Application of Coulomb wave functions to an orthogonal series associated with steady axisymmetric Euler flows

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Abstract

The regular Coulomb wave function of order zero is applied to an orthogonal series which is associated with the steady axisymmetric Euler equations. The series expansion of a function of bounded variation is proved to converge to the mean value of the left- and the right-hand limits of the original function at each point. In this proof, some complex functions related to the regular and the irregular Coulomb wave functions of orders zero and one are used.

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1. Introduction

1.1. Coulomb wave functions

The regular Coulomb wave function $F_L(\eta, \rho)$ for $L \in \mathbb{N} \cup \{0\}$, $\eta \in \mathbb{R}$, and $\rho > 0$ is defined by

$$F_L(\eta, \rho) = C_L(\eta)\rho^{L+1}\Phi_L(\eta, \rho),$$

$$\Phi_L(\eta, \rho) = e^{-i\rho} \frac{1}{(2i\rho)^{L+1}} M_{i\eta, L+1/2}(2i\rho).$$

(1)
where \( \mathbf{1}_F \) denotes Kummer’s regular confluent hypergeometric function, \( M \) denotes the regular Whittaker function, and

\[
C_L(\eta) = \frac{2^L |\Gamma(L + 1 + i\eta)|}{e^{\pi\eta/2}(2L + 1)!} = \begin{cases} 
\frac{2^L}{(2L + 1)!} \sqrt{2\pi \prod_{k=0}^{L} (k^2 + \eta^2)} \text{ for } \eta \neq 0, \\
\frac{2^L}{(2L + 1)!} \frac{\eta(e^{2\pi\eta} - 1)}{\eta(e^{2\pi\eta} - 1)} \text{ for } \eta = 0,
\end{cases}
\]

[2, Chapter 14; 4, Appendix I.A.14]. Although (1) contains the imaginary unit, \( F_L(\eta, \rho) \) is real because of

\[
e^{-i\rho} F_1(L + 1 - i\eta; 2L + 2; 2i\rho) = e^{i\rho} F_1(L + 1 + i\eta; 2L + 2; -2i\rho),
\]

which is verified by using the Kummer transformation in [2, Eq. (13.1.27)]. In addition, \( F_L(\eta, \rho) \) is represented without the imaginary unit as

\[
F_L(\eta, \rho) = \frac{e^{-\pi\eta} \rho^{L+1}}{C_L(\eta)(2L + 1)!} \int_0^\infty \frac{\cos(2\eta t - \rho \tanh t)}{\cosh^{2(L+1)} t} dt.
\]

If the number \( \eta \), called the Coulomb or Sommerfeld parameter, is constant, then \( w(\rho) = F_L(\eta, \rho) \) is a solution to the differential equation

\[
\frac{d^2 w}{d\rho^2} + \left[ 1 - \frac{2\eta}{\rho} - \frac{L(L + 1)}{\rho^2} \right] w = 0.
\]

As another solution to this equation that is independent of \( F_L(\eta, \rho) \), the irregular Coulomb wave function \( G_L(\eta, \rho) \) is defined by

\[
G_L(\eta, \rho) = \frac{(\pm 2i)^{2L+1} \rho^{L+1} e^{\mp i\rho}}{C_L(\eta)(2L + 1)!} \Gamma(L + 1 \mp i\eta) U(L + 1 \mp i\eta, 2L + 2, \pm 2i\rho) \pm i F_L(\eta, \rho)
\]

so that

\[
F_L'(\eta, \rho) G_L(\eta, \rho) - F_L(\eta, \rho) G_L'(\eta, \rho) = 1,
\]

\[
F_L(\eta, \rho) G_{L+1}(\eta, \rho) - F_{L+1}(\eta, \rho) G_L(\eta, \rho) = \frac{L + 1}{\sqrt{(L + 1)^2 + \eta^2}}.
\]

Here \( U \) denotes Kummer’s irregular confluent hypergeometric function, \( W \) denotes the irregular Whittaker function, and \( F_L'(\eta, \rho) \) and \( G_L'(\eta, \rho) \) stand for the derivatives with respect to \( \rho \) of \( F_L(\eta, \rho) \) and \( G_L(\eta, \rho) \), respectively. We can represent \( G_L(\eta, \rho) \) without the imaginary unit as

\[
G_L(\eta, \rho) = \frac{e^{-\pi\eta} \rho^{L+1}}{C_L(\eta)(2L + 1)!} \int_0^\infty \left[ \frac{\sin(2\eta t - \rho \tanh t)}{\cosh^{2(L+1)} t} + (1 + t^2) e^{2\eta \arctan t - \rho t} \right] dt.
\]

As \( \rho \to 0 \), \( G_0(\eta, \rho) \to 1/C_0(\eta) \) and \( G_L(\eta, \rho) = O(\rho^{-L}) \), while \( F_L(\eta, \rho) = O(\rho^{L+1}) \). There are various formulas for \( F_L(\eta, \rho) \) and \( G_L(\eta, \rho) \) in [2,7]. In particular, the case \( L = \eta = 0 \) is easy:

\[
F_0(0, \rho) = \sin \rho \text{ and } G_0(0, \rho) = \cos \rho.
\]

Coulomb wave functions are mainly used in quantum physics, especially in scattering theories (see [14], and references therein, e.g., [13, Chapter III]). They are also used for discussing black
holes in astrophysics [11,12]. In the field of fluid dynamics, Pekeris [17] discussed a function equivalent to \( \rho \Phi_0(\eta, \rho) \) with complex \( \eta \) and \( \rho \) to obtain the stability of the Poiseuille flow in a pipe. In some references (e.g., [5,6,8,18]), Coulomb wave functions of complex orders for complex arguments were investigated.

1.2. Orthogonal series associated with steady Euler flows

The motion of an inviscid incompressible fluid is described by the Euler equations. If it is two-dimensional and in a steady state, then the Euler equations are rewritten in terms of a stream function \( \psi(x, y) \) as

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -g(\psi),
\]

where \( g \) is an arbitrary differentiable function [3, Section 7.4]. The flow velocity is given by \( \frac{\partial \psi}{\partial y} \) and \( -\frac{\partial \psi}{\partial x} \) in the \( x \)- and \( y \)-directions, respectively. It is clear that each basis function of the two-dimensional Fourier series satisfies the above equation with \( g \) linear. Therefore, the two-dimensional Fourier series can be regarded as a superposition of steady planar Euler flows.

The steady Euler equations in three dimensions with axisymmetry are rewritten in terms of a Stokes stream function \( \phi(r, z) \) on the meridian plane \( \{ (r, z) \mid r > 0, \, z \in \mathbb{R} \} \) as

\[
r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial z^2} = -r^2 h(\phi)
\]

if the azimuthal component of the flow velocity is equal to zero [3, Section 7.5]. Here \( h \) is an arbitrary differentiable function. The flow velocity on the meridian plane is given by \( -r^{-1} \frac{\partial \phi}{\partial z} \) and \( r^{-1} \frac{\partial \phi}{\partial r} \) in the \( r \)- and \( z \)-directions, respectively. If \( h \) is linear, then

\[
r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial z^2} = -\mu r^2 \phi \tag{8}
\]

with a constant \( \mu \). What is a correspondent to the two-dimensional Fourier series, that is, an orthogonal series whose basis functions mean steady axisymmetric Euler flows?

Let \( \phi = R(r) e^{2\pi n z/\beta} \) \((n \in \mathbb{Z}, \, \beta > 0)\) in (8). Then \( R(r) \) should satisfy

\[
\mathcal{L}_n R = -\mu r^2 R, \tag{9}
\]

where

\[
\mathcal{L}_n = r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} \right) - \left( \frac{2n \pi}{\beta} \right)^2.
\]

The author [15,16] pointed out that

\[
R(r) = R^n_0(\sqrt{\mu}; r) \equiv F_0 \left( \frac{1}{\sqrt{\mu}} \left( \frac{n \pi}{\beta} \right)^2, \frac{\sqrt{\mu} r^2}{2} \right) \tag{10}
\]

is a solution of (9) with \( R(0) = 0 \) and there exists a set \( \{ \mu_{m,n} \} \) \((m \in \mathbb{N})\) for each fixed \( n \in \mathbb{Z} \) and a constant \( a > 0 \) such that \( R^n_0(\sqrt{\mu_{m,n}}, a) = 0 \) and

\[
\left( \frac{2n \pi}{ab} \right)^2 < \mu_{1,n} < \mu_{2,n} < \mu_{3,n} < \cdots, \quad \mu_{m,n} \sim \frac{[2(m + l)\pi]^2}{a^2}. \tag{11}
\]
Here $l$ is a positive integer independent of $m$. Furthermore, using the Hilbert–Schmidt theory, he deduced that $\{ R_{0}^{m}(\sqrt{m_{n}}, r) \} \ (m \in \mathbb{N})$ for each fixed $n$ is a complete orthogonal system on $(0, a)$ with the weight function $r$. In other words, every function $f(r)$ that satisfies $\int_{0}^{a} [ f(r) ]^{2} r \ dr < \infty$ can be represented in the form

$$f(r) \sim \sum_{m=1}^{\infty} R_{0}^{m}(\sqrt{m_{n}}, r) \int_{0}^{a} R_{0}^{m}(\sqrt{m_{n}}, t) f(t) \ t \ dt$$

in the square integrable space with the weight $r$. Particularly, the case $n = 0$ is easy because

$$R_{0}^{m}(\sqrt{m_{0}}, r) = \sin \frac{\sqrt{m_{0}} \ r^{2}}{2} = \sin \frac{m \pi r^{2}}{a^{2}}$$

for all $m \in \mathbb{N}$ and thus (12) with $n = 0$ is the Fourier sine series of $f(r)$ with respect to $r^{2}$.

In consequence, the set $\{ \phi_{m,n}(r, z) \} = \{ R_{0}^{m}(\sqrt{m_{n}}, r) e^{2i\pi z/b} \} \ (m \in \mathbb{N}, n \in \mathbb{Z})$ is a complete orthogonal system with the weight function $r$ on $(0, a) \times (-b/2, b/2)$ such that each basis means a steady axisymmetric Euler flow. This is useful, for example, when we derive an unknown Stokes stream function $\tilde{\phi}(r, z)$ from given vorticity $\omega(r, z)$ by solving

$$r \frac{\partial}{\partial r} \left( 1 \frac{\partial \tilde{\phi}}{\partial r} \right) + \frac{\partial^{2} \tilde{\phi}}{\partial z^{2}} = -r \omega.$$

Indeed, by expanding $\omega/r$ in the form $\sum_{m,n} \kappa_{m,n} \phi_{m,n}(r, z)$, we can formally write $\tilde{\phi}$ as $\sum_{m,n} (\kappa_{m,n}/\mu_{m,n}) \phi_{m,n}(r, z)$.

It is convenient for practical use of $\{ R_{0}^{m}(\sqrt{m_{n}}, r) \}$ or $\{ \phi_{m,n}(r, z) \}$ to know the value of $\mu_{m,n}$ for non-large $m$ (cf. (11) for large $m$). See Appendix A for an approximation of $\mu_{m,n}$ for non-large $m$.

Lastly, it should be noted that the author [16] derived an integral transform whose kernel is a Stokes stream function of a steady axisymmetric Euler flow by letting $a \to \infty$ and $b \to \infty$.

1.3. Aim of this paper

The author [16, Eq. (4.1)] derived

$$\int_{0}^{a} [ R_{0}^{m}(\sqrt{m_{n}}, r) ]^{2} r \ dr = \frac{A_{m,n} B_{m,n}}{2 \sqrt{m_{n} \ a}},$$

where

$$A_{m,n} = \frac{d}{dr} R_{0}^{m}(\sqrt{m_{n}}, r) \bigg|_{r=a} = \sqrt{m_{n} \ a} F_{0}^{\prime} \left( 1, 0, \frac{n \pi}{b}, \frac{\sqrt{m_{n} \ a^{2}}}{2} \right),$$

$$B_{m,n} = \frac{d}{du} R_{0}^{m}(u; a) \bigg|_{u=\sqrt{m_{n}}}. $$

See Appendix B for its derivation.

The aim of this paper is to prove the following theorem, which is a more specific result than (12).

**Theorem.** If $\int_{0}^{a} | f(t) | t \ dt < \infty$ and the total variation of $f$ is bounded on $[z_{1}, z_{2}] \subset (0, a)$, then

$$f(r - 0) + f(r + 0) 2 = 2a \sum_{m=1}^{\infty} \sqrt{\mu_{m,n}} A_{m,n} B_{m,n} R_{0}^{m}(\sqrt{m_{n}}, r) \int_{0}^{a} R_{0}^{m}(\sqrt{m_{n}}, t) f(t) \ t \ dt$$

for all fixed $r \in (z_{1}, z_{2})$ and $n \in \mathbb{Z} \setminus \{0\}$. 
Here we exclude the case \( n = 0 \) because it is easy, as was mentioned above. The right member of (14) is rewritten as

\[
\lim_{k \to \infty} \int_0^a \Sigma_k(r,t) f(t) dt
\]

with

\[
\Sigma_k(r,t) = 2a \sum_{m=1}^k \sqrt{\mu_{m,n}} R_0^n(\sqrt{\mu_{m,n}}; r) R_0^n(\sqrt{\mu_{m,n}}; t), \quad k \in \mathbb{N}.
\]

In Section 3, we prove Theorem by investigating \( \Sigma_k(r,t) \) by a method of complex analysis which is much more specific and fundamental than the Hilbert–Schmidt theory. Our method is similar to the method in [19, Sections 18.21–18.24] about the Fourier–Bessel series, the best-known orthogonal series with the weight function \( r \). Because of \( CL(\eta) \), however, Coulomb wave functions are more awkward to deal with for complex arguments than Bessel functions. In Section 2, we introduce some complex functions related to \( FL(\eta, \rho) \) and \( GL(\eta, \rho) \) with complex \( \eta \) and \( \rho \) and discuss their properties for \( L = 0 \) or 1.

2. Preliminaries

2.1. Definitions of some complex functions

If (2) is replaced by

\[
C_L(\eta) = \frac{2^L \Gamma(L + 1 + i\eta)}{(2L + 1)!} \exp\left[-\frac{\pi \eta^2}{2} - i\sigma_L(\eta)\right]
\]

\[
= \frac{2^L \Gamma(L + 1 - i\eta)}{(2L + 1)!} \exp\left[-\frac{\pi \eta^2}{2} + i\sigma_L(\eta)\right]
\]

\[
= \frac{2^L}{(2L + 1)!} \exp\left\{-\frac{\pi \eta^2}{2} + \frac{1}{2} \left[\ln \Gamma(L + 1 + i\eta) + \ln \Gamma(L + 1 - i\eta)\right]\right\},
\]

\[
\sigma_L(\eta) = \frac{1}{2i} \left[\ln \Gamma(L + 1 + i\eta) - \ln \Gamma(L + 1 - i\eta)\right]
\]

(15)

for every complex \( \eta \) that satisfies

\[
\text{Re } \eta \neq 0 \quad \text{or} \quad |\text{Im } \eta| < 1,
\]

(16)

then the definition (1) of \( FL(\eta, \rho) \) (\( L \in \mathbb{N} \cup \{0\} \)) remains valid for that \( \eta \) and every complex \( \rho \) (see [18]). Therefore,

\[
R^n_L(u; r) = FL\left(\frac{1}{u} \left(\frac{n \pi}{b}\right)^2, \frac{ur^2}{2}\right)
\]

(0 < \( r \leq a \)) is uniquely determined for every complex \( u \) that satisfies

\[
\text{Re } u \neq 0 \quad \text{or} \quad |\text{Im } u| > \left(\frac{n \pi}{b}\right)^2.
\]

(17)
However, (17) is inconvenient for our discussion. Instead of \( R_L^n(u; r) \), we will often use
\[
Q_L^n(u; r) = \left( \frac{ur^2}{2} \right)^{L+1} \Phi_L \left( \frac{1}{u} \left( \frac{n \pi}{b} \right)^2, \frac{ur^2}{2} \right)
\]
with \( \Phi_L \) introduced in (1). This function \( Q_L^n(u; r) \) is holomorphic in \( u \in \mathbb{C} - \{0\} \). If (17) is added, then
\[
C_L \left( \frac{1}{u} \left( \frac{n \pi}{b} \right)^2 \right) Q_L^n(u; r) = R_L^n(u; r).
\]
It is clear that, for arbitrary \( r_1 \) and \( r_2 \) belonging to \((0, a)\),
\[
\frac{Q_L^n(u; r_1)}{Q_L^n(u; r_2)} = \frac{R_L^n(u; r_1)}{R_L^n(u; r_2)},
\]
which will be used in Section 3.

Next, we define \( \chi_L^\pm(\eta, \rho) (L \in \mathbb{N} \cup \{0\}) \) by
\[
\chi_L^+(\eta, \rho) = \frac{(-2i)^{2L+1} \rho^{L+1} e^{i\rho}}{(2L+1)!} \Gamma(L + 1 + i\eta)U(L + 1 + i\eta, 2L+2, 2\rho e^{-i\pi/2}),
\]
where \( \eta \rho \neq 0, \text{Im} \eta \leq 0, \) and \(-\pi/2 < \arg \rho < 3\pi/2, \) and
\[
\chi_L^-(\eta, \rho) = \frac{(2i)^{2L+1} \rho^{L+1} e^{-i\rho}}{(2L+1)!} \Gamma(L + 1 - i\eta)U(L + 1 - i\eta, 2L+2, 2\rho e^{i\pi/2}),
\]
where \( \eta \rho \neq 0, \text{Im} \eta \geq 0, \text{and} -3\pi/2 < \arg \rho < \pi/2. \) Using \( \chi_0^\pm(\eta, \rho) \), we further define \( X_0^n(u; r) \) for \( u \in \mathbb{C} - \{0\} \) and \( r \in (0, a) \) by
\[
X_0^n(u; r) = \begin{cases} 
\chi_0^+(\frac{1}{u} \left( \frac{n \pi}{b} \right)^2, \frac{ur^2}{2}) & \text{when } 0 < \arg u \leq \pi, \\
\chi_0^-(\frac{1}{u} \left( \frac{n \pi}{b} \right)^2, \frac{ur^2}{2}) & \text{when } -\pi < \arg u \leq 0.
\end{cases}
\]
Here, and from now on, through this paper, \( u \) denotes a non-zero complex variable whose argument is restricted to the principal value: \(-\pi < \arg u \leq \pi. \)

To use \( \chi_L^\pm(\eta, \rho) \), we do not need (16). Thus the above definition of \( X_0^n(u; r) \) does not require (17). If (16) is added, then the definition (5) of \( G_L(\eta, \rho) \) is extended to complex \( \eta \) and \( \rho \) for which either \( \chi_L^+(\eta, \rho) \) or \( \chi_L^-(\eta, \rho) \) is defined:
\[
G_L(\eta, \rho) = \frac{\chi_L^+(\eta, \rho)}{C_L(\eta)} - iF_L(\eta, \rho) \quad \text{or} \quad G_L(\eta, \rho) = \frac{\chi_L^-(\eta, \rho)}{C_L(\eta)} + iF_L(\eta, \rho)
\]
(see [6, Eqs. (1.2) and (2.1)], in which \( \psi^\pm(L, \eta, \rho) \) corresponds to our \( \chi_L^\pm(\eta, \rho)/C_L(\eta) \)). Both \( G_L(\eta, \rho) \) are identical for \( \eta \) and \( \rho \) for which both \( \chi_L^+(\eta, \rho) \) and \( \chi_L^-(\eta, \rho) \) are defined. We set
\[
S_L^n(u; r) = G_L \left( \frac{1}{u} \left( \frac{n \pi}{b} \right)^2, \frac{ur^2}{2} \right)
\]
for \( r \in (0, a] \). It will be convenient to note that, if (17) is added, then
\[
Q_{n}^{0}(u; r_{1})X_{0}^{n}(u; r_{2}) = \begin{cases} 0 \quad &\text{when } 0 < \arg u \leq \pi, \\
R_{0}^{0}(u; r_{1}) \left[ S_{0}^{n}(u; r_{2}) + i R_{0}^{0}(u; r_{2}) \right] \quad &\text{when } -\pi < \arg u \leq 0
\end{cases}
\]
for arbitrary \( r_{1} \) and \( r_{2} \) belonging to \((0, a]\), and thus
\[
Q_{n}^{0}(u; r_{1})X_{0}^{n}(u; r_{2}) - Q_{n}^{0}(u; r_{2})X_{0}^{n}(u; r_{1}) = R_{0}^{0}(u; r_{1})S_{0}^{n}(u; r_{2}) - R_{0}^{0}(u; r_{2})S_{0}^{n}(u; r_{1}) \quad (19)
\]
is valid in both of the cases \( 0 < \arg u \leq \pi \) and \(-\pi < \arg u \leq 0 \). The left member of (19) is holomorphic in \( u \) except at the origin.

Since (3), that is, \( \Phi_{L}(\eta, \rho) = \Phi_{L}(-\eta, -\rho) \) is also valid for complex \( \eta \) and \( \rho \), we have
\[
Q_{n}^{0}(u; r) = (-1)^{L+1}Q_{L}^{n}(-u; r). \quad (20)
\]
It is also easy to see that
\[
X_{n}^{0}(u; r) = X_{0}^{n}(-u; r). \quad (21)
\]

2.2. Asymptotic behavior of complex functions

We will need the behavior of \( Q_{n}^{0}(u; r) \) and \( X_{n}^{0}(u; r) \) for \( u \to 0 \) in Section 3. For this, let us consider the behavior of \( \rho \Phi_{0}(\eta, \rho) \) and \( \chi_{0}^{\pm}(\eta, \rho) \) for
\[
\eta \to \infty \quad \text{and} \quad \rho \to 0 \quad \text{with } \eta \rho = (\text{real positive constant}). \quad (22)
\]

By the expansion of Buchholz [4, Eq. (7.16)], we have
\[
\rho \Phi_{0}(\eta, \rho) \sim 2\rho \left[ \frac{I_{1}(\sqrt{\eta \rho})}{\sqrt{\eta \rho}} - \frac{2\rho^{2} I_{2}(\sqrt{\eta \rho})}{3 (\sqrt{\eta \rho})^{2}} + \frac{2\rho^{4} I_{3}(\sqrt{\eta \rho})}{9 (\sqrt{\eta \rho})^{3}} + \cdots \right]
\]
in the limit (22), where \( I_{v} \) denotes the modified Bessel function of the first kind of order \( v \). This means that
\[
Q_{n}^{0}(u; r) \overset{u \to 0}{\sim} \frac{br}{2|n|\pi} u I_{1} \left( \frac{2|n|\pi r}{b} \right), \quad (23)
\]
which is valid even if \( r \) moves in \((0, a]\) because \( I_{v}(\zeta) = O(\zeta^{-v}) \) for \( v \in \mathbb{N} \) as \( \zeta \to 0 \).

In [1], an expansion of Kummer’s irregular confluent hypergeometric function for real arguments was derived. It is easily extended to our case of complex arguments. As a result, for \( \eta \) and \( \rho \) which satisfy the conditions in the definitions of \( \chi_{0}^{\pm}(\eta, \rho) \), we have
\[
\chi_{0}^{\pm}(\eta, \rho) \sim 8\eta \rho \left[ \frac{K_{1}(\sqrt{\eta \rho})}{\sqrt{\eta \rho}} + \frac{2\rho^{2} K_{2}(\sqrt{\eta \rho})}{3 (\sqrt{\eta \rho})^{2}} + \frac{2\rho^{4} K_{3}(\sqrt{\eta \rho})}{9 (\sqrt{\eta \rho})^{3}} + \cdots \right]
\]
in the limit (22), where \( K_{v} \) denotes the modified Bessel function of the second kind of order \( v \). It leads to
\[
X_{n}^{0}(u; r) \overset{u \to 0}{\sim} \frac{2|n|\pi r}{b} K_{1} \left( \frac{2|n|\pi r}{b} \right) + \frac{r^{4}}{6} u^{2} K_{2} \left( \frac{2|n|\pi r}{b} \right), \quad (24)
\]
which is valid even if \( r \) moves in \((0, a]\) because \( K_{v}(\zeta) = O(\zeta^{-v}) \) for \( v \in \mathbb{N} \) as \( \zeta \to 0 \).
Next, we consider the behavior of $R^n_L(u; r)$, $S^n_L(u; r)$, and $Q^n_L(u; r)$ ($L = 0, 1$) for $|u| \to \infty$. According to [6, Eqs. (1.2) and (8.5a,b)], we have

$$G_L(\eta, \rho) \pm i F_L(\eta, \rho) \sim \left[ 1 + \sum_{j=1}^{\infty} \left( \frac{L \mp i \eta}{j} \right) \left( \pm \frac{1}{2i\rho} \right)^j \frac{\Gamma(L + 1 + j \pm i\eta)}{\Gamma(L + 1 \pm i\eta)} \right] \times \exp \left\{ \pm i \left[ \rho - \eta \ln 2\rho - \frac{L\pi}{2} + \sigma_L(\eta) \right] \right\}$$

for $|\rho| \to \infty$, where $\eta$ satisfies (16) and $|\arg \rho| < \pi$. It reads

$$F_0(\eta, \rho) \sim \left[ 1 + \frac{\eta}{2\rho} + \frac{\eta(\cdots)}{\rho^2} + \cdots \right] \sin[\rho - \eta \ln 2\rho + \sigma_0(\eta)]$$

$$+ \left[ \frac{\eta^2}{2\rho} + \frac{\eta(\cdots)}{\rho^2} + \cdots \right] \cos[\rho - \eta \ln 2\rho + \sigma_0(\eta)],$$

$$G_0(\eta, \rho) \sim \left[ 1 + \frac{\eta}{2\rho} + \frac{\eta(\cdots)}{\rho^2} + \cdots \right] \cos[\rho - \eta \ln 2\rho + \sigma_0(\eta)]$$

$$+ \left[ -\frac{\eta^2}{2\rho} + \frac{\eta(\cdots)}{\rho^2} + \cdots \right] \sin[\rho - \eta \ln 2\rho + \sigma_0(\eta)],$$

$$F_1(\eta, \rho) \sim - \left[ 1 + \frac{\eta}{2\rho} + \frac{\eta(\cdots)}{\rho^2} + \cdots \right] \cos[\rho - \eta \ln 2\rho + \sigma_1(\eta)]$$

$$+ \left[ \frac{2 + \eta^2}{2\rho} + \frac{\eta(\cdots)}{\rho^2} + \cdots \right] \sin[\rho - \eta \ln 2\rho + \sigma_1(\eta)],$$

$$G_1(\eta, \rho) \sim \left[ 1 + \frac{\eta}{2\rho} + \frac{\eta(\cdots)}{\rho^2} + \cdots \right] \sin[\rho - \eta \ln 2\rho + \sigma_1(\eta)]$$

$$+ \left[ \frac{2 + \eta^2}{2\rho} + \frac{\eta(\cdots)}{\rho^2} + \cdots \right] \cos[\rho - \eta \ln 2\rho + \sigma_1(\eta)].$$

Since

$$C_0(\eta) = 1 + O(\eta), \quad C_1(\eta) = \frac{1}{2} + O(\eta),$$

$$\sigma_0(\eta) = -\gamma \eta + O(\eta^2), \quad \sigma_1(\eta) = (1 - \gamma) \eta + O(\eta^2) \quad (\gamma: \text{Euler’s constant})$$

follow from (15) and the power series of $\ln \Gamma$ in [2, Eq. (6.1.33)], we deduce

$$R^0_0(u; r) = [1 + O(u^{-2}(\ln u)^2)] \sin \frac{ur^2}{2} + O(u^{-1} \ln u) \cos \frac{ur^2}{2},$$

$$S^0_0(u; r) = [1 + O(u^{-2}(\ln u)^2)] \cos \frac{ur^2}{2} + O(u^{-1} \ln u) \sin \frac{ur^2}{2},$$

$$R^1_0(u; r) = [-1 + O(u^{-2}(\ln u)^2)] \cos \frac{ur^2}{2} + O(u^{-1} \ln u) \sin \frac{ur^2}{2},$$

$$S^1_0(u; r) = [1 + O(u^{-2}(\ln u)^2)] \sin \frac{ur^2}{2} + O(u^{-1} \ln u) \cos \frac{ur^2}{2},$$

$$Q^0_0(u; r) = [1 + O(u^{-1})] \sin \frac{ur^2}{2} + O(u^{-1} \ln u) \cos \frac{ur^2}{2},$$

$$Q^1_0(u; r) = [-3 + O(u^{-1})] \cos \frac{ur^2}{2} + O(u^{-1} \ln u) \sin \frac{ur^2}{2}$$

(25)
for $|u| \to \infty$. Here $|\arg u| < \pi - \delta$ with $\delta$ being an arbitrarily small positive number and $\Delta_{|u|} \leq r \leq a$ with $\Delta_{|u|}$ being a positive and monotonically decreasing function of $|u|$ such that $
abla_{|u|} \to \infty \Delta_{|u|} = 0$ and $
abla_{|u|} \to \infty |u| \Delta_{|u|})^2 = \infty$.

2.3. Relation between $Q^n_0$ and $Q^n_1$

For $L \in \mathbb{N}$, $\eta \in \mathbb{R}$, and $\rho > 0$, the recurrence relation

$$
\frac{d w_L}{d \rho} = \frac{L^2 + \eta^2}{L} w_{L-1} - \left( \frac{\eta}{L} + \frac{L}{\rho} \right) w_L
$$

holds, where $w_L(\rho) = F_L(\eta, \rho)$ or $G_L(\eta, \rho)$ [2, Eq. 14.2.1]. An elementary derivation of it was introduced in [7]. As Infeld [10] (and [9] for details of his method) pointed out by using the operators

$$
K^{L \pm} = \frac{L}{\sqrt{L^2 + \eta^2}} \left( \eta \frac{L}{\rho} \pm \frac{d}{d \rho} \right),
$$

the above recurrence relation (rewritten as $K^{L} w_L = w_{L-1}$) is directly deduced from

$$
K^{(L+1)+} K^{(L+1)-} w_L = w_L, \quad K^{(L-1)-} K^{L+} w_L = w_L,
$$

each of which is equivalent to (4). This implies that the reality of $\eta$ and $\rho$ is not necessary if we replace

$$
\frac{L}{\sqrt{L^2 + \eta^2}} \text{ by } \frac{C_{L-1}(\eta)}{(2L + 1)C_L(\eta)}
$$

(with $C_L(\eta)$ defined by (15)) in the definition of $K^{L \pm}$. As a result, we obtain

$$
\frac{d w_L}{d \rho} = \frac{(2L + 1)C_L(\eta)}{C_{L-1}(\eta)} w_{L-1} - \left( \frac{\eta}{L} + \frac{L}{\rho} \right) w_L
$$

(26)

($L \in \mathbb{N}$) for complex $\eta$ and $\rho$ for which $w_L$ and $w_{L-1}$ are defined.

From (26) with $w_L = F_L(\eta, \rho),$

$$
\frac{d}{d \rho} \left[ \rho^{L+1} \Phi_L(\eta, \rho) \right] = (2L + 1) \rho^L \Phi_{L-1}(\eta, \rho) - \left( \frac{\eta}{L} + \frac{L}{\rho} \right) \rho^{L+1} \Phi_L(\eta, \rho)
$$

follows. This is valid for all $\eta \in \mathbb{C}$ and $\rho \in \mathbb{C}$ because it does not contain $C_L(\eta)$ or $C_{L-1}(\eta)$. For an arbitrary function $P(\rho)$, we get

$$
\frac{d}{d \rho} \left[ P(\rho) \rho^{L+1} \Phi_L(\eta, \rho) \right] = \left[ \frac{d P}{d \rho} - \left( \frac{\eta}{L} + \frac{L}{\rho} \right) P(\rho) \right] \rho^{L+1} \Phi_L(\eta, \rho) + (2L + 1) P(\rho) \rho^L \Phi_{L-1}(\eta, \rho).
$$

When $P(\rho) = \rho^L e^{\eta \rho / L}$, it leads to

$$
\frac{d}{d \rho} \left[ \rho^L e^{\eta \rho / L} [\rho^{L+1} \Phi_L(\eta, \rho)] \right] = (2L + 1) \rho^L e^{\eta \rho / L} [\rho^L \Phi_{L-1}(\eta, \rho)],
$$
or

\[ \hat{\rho}^L e^{\frac{\rho}{L}} [\hat{\rho}^{L+1} \Phi_L(\eta, \hat{\rho})] = (2L + 1) \int_0^{\hat{\rho}} \rho^L e^{\frac{\rho}{L}} [\rho^L \Phi_{L-1}(\eta, \rho)] d\rho. \]

Let \( L = 1, \eta = u^{-1}(n\pi/b)^2, \rho = ur^2/2, \) and \( \hat{\rho} = ur^2/2 \) with \( u \) being a parameter. Then, for

\[ \Pi(r) = r^2 \exp \left[ \frac{1}{2} \left( \frac{n\pi r}{b} \right)^2 \right], \]

we obtain

\[ \Pi(r) Q^0_1(u; r) = 3u \int_0^r \Pi(t) Q^0_0(u; t) t dt. \quad (27) \]

If (17) is added, then

\[ \Pi(r) R^0_1(u; r) = \frac{3uC_1(u^{-1}(n\pi/b)^2)}{C_0(u^{-1}(n\pi/b)^2)} \int_0^r \Pi(t) R^0_0(u; t) t dt. \quad (28) \]

In the same way, from (26), we can derive

\[ \Pi(r) S^n_1(u; r) - \Pi(\tau) S^n_1(u; \tau) = \frac{3uC_1(u^{-1}(n\pi/b)^2)}{C_0(u^{-1}(n\pi/b)^2)} \int_\tau^r \Pi(t) S^n_0(u; t) t dt \quad (29) \]

for \( \tau \in (0, a) \).

\section{3. Proof of Theorem}

\subsection{3.1. Integral representation of \( \Sigma_k \)}

Now, we define \( \Theta(u; r, t) \) for \( r \in (0, a), t \in (0, a) \), and \( u \in \mathbb{C} - \{0\} \) (with \( -\pi < \arg u \leq \pi \)) by

\[ \Theta(u; r, t) = \begin{cases} 2[Q^0_0(u; r)X^0_0(u; a) - Q^0_0(u; a)X^0_0(u; r)] \frac{Q^n_0(u; t)}{Q^n_0(u; a)} & \text{when } t \leq r, \\ 2[Q^0_0(u; t)X^0_0(u; a) - Q^0_0(u; a)X^0_0(u; t)] \frac{Q^n_0(u; r)}{Q^n_0(u; a)} & \text{when } r \leq t. \end{cases} \]

(30)

It is important that, while \( X^n_0(u; s) \) \((s = r, t, a)\) changes discontinuously in crossing the real-\( u \)-axis, \( \Theta(u; r, t) \) does not have a jump anywhere on the \( u \)-plane except its poles and the origin (see (19)). In virtue of (20) and (21), we have

\[ \Theta(u; r, t) = -\Theta(-u; r, t). \]

(31)

Furthermore, from (23) and (24), it follows that

\[ \lim_{u \to 0} \Theta(u; r, t) = 0 \quad \text{uniformly for } r \in (0, a) \text{ and } t \in (0, a). \]

(32)

Let \( u = u_0 \) be a zero of \( Q^n_0(u; a) \). Then, since \( Q^n_0(u_0; r) \) (as well as \( R^n_0(u_0; r) \)) is a solution to (9) with \( \mu = u_0^2 \), we have

\[ u_0^2 \int_0^a |Q^n_0(u_0; r)|^2 r dr = -\int_0^a \left[ Q^n_0(u_0; r) \right]^r \frac{1}{r} L_n Q^n_0(u_0; r) dr \]

\[ = \int_0^a \frac{1}{r} \left[ \left| \frac{d}{dr} Q^n_0(u_0; r) \right|^2 + \left( \frac{2n\pi}{b} \right)^2 |Q^n_0(u_0; r)|^2 \right] dr, \]
where * denotes the complex conjugate. This implies that $u_0^2$ is real and positive. In other words, there is no zero of $Q_n^0(u; a)$ except $u = \pm \sqrt{\mu_{m,n}} \in \mathbb{R}$.

Noting that (6) with $R_n^0(\sqrt{\mu_{m,n}}; a) = 0$ yields

$$S_n^0(\sqrt{\mu_{m,n}}; a) = \left[ F_0^0 \left( \frac{1}{\sqrt{\mu_{m,n}}} \left( \frac{n\pi}{b}, \sqrt{\mu_{m,n}} a^2 \right) \right) \right]^{-1} = \frac{\sqrt{\mu_{m,n}} a}{A_{m,n}} \quad (33)$$

and using (18) and (19), we deduce that the residue of $\Theta(u; r, t)$ at $u = \sqrt{\mu_{m,n}}$ is given by

$$\text{Res} \left[ \Theta(\cdot; r, t); \sqrt{\mu_{m,n}} \right] = \frac{2a}{A_{m,n} B_{m,n}} \sqrt{\mu_{m,n}} R_n^0(\sqrt{\mu_{m,n}}; r) R_n^0(\sqrt{\mu_{m,n}}; t).$$

Therefore, $\Sigma_k(r, t)$ is represented in the integral form

$$\Sigma_k(r, t) = \frac{1}{2\pi i} \oint_{C_k} \Theta(u; r, t) du,$$

where $C_k$ is the positively oriented rectangular contour on the complex plane whose vertices are at $\pm i q_k$ and $p_k \pm i q_k$ with

$$p_k = \frac{\sqrt{\mu_{k,n}} + \sqrt{\mu_{k+1,n}}}{2} \quad \text{and} \quad q_k \gg p_k.$$

That is to say,

$$\Sigma_k(r, t) = \frac{1}{2\pi i} \left[ \int_0^{p_k} \Theta(v - i q_k; r, t) dv + \int_{-q_k}^{q_k} \Theta(p_k + iv; r, t) idv \\
+ \int_0^{p_k} \Theta(v + i q_k; r, t) dv + \int_{q_k}^{0} \Theta(iv; r, t) idv \right].$$

Here, in the right member, the integral $\int_{q_k}^{0} \Theta(iv; r, t) idv$ is equal to zero because of (31) and (32). Thus

$$\Sigma_k(r, t) = \frac{1}{2\pi i} \int_{D_k} \Theta(u; r, t) du, \quad (34)$$

where $D_k = C_k - (iq_k, -iq_k)$.

Let us consider upper bounds of $|\Theta(u; r, t)|$ and $|\Sigma_k(r, t)|$ by following the procedure in [19, Section 18.21]. Recalling (19) and (25), we get

$$\Theta(u; r, t) \mid_{|u| \to \infty} \sim \begin{cases} \frac{2 \sin[u(r^2 - a^2)/2] \sin(u t^2/2)}{\sin(u a^2/2)} & \text{when } A_{|u|} \leq t \leq r < a, \\
\frac{2 \sin[u(t^2 - a^2)/2] \sin(u r^2/2)}{\sin(u a^2/2)} & \text{when } A_{|u|} \leq r \leq t < a. \end{cases}$$

It is easy to see that the absolute values of these asymptotic forms are written as $O(\exp[-(t^2 - r^2)\Im u]/2])$ if $|\Im u|$ is large. In addition, (11) leads to

$$p_k \sim \frac{2(k + l + 1/2)\pi}{a^2}, \quad (35)$$
which implies that the infimum of $|\sin(ua^2/2)|$ for any $u \in \mathcal{D}_k$ with $k$ large is nearly equal to one. Therefore,

$$|\Theta(u; r, t)| \leq c_1 \exp \left( -\frac{|t^2 - r^2| \Im u}{2} \right)$$

holds for any $u \in \mathcal{D}_k$ with $k$ large. Here, and from now on, $c_1, c_2, c_3, \ldots$ denote positive constants independent of estimated functions. Using (34) and $\max_{x>0} xe^{-\lambda x} = (\lambda e)^{-1}$ for $\lambda > 0$, we derive

$$|\Sigma_k(r, t)| \leq c_1 \left( \int_0^{q_k} \exp \left( -\frac{|t^2 - r^2| v}{2} \right) dv + p_k \exp \left( -\frac{|t^2 - r^2| q_k}{2} \right) \right) \leq \frac{c_2}{|t^2 - r^2|}$$

if $k$ is large and $r \in [\Delta_k, a)$ is not equal to $t \in [\Delta_k, a)$, where $\Delta_k$ is defined in the same way as $\Delta[u]$; a positive and monotonically decreasing function of $k$ such that $\lim_{k \to \infty} \Delta_k = 0$ and $\lim_{k \to \infty} k \Delta_k^2 = \infty$.

### 3.2. Some integrals for $\Sigma_k$ and $\Pi$

Let us prove the validity of a special case of (14):

$$\lim_{k \to \infty} \int_0^a \Sigma_k(r, t) \Pi(t) t dt = \Pi(r) \quad \text{for all fixed } r \in (0, a).$$

First, from (28) and the equality $3C_1(\eta)/C_0(\eta) = \sqrt{1 + \eta^2}$ for real $\eta$, we obtain

$$\int_0^a R_0^n(\sqrt{\mu_{m,n}}; t) \Pi(t) t dt = \frac{\Pi(a)}{\sqrt{\mu_{m,n} + (n\pi/b)^4}} R_1^n(\sqrt{\mu_{m,n}}; a).$$

It follows from (7) with $R_0^n(\sqrt{\mu_{m,n}}; a) = 0$ and (33) that

$$-\sqrt{1 + \frac{1}{\mu_{m,n}} \left( \frac{n\pi}{b} \right)^4} R_1^n(\sqrt{\mu_{m,n}}; a) = \frac{1}{S_0^n(\sqrt{\mu_{m,n}}; a)} = \frac{A_{m,n}}{\sqrt{\mu_{m,n} a}}.$$

They yield

$$\int_0^a \Sigma_k(r, t) \Pi(t) t dt = 2a \sum_{m=1}^k \sqrt{\mu_{m,n}} R_0^n(\sqrt{\mu_{m,n}}; r) \frac{A_{m,n} B_{m,n}}{A_{m,n} B_{m,n}} \int_0^a R_0^n(\sqrt{\mu_{m,n}}; t) \Pi(t) t dt$$

$$= -2\Pi(a) \sum_{m=1}^k \frac{\sqrt{\mu_{m,n}} R_0^n(\sqrt{\mu_{m,n}}; r)}{[\mu_{m,n} + (n\pi/b)^4] B_{m,n}}.$$

Taking this into account, we introduce

$$A(u; r) = \frac{-2\Pi(a) u Q_0^n(u; r)}{[u^2 + (n\pi/b)^4] Q_0^n(u; a)}.$$

In virtue of (18), it is easy to see that

$$\text{Res}[A(\cdot; r); \sqrt{\mu_{m,n}}] = \frac{-2\Pi(a) \sqrt{\mu_{m,n}} R_0^n(\sqrt{\mu_{m,n}}; r)}{[\mu_{m,n} + (n\pi/b)^4] B_{m,n}}.$$
Moreover,

\[ \text{Res} \left[ A(\cdot, r); -i \left( \frac{n\pi}{b} \right)^2 \right] = -\Pi(a) \frac{Q_0^n(-i(n\pi/b)^2; r)}{Q_0^n(-i(n\pi/b)^2; a)} = -\Pi(r) \]

because

\[ Q_0^n \left( -i \left( \frac{n\pi}{b} \right)^2; r \right) = -\frac{i}{2} \left( \frac{n\pi r}{b} \right)^2 \exp \left[ -\frac{i}{2} \left( \frac{n\pi r}{b} \right)^2 \right] \Gamma_1 \left( 2; 2; \left( \frac{n\pi r}{b} \right)^2 \right) = -\frac{i}{2} \left( \frac{n\pi}{b} \right)^2 \Pi(r) \]

follows from \( \Gamma_1 (\xi; \zeta; \xi) = e^{\xi} \). Let \( \tilde{C}_k \) be the same rectangular contour as \( C_k \) except a small indentation on the right at \( i(n\pi/b)^2 \) and a small protrusion on the left at \( -i(n\pi/b)^2 \). Then, noting that (20) yields \( A(u; r) = -A(-u; r) \) and (23) yields \( \lim_{u \to 0} A(u; r) = 0 \), we obtain

\[ \int_0^a \Sigma_k(r, t) \Pi(t) dt = \sum_{m=1}^k \text{Res} \left[ A(\cdot, r); \sqrt{\mu_{m,n}} \right] \]

\[ = \frac{1}{2}\pi i \int_{\tilde{C}_k} A(u; r) du - \text{Res} \left[ A(\cdot, r); -i \left( \frac{n\pi}{b} \right)^2 \right] \]

\[ = \frac{1}{2}\pi i \int_{D_k} A(u; r) du + \Pi(r). \]

In the same way as (36) and (37),

\[ |A(u; r)| \leq \frac{c_3 \Pi(a)}{p_k} \exp \left( \frac{r^2 - a^2}{2} \right) \text{Im} u \]

for \( u \in D_k \) with \( k \) large,

\[ \lim_{k \to \infty} \left| \int_{D_k} A(u; r) du \right| \leq \lim_{k \to \infty} \frac{c_4 \Pi(a)}{p_k (a^2 - r^2)} = 0 \]

are derived. Hence (38) follows.

Next, let us prove

\[ \lim_{k \to \infty} \int_0^r \Sigma_k(r, t) \Pi(t) dt = \frac{1}{2} \Pi(r) \quad \text{for all fixed } r \in (0, a). \]

(39)

Applying (27) to (30) for \( t \leq r \), we obtain

\[ \int_0^r \Theta(u; r, t) \Pi(t) dt = 2 \left[ Q_0^n(u; r)X_0^n(u; a) - Q_0^n(u; a)X_0^n(u; r) \right] \frac{\Pi(r)Q_0^n(u; r)}{3u Q_0^n(u; a)}. \]

It is equal to

\[ -\frac{2\Pi(r) \sin[u(r^2 - a^2)/2] \cos(\sqrt{2}/2)}{u \sin(\sqrt{2}/2)} + O(u^{-2} \ln u) \]

\[ = -\frac{\Pi(r) \sin[u(2r^2 - a^2)/2]}{u \sin(\sqrt{2}/2)} + \frac{\Pi(r)}{u} + O(u^{-2} \ln u) \]

for \( |u| \to \infty \), as is verified by using (19) and (25). Consequently, (39) follows from (34),

\[ \int_{D_k} \frac{1}{u} du = \int_{-\pi/2}^{\pi/2} \frac{1}{\varepsilon i \varepsilon^{i\theta}} i d\theta = \pi i, \]
and
\[ \lim_{k \to \infty} \int_{D_k} \frac{\sin[u(2r^2 - a^2)/2]}{u \sin(ua^2/2)} \, du = 0, \]

which is proved by using (35) in the same way as Lemma with \( v = \frac{1}{2} \) in [19, Section 18.22, p. 587].

Clearly, from (38) and (39), we get
\[ \lim_{k \to \infty} \int_{r}^{a} \Sigma_k(r, t) \Pi(t) \, dt = \frac{1}{2} \Pi(r) \quad \text{for all fixed } r \in (0, a). \quad (40) \]

### 3.3. Decay and uniform boundedness of integrals for \( \Sigma_k \) and \( \Pi \)

Let \( \beta_1 \) and \( \beta_2 \) be arbitrary bounded constants such that \( 0 < \beta_1 < \beta_2 < a \). Applying (19), (28), and (29) to (30) for \( r \leq t \) and using (25), we derive that, for \( r \in [A|u|, \beta_1] \) with \( |u| \to \infty \),
\[ \int_{\beta_1}^{\beta_2} \Theta(u; r, t) \Pi(t) \, dt \]
\[ = \frac{2C_0(\omega^{-1}((n \pi/b)^2))}{3uC_1(\omega^{-1}((n \pi/b)^2))} \left[ (\Pi(\beta_2) R_0^n(u; \beta_2) - \Pi(\beta_1) R_0^n(u; \beta_1)) S_0^n(u; a) \right. \]
\[ \left. - R_0^n(u; a) (\Pi(\beta_2) S_0^n(u; \beta_2) - \Pi(\beta_1) S_0^n(u; \beta_1)) \right] Q_0^n(u; r) \]
\[ = \frac{2}{u} \left[ \left( -\Pi(\beta_2) \cos \frac{u(a^2 - \beta_2^2)}{2} + \Pi(\beta_1) \cos \frac{u(a^2 - \beta_1^2)}{2} \right) \sin(\pi u^2/2) + O(u^{-2} \ln u) \right] \]
\[ = - \frac{\Pi(\beta_2)}{u \sin(ua^2/2)} \left[ \sin \frac{u(a^2 - \beta_2^2 + r^2)}{2} - \sin \frac{u(a^2 - \beta_1^2 + r^2)}{2} \right] \]
\[ + \frac{\Pi(\beta_1)}{u \sin(ua^2/2)} \left[ \sin \frac{u(a^2 - \beta_2^2 + r^2)}{2} - \sin \frac{u(a^2 - \beta_1^2 - r^2)}{2} \right] + O(u^{-2} \ln u). \]

From this, (34), (35), and Lemma with \( v = \frac{1}{2} \) in [19, Section 18.22, p. 587], we can deduce that
\[ \lim_{k \to \infty} \int_{\beta_1}^{\beta_2} \Sigma_k(r, t) \Pi(t) \, dt = 0 \quad \text{for all fixed } r \in (0, \beta_1), \quad (41) \]
\[ \left| \int_{\beta_1}^{\beta_2} \Sigma_k(r, t) \Pi(t) \, dt \right| \leq c_0 \quad \text{for all } k \geq k_0 \text{ and } r \in [A_k, \beta_1], \quad (42) \]

where \( k_0 \) is a sufficiently large integer and \( c_0 \) is a positive constant independent of \( k, r, \beta_1, \) and \( \beta_2 \).

In the same way, from (30) for \( t \leq r \), it can be derived that
\[ \lim_{k \to \infty} \int_{\beta_1}^{\beta_2} \Sigma_k(r, t) \Pi(t) \, dt = 0 \quad \text{for all fixed } r \in (\beta_4, a), \quad (43) \]
\[ \left| \int_{\beta_1}^{\beta_2} \Sigma_k(r, t) \Pi(t) \, dt \right| \leq c_0 \quad \text{for all } k \geq k_0 \text{ and } r \in [\beta_4, a], \quad (44) \]
where \( \beta_3 \) and \( \beta_4 \) are arbitrary constants satisfying \( 0 \leq \beta_3 < \beta_4 < a \) and \( k_0 \) and \( c_0 \) are the same as in (42) but \( c_0 \) is also independent of \( \beta_3 \) and \( \beta_4 \).

### 3.4. Localization principle and the rest of proof

Let us prove an analogue of the localization principle of the Fourier series:

\[
\lim_{k \to \infty} \int_\alpha^\beta \sigma_k(r, t) f(t) t \, dt = 0 \quad \text{for all fixed} \quad r \in (0, \alpha) \cup (\beta, a),
\]

(45)

where \( \alpha \) and \( \beta \) are arbitrary constants such that \( 0 \leq \alpha < \beta \leq a \). First, by \( f_\delta \in C^\infty[\alpha, \beta] \) with a parameter \( \delta > 0 \), we define a mollification of \( f \) such that

\[
\lim_{\delta \to 0} \int_{\alpha}^{\beta} |f_\delta(t) - f(t)| t \, dt = 0.
\]

We further define an approximation of \( f_\delta \) by

\[
g_{\delta, N}(t) = \frac{f_\delta(t_j)}{\Pi(t_j)} \Pi(t) \quad \text{for} \quad t \in (t_{j-1}, t_j),
\]

where \( t_j = \alpha + j(\beta - \alpha)/N \) for \( j = 0, 1, 2, \ldots, N \). Since \( \Pi(t_j) \) with \( j \geq 1 \) is larger than \( (\beta - \alpha)^2/N^2 \),

\[
\left| \int_\alpha^\beta \sigma_k(r, t) g_{\delta, N}(t) t \, dt \right| \leq \frac{N^2}{(\beta - \alpha)^2} \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \sigma_k(r, t) \Pi(t) t \, dt \leq \frac{N^2}{(\beta - \alpha)^2} \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \sigma_k(r, t) \Pi(t) t \, dt
\]

is obtained. In virtue of this, (37), (41), and (43), we can make

\[
\int_\alpha^\beta \sigma_k(r, t) [f_\delta(t) - f(t)] t \, dt, \int_\alpha^\beta \sigma_k(r, t) [g_{\delta, N}(t) - f_\delta(t)] t \, dt, \int_\alpha^\beta \sigma_k(r, t) g_{\delta, N}(t) t \, dt
\]

arbitrarily close to zero by choosing small \( \delta \), large \( N \), and large \( k \) in this order. Thus (45) is proved.

Lastly, the equality

\[
\lim_{k \to \infty} \int_0^a \sigma_k(r, t) f(t) t \, dt = \frac{f(r - 0) + f(r + 0)}{2}
\]

\[
= \lim_{k \to \infty} \int_0^r \sigma_k(r, t) \Pi(t) \left[ \frac{f(t)}{\Pi(t)} - \frac{f(r - 0)}{\Pi(r)} \right] t \, dt
\]

\[
+ \lim_{k \to \infty} \int_r^a \sigma_k(r, t) \Pi(t) \left[ \frac{f(t)}{\Pi(t)} - \frac{f(r + 0)}{\Pi(r)} \right] t \, dt
\]

follows from (39) and (40). Its right member can be proved equal to zero by using (42), (44), and (45) in the same manner as in [19, Section 18.24]. The proof of Theorem is completed.
Appendix A.

In this appendix, an approximation of $\mu_{m,n}$ for non-large $m$ is derived. Let $a_m$ be the $m$th zero of the Airy function $Ai$, for example,

$$a_1 = -2.3381 \ldots, \quad a_2 = -4.0879 \ldots, \quad a_3 = -5.5205 \ldots$$

[2, Table 10.13].

**Lemma.** If $\varepsilon_n = (n \pi a / b)^{-2/3}$ is small enough, then

$$\mu_{m,n} = \left( \frac{2n \pi}{ab} \right)^2 \left[ 1 - a_m \varepsilon_n + \frac{8}{15} a_m^2 \varepsilon_n^2 + \left( \frac{6}{35} - \frac{32}{175} a_m^3 \right) \varepsilon_n^3 + O(\varepsilon_n^4) \right].$$

**Proof.** Set $x = \lambda - 2 \eta \rho \lambda^{-2}$ with $\lambda = (2 \eta)^{2/3}$ and assume that $\lambda$ is large and $2 \eta \sim \rho$ (i.e., $2 \eta \rho \sim \lambda^2$ and $x = o(\lambda)$). Then, according to [2, Eq. (14.4.9); 7, Eq. (11.6)],

$$\frac{F_0(\eta, \rho)}{\sqrt{\pi \lambda^{1/4}}} = Ai(x) \left[ 1 - \frac{x}{5} \lambda^{-1} + O(\lambda^{-2}) \right] + Ai'(x) \lambda^{-1} \left[ \frac{x^2}{5} + \frac{2x^3 + 6}{35} \lambda^{-1} + O(\lambda^{-2}) \right]$$

is valid, where $Ai'$ denotes the derivative of $Ai$. Expanding $Ai(x)$ and $Ai'(x)$ about the point $x = a_m$ and noting that $Ai''(x) = x Ai(x)$, we have

$$\frac{F_0(\eta, \rho)}{\sqrt{\pi \lambda^{1/4}}} = [Ai'(a_m) + O((x - a_m)^2)](x - a_m) \left[ 1 - \frac{x}{5} \lambda^{-1} + O(\lambda^{-2}) \right]$$

$$+ [Ai'(a_m) + O((x - a_m)^2)] \lambda^{-1} \left[ \frac{x^2}{5} + \frac{2x^3 + 6}{35} \lambda^{-1} + O(\lambda^{-2}) \right].$$

This is equal to

$$\lambda^{-1} Ai'(a_m) \left[ \sigma + \frac{a_m^2}{5} + \left( \tau + \frac{a_m \sigma}{5} + \frac{2a_m^3 + 6}{35} \right) \lambda^{-1} \right] + O(\lambda^{-3})$$

for $x = a_m + \sigma \lambda^{-1} + \tau \lambda^{-2} + O(\lambda^{-3})$. Therefore, $F_0(\eta, \rho) = O(\lambda^{1/4 - 3})$ for

$$x = \lambda - 2 \eta \rho \lambda^{-2} = a_m - \frac{a_m^2}{5} \lambda^{-1} - \left( \frac{3a_m^3}{175} + \frac{6}{35} \right) \lambda^{-2} + O(\epsilon_n^3).$$

Recall that $\mu_{m,n}$ is derived from $F_0(\eta, \rho) = 0$ with $\eta = (n \pi / b)^2 / \sqrt{\mu_{m,n}}$ and $\rho = \sqrt{\mu_{m,n}} a^2 / 2$. In this case, $2 \eta \rho = \varepsilon_n^{-3}$, and then $\lambda \sim \varepsilon_n^{-1}$. Solving

$$\lambda = \varepsilon_n^{-3} \lambda^{-2} = a_m - \frac{a_m^2}{5} \lambda^{-1} - \left( \frac{3a_m^3}{175} + \frac{6}{35} \right) \lambda^{-2} + O(\varepsilon_n^3),$$

or

$$\lambda^3 - [a_m + O(\varepsilon_n^3)] \lambda^2 + \frac{a_m^2}{5} \lambda - \varepsilon_n^{-3} + \frac{3a_m^3}{175} + \frac{6}{35} = 0$$

by Cardano’s formula, we obtain

$$\lambda = \frac{1}{\varepsilon_n} \left\{ 1 + \frac{a_m}{5} \varepsilon_n + \frac{2a_m^2}{45} \varepsilon_n^2 - \left[ \left( \frac{1}{175} - \frac{1}{405} \right) a_m^3 + \frac{2}{35} \right] \varepsilon_n^3 + O(\varepsilon_n^4) \right\}.$$
From this and
\[ \mu_{m,n} = \left( \frac{2n\pi}{ab} \right)^2 \frac{2\eta}{(2\eta)^2} = \left( \frac{2n\pi}{ab} \right) \frac{1}{(\epsilon_n L)^3}, \]
Lemma follows. □

Appendix B.

In this appendix, we verify the validity of (13). Its derivation is similar to that of Lommel’s integrals for Bessel functions in [19]. Using (9) with (10) and noting that
\[ R_0^n(\cdot; 0) = R_0^n(\sqrt{\mu_{m,n}}; a) = 0, \quad \lim_{r \searrow 0} \frac{1}{r} \frac{d}{dr} R_0^n(\cdot; r) < \infty, \]
we have
\[
\begin{align*}
\varepsilon \int_0^a R_0^n(\sqrt{\mu_{m,n}}; r) R_0^n(\sqrt{\mu_{m,n}} + \varepsilon; r) r \, dr \\
= \int_0^a \frac{1}{r} \left[ R_0^n(\sqrt{\mu_{m,n}} + \varepsilon; r) \mathcal{L}_n R_0^n(\sqrt{\mu_{m,n}}; r) \right. \\
- \left. R_0^n(\sqrt{\mu_{m,n}}; r) \mathcal{L}_n R_0^n(\sqrt{\mu_{m,n}} + \varepsilon; r) \right] dr \\
= \int_0^a \left\{ R_0^n(\sqrt{\mu_{m,n}} + \varepsilon; r) \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} R_0^n(\sqrt{\mu_{m,n}}; r) \right] \\
- \frac{d}{dr} R_0^n(\sqrt{\mu_{m,n}}; r) \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} R_0^n(\sqrt{\mu_{m,n}} + \varepsilon; r) \right] \right\} dr \\
= \frac{A_{m,n}}{a} R_0^n(\sqrt{\mu_{m,n}} + \varepsilon; a)
\end{align*}
\]
when \( 0 < |\varepsilon| \leq \mu_{1,n} \). Therefore,
\[ \int_0^a \left[ R_0^n(\sqrt{\mu_{m,n}}; r) \right]^2 r \, dr = \frac{A_{m,n}}{a} \lim_{\varepsilon \to 0} \frac{R_0^n(\sqrt{\mu_{m,n}} + \varepsilon; a)}{\varepsilon} = \frac{A_{m,n}}{a} \lim_{u \to \sqrt{\mu_{m,n}}} \frac{R_0^n(u; a)}{u^2 - \mu_{m,n}} \]
and this leads to (13).

References