n(r_2 - r_1)^n \binom{(x - r_1)/(r_2 - r_1)}{n} / k$, *which is Sheffer*
  —*relative to* $q_n(x) = c^n(r_2 - r_1)^n \binom{x/(r_2 - r_1)}{n} / k$,
  —*with delta operator* $f(D) = (E^{r_2 - r_1} - 1)/c(r_2 - r_1)$,
  —*invertible operator* $a(D) = E^{r_1}/k$, *and*
  —*with roots* $r_n = (n - 1)(r_2 - r_1) + r_1$.

Proof. By Proposition 2.2, we have $a(D) = E^{r_1}/k$. Now, apply $a(D)^{-1} = kE^{-r_1}$ to the sequences of polynomials mentioned in Proposition 3.1. ∎

4. DIVIDED DIFFERENCE UMBRAL CALCULUS

4.1. *Umbral Calculus*

In the divided difference umbral calculus, $b_n = 1$. Thus, $D_b x^n = x^{n-1}$ for $n > 0$, and $D_b x^0 = 0$. Thus, by linearity, $D_b p(x) = (p(x) - p(0))/x$. The divided difference shift is given by $E_b^y x^n = x^n + yx^{n-1} + \cdots + y^n = (x^{n+1} - y^{n+1})/(x - y)$. Thus, $E_b^y p(x) = (xp(x) - yp(y))/(x - y)$, and $E_b^y = 1/(1 - yb)$, or equivalently, $\Phi(t) = 1/(1 - t)$. Note however, that the inverse of E^y is not $E_b^{-y} = 1/(1 + yb)$, but rather

$$\left(E_b^y\right)^{-1} = 1 - yD_b$$

$$\left(E_b^y\right)^{-1} p(x) = ((x - y)p(x) + yp(0))/x.$$

4.2. *Convolution Sequences*

Up to scaling (vertical and horizontal), the only persistent convolution sequences in the divided difference umbral calculus are x^n and $x(x - 1)^{n-1}$.

PROPOSITION 4.1. *Let* $(q_n)_{n \geq 0}$ *be a divided difference persistent convolution sequence. Then either*:

- $q_n(x) = c^n x^n$ *with*
  —*delta operator* $f(D_b) = D_b/c$, *and*
  —*roots* $r_n = 0$ *for all* $n \geq 1$, *or else*

- $q_n = c^n \prod_{i=0}^{n-1}(x - ir_2)$ *with*
  —*delta operator* $f(D_b) = D_b/c(1 - r_2 D_b)$ *given by* $f(D_b)p(x) = (p(x) - p(r_2))/c(x - r_2)$, *and*
  —*roots* $r_n = r_2$ *for* $n \geq 2$.

The divided difference forward difference operator is thus the finite difference operator

$$\Delta_b p(x) = \frac{p(x) - p(1)}{x - 1}.$$

Proof. By Proposition 2.1, $f(D_b) = D_b/c(1 - r_2 D_b)$. Now,

$$\begin{aligned} f(D_b)p(x) &= \frac{E_b^{r_2} p(x) - p(x)}{cr_2} \\ &= \frac{xp(x) - r_2 p(r_2)}{cr_2(x - r_2)} - \frac{p(x)}{cr_2} \\ &= \frac{xp(x) - r_2 p(r_2) - xp(x) + r_2 p(x)}{cr_2(x - r_2)} \\ &= \frac{p(x) - p(r_2)}{c(x - r_2)}. \end{aligned}$$

In particular, for $p(x) = q_n(x)$ where $n \geq 2$, $p(r_2) = 0$, so

$$f(D_b)q_n(x) = c^{n-1}x(x - r_2)^{n-2} = q_{n-1}(x).$$

Moreover, $f(D_b)cx = 1$ and $f(D_b)1 = 0$. ∎

4.3. *Sheffer Sequences*

The result for Sheffer sequences is surprisingly similar. Other than those mentioned above, up to scaling (vertical and horizontal) the only persistent Sheffer sequences in the divided difference umbral calculus are of the form $(x - a)(x - 1)^n$.

PROPOSITION 4.2. *Let* $(s_n)_{n \geq 0}$ *be a divided persistent Sheffer sequence. Then* $s_n(x) = c^n(x - r_1)(x - r_2)^{n-1}/k$, *which is Sheffer*

- *relative to* $q_n(x) = c^n x(x - r_2)^{n-1}$,
- *with delta operator* $f(D_b) = D_b/c(1 - r_2 D_b)$,[1]

[1] Note that in this case $f(D_b)$ does not depend on r_1, and $f(D_b)$ simplifies to D_b/c if furthermore $r_2 = 0$.

- *with invertible operator* $a(D_b) = 1/k(1 - r_1 D_b)$ (*so that* $a(D_b)p(x) = (xp(x) - r_1 p(r_1))/(x - r_1)$), *and*
- *with roots* $r_n = r_1$ *for all* $n \geq 1$.

Proof. *Case* $r_1 = 0$. This specializes to Proposition 4.1.

Case $r_1 \neq 0$. It suffices to check that the operators $f(D_b)$ and $a(D_b)$ are indeed those required by Proposition 2.2, and that $f(D_b)s_n = s_{n-1}$. For $n > 1$,

$$\begin{aligned} f(D_b)s_n(x) &= \frac{s_n(x) - s_n(r_2)}{c(x - r_2)} \\ &= \frac{c^n(x - r_1)(x - r_2)^{n-1} - 0}{c(x - r_2)} \\ &= s_{n-1}(x). \end{aligned}$$

∎

Remark 1. The sequence $(x - a)^n$ is of binomial type in both the classical and the divided difference umbral calculus. However, $(x - a)^n/n!$ (resp. $(x - a)^n$) is a convolution sequence in the classical (resp. divided difference) umbral calculus, but not the other. As we will see in Lemma 6.1, convolution sequences characterize in some sense their umbral calculus. For this reason in part, we have chosen to work with convolution sequences instead of sequences of binomial type.

Remark 2. The sequence $(x - a)^n$ is Sheffer on the divided difference umbral calculus with respect to $x(x - a)^{n-1}$ and *not* x^n as its classical counterpart might seem to suggest.

5. HYPERBOLIC UMBRAL CALCULUS

5.1. *Umbral Calculus*

Consider the hyperbolic factorials $b_n = (2n)!$. The hyperbolic derivative is given by

$$D_b x^n = \frac{(2n)!}{(2n - 2)!} x^{n-1}.$$

Let $z = \sqrt{x}$.

$$\begin{aligned} D_b z^{2n} &= 2n(2n - 1) z^{2n-2} \\ &= \frac{d^2 z^{2n}}{dz^2}. \end{aligned}$$

Thus, the hyperbolic derivative is simply the second derivative with respect to $z = \sqrt{x}$. As a linear operator [6], the hyperbolic derivative can be expressed in terms of the ordinary derivative with respect to x, namely $D_b = Dx^2 + D$ [7, Proposition 2.4.2.1].

The hyperbolic shift is given by

$$E_b^y = \sum_{n=0}^{\infty} y^n D_b^n/(2n)! = \cosh\left(\sqrt{yD_b}\right),$$

justifying the designation "hyperbolic."

PROPOSITION 5.1. *The hyperbolic shift is given explicitly by*

$$E_b^y p(x) = \left[p\left(\left[\sqrt{x} + \sqrt{y}\right]^2\right) + p\left(\left[\sqrt{x} - \sqrt{y}\right]^2\right)\right]\Big/2.$$

Proof. By linearity it suffices to consider $p(x) = x^n$.

$$\begin{aligned} E_b^y x^n &= \sum_{k=0}^{n} \frac{(2n)!}{(2k)!(2n-2k)!} x^k y^{n-k} \\ &= \sum_{k=0}^{n} \binom{2n}{2k} \sqrt{x}^{2k} \sqrt{y}^{2n-2k} \\ &= \frac{1}{2} \sum_{k=0}^{2n} \binom{2n}{k} \left(\sqrt{x}^k \sqrt{y}^{2n-k} + (-1)^{2n-k} \sqrt{x}^k \sqrt{y}^{2n-k}\right) \\ &= \frac{1}{2}\left(\left(\sqrt{x} + \sqrt{y}\right)^{2n} + \left(\sqrt{x} - \sqrt{y}\right)^{2n}\right). \end{aligned}$$ ∎

Note, however, that the inverse of E_b^y is not $E_b^{-y} = \cosh(i\sqrt{yD_b})$, but

$$\left(E_b^y\right)^{-1} = \operatorname{sech}\left(\sqrt{yD_b}\right) = \sum_{n=0}^{\infty} \mathscr{E}_{2n}(aD_b)^n/(2n)!,$$

where $\mathscr{E}_{2n}$ are the Euler numbers.

5.2. *Convolution Sequences*

Up to scaling (vertical and horizontal) the only hyperbolic persistent convolution sequences are $x^n/(2n)!$ and $x(x-1)(x-4)(x-9)\cdots(x-(n-1)^2)/(2n)!$.

PROPOSITION 5.2. *Let* $(q_n)_{n \geq 0}$ *be a hyperbolic persistent convolution sequence. Then either*:

- $q_n(x) = c^n x^n/(2n)!$ *with*
  —*delta operator* $f(D_b) = D_b/c$, *and*
  —*roots* $r_n = 0$ *for all* $n \geq 0$, *or else*
- $q_n(x) = c^n \prod_{i=0}^{n-1}(x - i^2 r_2)$ *with*
  —*delta operator* $f(D_b) = (\cosh(\sqrt{r_2 D_b}) - 1)/cr_2$, *or more explicitly*,

$$f(D_b)p(x) = \left(p\left(\left(\sqrt{x} + \sqrt{r_2}\right)^2\right) + p\left(\left(\sqrt{x} - \sqrt{r_2}\right)^2\right) - 2p(x)\right)\Big/2cr_2,$$

—*and roots* $r_n = (n-1)^2 r_2$ *for* $n \geq 1$.

The hyperbolic forward difference operator is thus given by

$$\Delta_b p(x) = p\left(x + 2\sqrt{x} + 1\right) + p\left(x - 2\sqrt{x} + 1\right) - 2p(x).$$

Proof. By Proposition 2.1, $f(D_b) = (E_b^{r_2} - 1)/cr_2$. Now, by Proposition 5.1,

$$f(D_b)p(x) = \left(p\left(\left(\sqrt{x} + \sqrt{r_2}\right)^2\right) + p\left(\left(\sqrt{x} - \sqrt{r_2}\right)^2\right) - 2p(x)\right)\Big/2cr_2.$$

We now show that $f(D_b)q_n(x) = q_{n-1}(x)$ by verifying that $f(D_b)q_n(r)$ has the correct roots.

$$\begin{aligned} f(D_b)q_n\left((i-1)^2 r_2\right) &= \Big(q_n\left(\left((i-1+1)\sqrt{r_2}\right)^2\right) + q_n\left(\left((i-1-1)\sqrt{r_2}\right)^2\right) \\ &\qquad - 2q_n\left((i-1)r_2\right)\Big)/2cr_2 \\ &= \left(q_n(i^2 r_2) + q_n\left((i-2)^2 r_2\right) - 2q_n\left((i-1)^2 r_2\right)\right)/2cr_2 \\ &= 0 + 0 - 0. \end{aligned}$$

∎

5.3. *Sheffer Sequences*

The result for Sheffer sequences is surprisingly similar. Other than those mentioned above, up to scaling (vertical and horizontal) the only persistent Sheffer sequence in the divided difference umbral calculus is $(x-1)(x-9)(x-25)(x-49)\cdots(x-(2n-1)^2)/(2n)!$ (see Table II).

PROPOSITION 5.3. *Let* $(s_n)_{n \geq 0}$ *be a hyperbolic persistent Sheffer sequence. Then either*:

- $r_1 = 0$ *in which case the polynomials* $s_n(x) = c^k x(x - r_2) \times (x - 4r_2)\cdots(x-(n-1)^2 r_2)/k(2n)!$ *with roots* $r_n = (n-1)r_2$ *form in fact a convolution sequence as in Proposition* 5.2, *or else*

TABLE II
Partial List of Persistent Sheffer Sequences

Umbral calculus			Roots	Delta operator	Invertible operator	Polynomials
Name	b_n	Φ	r_n	$f(D_b)p(x)$	$a(D_b)p(x)$	$q_n(x)$
divided difference	1	$\frac{1}{1-t}$	$r, 1, 1, 1, \ldots$	$\frac{p(x)-p(1)}{x-1}$	$\frac{xp(x)-rp(r)}{x-r}$	$(x-r)(x-1)^{n-1}$
classical	$n!$	exp	$r, r+1, \ldots$	$p(x+1)-p(x)$	$p(x+r)$	$\binom{x-r}{n}$
classical	$n!$	exp	$r, r, r, \ldots$	$p'(x)$	$p(x+r)$	$\frac{(x-r)^n}{n!}$
hyperbolic	$(2n)!$	cosh	$1, 9, 25, 49, \ldots$	$\frac{1}{2}\Big[p\big([\sqrt{x}+2]^2\big)+p\big([\sqrt{x}-2]^2\big)-2p(x)\Big]$	$\frac{1}{2}\Big[p\big([\sqrt{x}+2]^2\big)+p\big([\sqrt{x}-2]^2\big)\Big]$	$\frac{1}{2n!}((x-1)(x-9)\cdots(x-(2n-1)^2)$

• $r_1 \neq 0$ *in which case, the polynomials* $s_n(x) = c^k(x - r_1)(x - 9r_1) \times (x - 25r_1) \cdots (x - (2n - 1)^2 r_1)/k(2n)!$ *are Sheffer*

—*relative to* $c^k x(x - 4r_1)(x - 16x_1) \cdots (x - (2n - 2)^2 r_1)/k(2n)!$,

—*with delta operator*

$$f(D_b) = \frac{\cosh\left(2\sqrt{r_1 D_b}\right) - 1}{2cr_1}$$

given by $f(D_b)p(x) = [p([\sqrt{x} + 2\sqrt{r_1}]^2) + p([\sqrt{x} - 2\sqrt{r_2}]^2) - 2p(x)]/2cr_1$,

—*invertible operator* $a(D_b) = \cosh(\sqrt{r_1 D_b})$ *given by Proposition* 5.1, *and*

—*roots* $r_n = (2n - 1)^2 r_1$.

Proof. By Proposition 2.2, either $q_n(x)$ is a convolution sequence in which case we refer to Proposition 5.2, or else $a(D_b) = E_b^{r_1} = \cosh(\sqrt{r_1 D_b})$ and $f(D_b)$ is equal to either

$$Q_0 = \frac{kD_b\Phi'(r_1 D_b)}{c\Phi(r_1 D_b)} \qquad \text{or} \qquad Q_1 = k\left(\frac{E_b^{r_2}}{E_b^{r_1}} - 1\right)\Big/ c(r_2 - r_1)$$

depending on whether or not $r_1 = r_2$.

Without loss of generality, we will take $r_1 = c = k = 1$.

Case $r_1 = r_2$. Application of the hyperbolic transfer formula [13] allows us to calculate the first few polynomials $s_n(x)$,

$$s_0(x) = 1$$

$$s_1(x) = (x - 1)/2$$

$$s_2(x) = (x - 1)^2/24$$

$$s_3(x) = (x-1)^2(x+7)/720$$

$$s_4(x) = (x-1)^2(3x^2+90x+163)/40320.$$

We note that -7 is a root of s_3, but not of s_4. This *contradicts* the assertion that s_n has persistent roots.

Case $r_1 \neq r_2$. Again by the hyperbolic transfer formula,

$$s_0(x) = 1$$

$$s_1(x) = (x-1)/2$$

$$s_2(x) = (x-1)(x-r_2)/24$$

$$s_3(x) = (x-1)(x-r_2)(x-4r_2+11)/720$$

$$s_4(x) = (x-1)(x-r_2)g(x)/(120960),$$

where $g(x) = 3x^2 + (129 - 39r_2)x + 108r_2^2 - 641r_2 + 696$. Since $4r_2 - 11$ is a root of s_3, it must also be a root of $g(x)$. Thus,

$$\begin{aligned} 0 &= g(4r_2 - 11) \\ &= 40(r_2 - 9). \end{aligned}$$

Hence, we must have $r_2 = 9$, and using the sum of cubes identity $\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$, we have

$$\begin{aligned} f(t) &= \left(\frac{\cosh(3\sqrt{t})}{\cosh(\sqrt{t})} - 1\right)\Big/2 \\ &= \frac{e^{3\sqrt{t}} + e^{-3\sqrt{t}}}{2\left(e^{\sqrt{t}} + e^{\sqrt{t}}\right)} - \frac{1}{2} \\ &= \frac{e^{2\sqrt{t}} + e^{-2\sqrt{t}}}{2} - 1 \\ &= \cosh(2\sqrt{t}) - 1. \end{aligned}$$

As in the proof of Proposition 5.3,

$$f(D_b)p(x) = \frac{p\left(\left[\sqrt{x}+2\right]^2\right) + p\left(\left[\sqrt{x}-2\right]^2\right) - 2p(x)}{2},$$

so that $f(D_b)(x-1)(x-9)(x-25)\cdots(x-(2n+1)^2)$ has roots $1, 9, 25, \ldots, (2n-1)^2$. ∎

Note that this proof distinguishes itself from the analysis of persistent Sheffer sequences in Propositions 3.2 and 4.2 in that the necessary conditions of Proposition 2.2 are not sufficient, so a certain amount of case analysis is required.

6. COMPLETENESS

In this section, we will show that as enumerated in Table I, the three umbral calculi treated above give all examples of persistent convolution sequences.

THEOREM 6.1. *Let $(q_n)_{n \geq 0}$ be a persistent convolution sequence in some umbral calculus b. Then without loss of generality, one of the following holds*:

$$\begin{aligned} &(\textit{Divided Difference})\ b_n &&= 1 \textit{ for all } n,\\ &(\textit{Classical})\ b_n &&= n! \textit{ for all } n,\\ &(\textit{Hyperbolic})\ b_n &&= (2n)! \textit{ for all } n, \textit{ or else}\\ &(\textit{Trivial})\ q_n(x) &&= (cx)^n/b_n. \end{aligned}$$

Note, for example, that there are no nontrivial persistent convolution sequences in the "ultrabolic" umbral calculus $b_n = (3n)!$.

We must first explain what we mean by "without loss of generality." The first three generalized factorials b_0, b_1, and b_2 do not affect the umbral calculus. In particular, the first three generalized factorials can be arbitrarily changed while keeping exactly the same sequences of binomial type.

LEMMA 6.1. *Let b be an umbral calculus. Then for any nonzero constants β, γ, there exists another umbral calculus b' where $b'_0 = 1$, $b'_1 = \beta$, $b'_2 = \gamma$ such that any polynomial sequence, $(q_n)_{n \geq 0}$ is a convolution sequence in b if and only if $(a_n q_n(x))_{n \geq 0}$ is in b' (where $a_0 = 1$ and $a_n = b_1/\beta$ for $n \geq 1$).*

Proof. We will consider successively b_0, b_1, and b_2. Suppose $q_n(x)$ is a convolution sequence, so $f(D_b)q_n(x) = q_{n-1}(x)$ and $q_n(0) = \delta_{n0}$.

Let $b_n^{(0)} = b_n/b_0$ for all n, then $b_n^{(0)}/b_{n-1}^{(0)} = b_n/b_{n-1}$, and hence the generalized derivative is unchanged $D_{b^{(0)}} = D_b$. Thus, $f(D_{b^{(0)}})q_n(x) = f(D_b)q_n(x) = q_{n-1}(x)$. (Note that $E^y_{b^{(0)}} = E^y_b/b_0$. It is conventional to take $b_0^{(0)} = 1$ as here and in the examples above so that E^0_b is the identity.)

Let $b_n^{(1)} = c^n b_n^{(0)}$. (In particular, $c = \gamma b_2/\beta b_1$.) Obviously, the generalized derivative is rescaled $D_{b^{(1)}} = cD_b$. Thus, $f(D_b/c)q_n(x) = q_{n-1}(x)$. (Note that $E^y_{b^{(1)}} = E^{cy}_b/b_0$.)

Finally, let $b'_n = kb_n^{(1)}$ for $k \geq 1$ and $b'_0 = 1$. (In particular, $k = \beta/c$.) $D_{b'}x^n$ is equal to $D_{b^{(1)}}x^n$ unless $n = 1$ in which case the former is k times the latter. Thus, if $r(x) = D_{b^{(1)}}p(x)$, then $D_{b'}p(x) = r(x) + (k-1)r(0)$.

More generally, if $r = D^k_{b^{(1)}}p$, then by induction, $D^k_{b'}p(x) = r(x) + (k-1)r(0)$. Thus, since $q_{n-1} = f(D_{b^{(1)}})/cq_n$, we also have $f(D_{b'}/c)q_n(x) = q_{n-1}(x) + (k-1)q_{n-1}(0) = q_{n-1}(x)$ for $n > 1$ since $q_{n-1}(0) = 0$. ∎

The root r_1 is automatically zero, since $q_n(0) = 0$ for $n \geq 1$. By Proposition 2.1, r_2 cannot be zero unless q_n is trivial, so we may without loss of generality set $r_2 = 1$ (replacing x with x/r_2 otherwise). Similarly, the generalized factorials b_0, b_1, b_2 have just been shown to be arbitrary.

To complete the proof of Theorem 6.1, we will show how b_n for $n \geq 3$ is determined by r_3. Conversely, we will show how b_n determines r_n for $n \geq 3$. At this point, the only variable left to determine is r_3.

Recall that by Proposition 2.1, the delta operator $Q = f(D_b)$ is determined by b. Thus, by the general transfer formula [13], q_n is determined by b. Requiring that q_5 has the correct root imposes conditions on r_3. These conditions have six solutions of which three can be eliminated by considering q_6. The other three lead to the classical, divided difference, and hyperbolic umbral calculi, respectively.

LEMMA 6.2. *Let $(q_n)_{n\geq 0}$ be a persistent convolution sequence with roots r_n in the b umbral calculus. Without loss of generality, $b_0 = b_1 = b_2 = 1$, $q_1(x) = x$, and $q_2(x) = x(x-1)$. For $n \geq 3$, each generalized factorial b_n is determined by the root r_3. In particular,*

$$b_3 = \frac{r_3 + 1}{2}$$

$$b_4 = \frac{(r_3+1)\left(r_3^2 + r_3 + 1\right)}{r_3 + 5}$$

$$b_5 = \frac{(r_3+1)\left(r_3^2 + r_3 + 1\right)\left(r_3^3 + r_3^2 + r_3 + 1\right)}{4r_3^2 + 6r_3 + 7}$$

$$b_6 = \frac{(r_3+1)\left(r_3^2 + r_3 + 1\right)\left(r_3^3 + r_3^2 + r_3 + 1\right)\left(r_3^4 + r_3^3 + r_3^2 + r_3 + 1\right)}{3r_3^4 + 8r_3^3 + 14r_3^2 + 14r_3 + 21}.$$

Note that the numerators are the Gaussian factorials.

Proof. As in Proposition 2.1, we use the generating function $\sum_{n=0}^{\infty} q_n(x)t^n = \Phi(xf^{(-1)}(t))$ evaluating now at $x = r_3$

$$1 + r_3 t + r_3(r_3 - 1)t^2 = \Phi\left(r_3 f^{(-1)}(t)\right).$$

Now, substitute $t = f(u) = \Phi(u) - 1$.

$$\Phi(r_3 u) = A + B\Phi(u) + C\Phi(u)^2, \tag{5}$$

where $A = (r_3 - 1)^2$, $B = r_3(3 - 2r_3)$, and $C = r_3(r_3 - 1)$.

Let $\phi_n = 1/b_n$ so that $\Phi(u) = \sum_{n=0}^{\infty} \phi_u u^n$. Identify coefficients of u^n in Eq. (5) for $n \geq 3$

$$\begin{aligned}\phi_n r_3^n &= B\phi_n + C\sum_{i=0}^{n} \phi_i \phi_{n-i} \\ &= (B + 2C)\phi_n + C\sum_{i=1}^{n-1} \phi_i \phi_{n-i} \\ \phi_n &= \frac{C}{r_3^n - B - 2C} \sum_{i=1}^{n-1} \phi_i \phi_{n-i}.\end{aligned}$$

By hypothesis,

$$\begin{aligned}q_0(x) &= 1 \\ q_1(x) &= x \\ q_2(x) &= x(x-1) \\ q_3(x) &= x(x-1)(x-r_3)/b_3.\end{aligned}$$

Using the general transfer formula [13] and Lemma 6.2, we compute

$$q_4(x) = x(x-1)(x-r_3)\left(x - \left(5r_3^2 + 1\right)/(r_3 + 5)\right)/b_4$$

$$q_5(x) = x(x-1)(x-r_3)\left(x - r_3^2\right)\left(x + \frac{3 + 2r_3 + 7r_3^3}{2r_3^2 + 3r_3 + 7}\right)\Bigg/ b_5$$

$$\begin{aligned}q_6(x) = x(x-1)(x-r_3)\left(x - r_3^2\right) \\ \times \left[\begin{array}{c}\left(3r_3^4 + 8r_3^3 + 14r_3^2 + 14r_3 + 21\right)x^2 \\ -\left(7r_3^6 + 14r_3^5 + 35r_3^4 + 24r_3^3 + 11r_3^2 + 15r_3 + 14\right)x \\ +\left(21r_3^7 + 7r_3^5 + 14r_3^4 + 14r_3^3 + 3r_3 + 1\right)\end{array}\right]\Bigg/ 3b_6.\end{aligned}$$

Now, the root $r_4 = (5r_3^2 + 1)/(r_3 + 5)$ must persist in $q_5(x)$. Thus, either $r_4 = r_3^2$ or $r_4 = -(3 + 2r_3 + 7r_3^3)/(2r_3^2 + 3r_3 + 7)$. Solving for r_3 yields the following solutions: $r_3 = 1, 2, 4, -1/3, (-1 \pm i\sqrt{3})/2$ which we shall consider individually.

$r_3 = 1$ leads to the divided difference umbral calculus.

$r_3 = 2$ leads to the classical umbral calculus.

$r_3 = 4$ leads to the hyperbolic umbral calculus.

$r_3 = 1/3$ leads to $r_4 = -1/3$, $q_4(x) = x(x-1)(x+1/3)(x-1/3)/b_4$, and $q_5(x) = x(x-1)(x+1/3)(x-1/3)(x-1/9)/b_5$ so that $r_5 = 1/9$. Then $q_6(x) = x(x-1)(x+1/3)(x-1/9)(12852x^2 - 7084x - 280)/12852b_6$. Note that $r_4 = -1/3$ is not a root of q_6, a contradiction.

$r_3 = (-1 \pm i\sqrt{3})/2$ leads to $r_4 = (-1 \mp i\sqrt{3})/2$ and $q_5(x) = x(x-1) \times (x^2 + x + 1)(x-2)$ which implies $r_5 = 2$. However, $q_6(2) = 91 - 7i\sqrt{3})/3 \neq 0$. ∎

We now have completely classified all persistent convolution sequences (Table I).

Nevertheless, consider the following result communicated to us by H. Niederhausen:

> *Let q_n be the convolution sequence for $f(D_b)$ in some umbral calculus b. Then $q_{n+1}(x)/x$ is a Sheffer sequence in the umbral calculus b' where $b'_n = b_{n+1}$.*

Since $q_n(x) = x(x-1)\cdots(x-(n-1)^2)$ is a convolution sequence in the hyperbolic umbral calculus (see Subsection 5.3), it follows that the persistent sequence $s_n(x) = (x-1)(x-4)\cdots(x-n^2)$ is Sheffer. Note however that s_n does not appear in Table II.

We hope to continue working on the classification of persistent Sheffer sequences.

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