The Linear Part of a Discontinuously Acting Euclidean Semigroup

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Let \( \mathbb{R}^n \) be a Euclidean space and let \( S \) be a Euclidean semigroup, i.e., a subsemigroup of the group of isometries of \( \mathbb{R}^n \). We say that a semigroup \( S \) acts discontinuously on \( \mathbb{R}^n \) if the subset \( \{ s \in S : sK \cap K \neq \emptyset \} \) is finite for any compact set \( K \) of \( \mathbb{R}^n \). The main results of this work are

**Theorem.** If \( S \) is a Euclidean semigroup which acts discontinuously on \( \mathbb{R}^n \), then the connected component of the closure of the linear part \( \ell(S) \) of \( S \) is a reducible group.

**Corollary.** Let \( S \) be a Euclidean semigroup acting discontinuously on \( \mathbb{R}^n \); then the linear part \( \ell(S) \) of \( S \) is not dense in the orthogonal group \( O(n) \).

These results are the first step in the proof of the following

**Margulis’ Conjecture.** If \( S \) is a crystallographic Euclidean semigroup, then \( S \) is a group.

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1. INTRODUCTION

Let \( G = \text{Aff } \mathbb{R}^n \) be the group of all affine transformations of the \( n \)-dimensional real affine space. This group is the semidirect product of \( \text{GL}_n(\mathbb{R}) \) and the subgroup of all parallel translations which can be identified with \( \mathbb{R}^n \), i.e.,

\[
\text{Aff } \mathbb{R}^n = \mathbb{R}^n \rtimes \text{GL}_n(\mathbb{R}).
\]

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We will consider the natural homomorphism

$$\ell : \text{Aff} \mathbb{R}^n \to \text{GL}_n(\mathbb{R}^n),$$

and, because the group Aff $\mathbb{R}^n$ is a semidirect product, we have for every element $g \in \text{Aff}(\mathbb{R}^n)$ the decomposition

$$g = v_g \ell(g), \quad v_g \in \mathbb{R}^n, \quad \ell(g) \in \text{GL}(\mathbb{R}^n).$$

The element $\ell(g)$ is said to be the linear part of the transformation $g$. Therefore we can associate to any subgroup (subsemigroup) $S \subseteq \text{Aff} \mathbb{R}^n$ the subgroup (subsemigroup) $\ell(S) = \{\ell(s), s \in S\}$. Let $q : \mathbb{R}^n \to \mathbb{R}$ be the quadratic form $q(x) = x_1^2 + \cdots + x_n^2$, where $x = (x_1, \ldots, x_n)$. Then

$$\text{Isom} \mathbb{R}^n = \{g \in \text{Aff} \mathbb{R}^n : q(\ell(x)) = q(x), \ \forall x \in \mathbb{R}^n\}$$

is the group of isometries of the Euclidean affine space $\mathbb{R}^n$. We have

$$\text{Isom} \mathbb{R}^n = \mathbb{R}^n \ltimes O(q),$$

where $O(q)$ is the orthogonal group of the form $q$.

We say that a semigroup $S$, $S \subseteq \text{Aff} \mathbb{R}^n$ acts properly discontinuously if the set $\{s \in S : sK \cap K \neq \emptyset\}$ is finite for every compact set $K$ in $\mathbb{R}^n$ (we consider $\mathbb{R}^n$ with the Euclidean topology $d(x, y) = \sqrt{q(x - y)}$).

We say that a semigroup $S$ is crystallographic if it acts properly discontinuously and there exists a compact set $\tilde{K}$, $\tilde{K} \subseteq \mathbb{R}^n$, such that

$$\bigcup_{\gamma \in \Gamma} \gamma \tilde{K} = \mathbb{R}^n.$$

The main goal of our work is to prove

**Theorem.** If $S$ is a subsemigroup of $\text{Isom} \mathbb{R}^n$ and acts discontinuously on $\mathbb{R}^n$, then the connected component of the closure of the linear part $\ell(S)$ of $S$ is a reducible semigroup.

**Corollary.** Let $S$ be a Euclidean semigroup (i.e., a subsemigroup of $\text{Isom} \mathbb{R}^n$) acting discontinuously on $\mathbb{R}^n$; then the linear part $\ell(S)$ of $S$ is not Zariski dense in the orthogonal group $O(n)$.

These results are the first step in the proof of the following:

**Margulis’ Conjecture.** If $S$ is a crystallographic subsemigroup of $\text{Isom}(\mathbb{R}^n)$ then $S$ is a group.
The main theorem will be proved in Section 4. In Section 2 we explain the basic properties of proximal elements. This section as well as Section 3 contains several notations and definitions which we use throughout the paper. In Section 3 we show how to construct a set of proximal elements and, using them, how to construct a free subsemigroup. We introduce in Section 4 the important definition of the cone $C(S)$ for the subsemigroup $S$ of Euclidean transformations and prove that there is a strong connection between some properties of the cone with the fact that $S$ acts properly and that $S$ is a crystallographic semigroup. Then based on these facts, we prove the main theorem.

We would like to mention the following two steps in our work. The first one is the definition of the convex cone $C(S)$, which is the convex hull of “all possible” directions coming from the acting semigroup $S$. The idea of this definition is based on the following observation: for a crystallographic semigroup $S$ the cone $C(S)$ contains almost all possible directions and is $S$-invariant. Now, if this cone is “big” and contains the zero vector, then there is a compact subset $K$ in $\mathbb{R}^n$ such that $KS = \{ks; k \in K, s \in S\} = \mathbb{R}^n$.

The next step is Proposition 4.5, which says that if the acting semigroup is not almost solvable (more exactly, the algebraic closure of $S$ is not almost abelian), then every $k$ elements $g_i, i = 1, \ldots, k$, in our semigroup $S$ with non-trivial translation vectors $v_0(g_i)$ can approximated by free generators in the following sense: for any given $\epsilon$ we can find $k$ free generators $g_i^*, i = 1, \ldots, k$ (i.e., elements which freely generate subsemigroup of $S$), such that the angles between the parallel translations $v_0(g_i)$ and $v_0(g_i^*), i = 1, \ldots, k$, are less than $\epsilon$. This statement can be considered as a Tits-type alternative.

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2. PROXIMALITY AND PROXIMAL ELEMENTS

In this section we explain the basic properties of proximal elements.

Let $V$ be a finite-dimensional vector space over a local field $k$ with absolute value $|\cdot|$ on $k$. We fix a norm $\| \cdot \|$ and a corresponding metric $d$ on $V$. Our considerations do not depend on the chosen norm because any two norms on $V$ are equivalent in the following sense: Two functions $f$ and $f'$ (i.e., norm, metric, etc.) on a set $X$ are equivalent if there is a positive constant $c \leq 1$ such that

$$cf(x) \leq f'(x) \leq c^{-1}f(x)$$

for every $x \in X$. 

Let $g$ be a linear transformation and let $\chi(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$, $n = \dim V$, be the characteristic polynomial of the transformation $g$. We put $\Omega(g) = \{\lambda_i : |\lambda_i| = \max_{1 \leq j \leq n} |\lambda_j|\}$.

The linear transformation $g$ is called proximal if $|\Omega(g)| = 1$. Suppose $g$ has a unique eigenvalue $\lambda_1$ of maximal modulus; then $\lambda_1 \in k$ and we have the two subspaces:

$$V_{\lambda_1} = \{v \in V : gv = \lambda_1 v, \lambda_1 \in \Omega(g)\}$$

and $\overline{V}_{\lambda_1} = \{v \in V : \chi_1(g)v = 0\}$, where $\chi_1(\lambda) = \chi(\lambda)/\lambda - \lambda_1$.

In general, let $P$ be the projective space corresponding to $V$. Then for every element $g \in \text{GL}(V)$ we will denote by $\hat{g}$ the projective transformation corresponding to $g$, and by $A(g)$ and $A'(g)$ the subspaces of $P$ corresponding to the kernels of the endomorphisms $\chi_1(g)$ and $\chi_2(g)$, respectively, where

$$\chi_1(\lambda) = \prod_{\lambda_i \in \Omega(g)} (\lambda - \lambda_i), \quad \chi_2(\lambda) = \prod_{\lambda_i \notin \Omega(g)} (\lambda - \lambda_i).$$

It is easy to check that $\chi_i(\lambda) \in k[\lambda]$, $i = 1, 2$ [7]. Then $g$ is proximal iff $A(g)$ is a point in $P$.

Now we consider the metric (see [1, 7] for an explanation about definition of this metric) $\hat{d}$ on the projective space $P$ induced by the metric $d$ on $V$. For every $g \in \text{GL}(V)$ we put

$$\|\hat{g}\| = \sup_{x,y \in P} \frac{\hat{d}(\hat{g}x, \hat{g}y)}{\hat{d}(x, y)}.$$

We will use the following criterion for proximality due to Tits [7].

**Lemma 2.1 (Tits’ Criterion).** Let $g \in \text{GL}(V)$ and let $K$ be a compact subset of $P$ such that $\hat{g}K \subset \hat{K}$ and $\|\hat{g}\|_K < 1$. Hence $\hat{K}$ denotes the interior of $K$ with respect to the topology of $P$. Then $g$ is proximal, $\hat{g}$ has a unique fixed point in $\hat{K}$, and $A'(g) \cap K = \emptyset$.

**Lemma 2.2.** Let $g_i \in \text{GL}(V)$ be proximal for $i = 1, \ldots, m$. Assume that

$$A(g_i) \in P \setminus A'(g_j), \quad i \neq j, 1 \leq i, j \leq m.$$

Then there is $N \in \mathbb{Z}$, $N > 0$, such that $g_1^N, \ldots, g_m^N$ freely generate a free subsemigroup $\hat{S} = \langle g_1^N, \ldots, g_m^N \rangle$ of $\text{GL}(V)$. 
Proof. Let us consider compact subsets \( K_i \) and open subsets \( U_i \) in \( P \) such that

1. \( A(g_i) \in U_i \subseteq K_i \subseteq P \setminus A'(g_j) \), \( 1 \leq i, j \leq m \);
2. \( K_i \cap K_j = \emptyset \), \( 1 \leq i \neq j \leq m \);
3. There is a point \( x_0 \), \( x_0 \in P \setminus A'(g_i) \), \( 1 \leq i \leq m \), \( x_0 \in P \setminus (\bigcup_{i=1}^{m} K_i) \cup (\bigcup_{i=1}^{m} A'(g_i)) \).

Then there is a positive \( N \) such that

1. \( \hat{g}^n_i(K_j) \subseteq U_i \), for all \( 1 \leq i, j \leq m \) and \( n \geq N \);
2. \( \hat{g}^n_i(x_0) \in U_i \), for all \( 1 \leq i \leq m \) and \( n \geq N \).

Assume now that \( \hat{g}^n_i \cdots \hat{g}^n_k = \hat{g}^n_{i_1} \cdots \hat{g}^n_{k_1} \). Then \( \hat{g}^n_i \cdots \hat{g}^n_k(x_0) \in U_{i_1} \), \( \hat{g}^n_i \cdots \hat{g}^n_k(x_0) \in U_{i_1} \). Since \( U_{i_1} \cap U_{k_1} = \emptyset \) if \( i_1 \neq k_1 \) because of (1) and (2), then we will have a contradiction. Then \( j_1 = i_2 \) and \( n_1 = s_1 \) and so on.

3. QUASI-PROJECTIVE TRANSFORMATIONS

In this section we shall recall the notion of quasi-projective transformation due to Furstenberg [4], and notions and results about contractions and proximal elements from [1, 5].

A map \( b : P \to P \) is called quasi-projective if there is a sequence of projective transformations of \( P \) converging pointwise to \( b \). Let \( M_0(b) \) be the closure of the set of points where \( b \) is discontinuous, and let \( M_0(b) \) be the \( b \)-image of the set of continuity points of \( b \).

We will call a sequence \( s = (B_n)_{n \in \mathbb{N}} \) in \( \text{GL}(V) \) contractive if

1. \( \hat{B}_n \) converges pointwise on \( P \),
2. there is a sequence \( (\alpha_n)_{n \in \mathbb{N}} \in \mathbb{R}^* \) such that \( (\alpha_n B_n)_{n \in \mathbb{N}} \) converges to a linear map of rank 1.

Let \( s \) be a contractive sequence and let \( b_1 \) be a linear rank 1 map such that \( (\alpha_n B_n)_{n \in \mathbb{N}} \) converges to \( b_1 \) (see [1]). Then we will put

\[ M_0(s) = \text{projective subspace corresponding to } \text{Im } b_1 \]

and

\[ L_1(s) = \text{projective subspace corresponding to } \text{ker } b_1. \]

These subspaces are well defined (see [1, 3.8]).

A quasi-projective transformation \( b \) is called a contraction if \( M_0(b) \) consists of one point only.
For a subsemigroup \( H \) of \( \text{GL}(V) \) let \( \overline{H} \) be the set of quasi-projective transformations which are pointwise limits of sequences of projective transformations induced by elements of \( H \).

Let \( H \) be a subsemigroup of \( \text{GL}(V) \). Then \( V \) (considered as an \( H \)-module) is called strongly irreducible if there is no finite union of linear subspaces that is invariant with respect to \( H \) other than 0 and \( V \). Let \( \overline{H} \) be the Zariski closure of \( H \) and let \( \overline{H}^0 \) be the connected component of \( \overline{H} \) in the Zariski topology. Then \( V \) is strongly irreducible as an \( H \) module, iff \( V \) is \( \overline{H}^0 \)-irreducible.

Let us also recall that the Zariski closure of a semigroup is a group. For the proof of the next statement we use the proof of Lemmas 3.7 and 3.10 in [1].

**Lemma 3.1.** Let \( h \) be a contractive sequence \( h = (h_m)_{m \in \mathbb{N}}, h_m \in \text{GL}(V), \) and \( M_0(h) \notin L_1(h) \). Then there is \( M, M' \in \mathbb{N} \) such that

1. \( h_m \) is proximal for \( m > M \);
2. for every \( \varepsilon, v > 0 \) there is \( M(\varepsilon) \) such that \( \hat{d}(A(h_m), M_0(h)) < \varepsilon, \hat{d}(A(h_m), L_1(h)) < \varepsilon \) if \( m > \max\{M, M(\varepsilon)\} \).

**Proof.** Follows immediately from Tits’ Criterion 2.1 and [1, Lemma 3.7, 3.10].

**Lemma 3.2.** Suppose that the subsemigroup \( H \) of \( \text{GL}(V) \) is Zariski-connected and acts irreducibly on \( V \). If \( H \) contains a contractive sequence then there exists a finite set of proximal elements \( h_1, \ldots, h_t \) in \( H \) with the following properties:

1. The points \( A(h_i) \) are distinct.
2. \( A(h_i) \notin P \setminus A'(h_j) \) for \( 1 \leq i, j \leq t \).
3. \( \bigcap_{1 \leq j \leq t} A'(h_j) = \emptyset \).
4. The lines corresponding to \( A(h_i), 1 \leq i \leq t, \text{span} V \).

**Proof.** By [1, Lemma 3.3], there is a finite set of contractive sequences \( s_1, \ldots, s_t \) in \( H \) with the following properties:

1'. The points \( M_0(S_i) \) are distinct.
2'. \( M_0(S_i) \notin L_1(S_j) \) for \( 1 \leq i, j \leq t \).
3'. \( \bigcap_{1 \leq j \leq t} L_1(S_j) = \emptyset \).
4'. The lines corresponding to \( M_0(S_i), 1 \leq i \leq t, \text{span} V \).

Then by Lemma 3.1, we will obtain for sufficiently small \( \varepsilon, \varepsilon > 0 \), a finite set of proximal elements \( h_1, \ldots, h_t \) in \( H \) which satisfy all of the conditions of the lemma.
Lemma 3.3. Let $H$ be a Zariski-connected subsemigroup of $GL(V)$ which acts irreducibly on $V$. Suppose that $h_1, \ldots, h_t$ is a finite set of proximal elements $H$, such that

(i) The points $A(h_i)$ are distinct, for all $i, 1 \leq i \leq t$.

(ii) $A(h_i) \in P \setminus A'(h_i)$, for all $i, j, 1 \leq i, j \leq t$.

(iii) $\cap_{0 \leq j < t} A(h_j) = \emptyset$.

(iv) The lines corresponding to $A(h_i)$, $1 \leq i \leq t$, span $V$.

Then for every element $h \in H$, and an infinite subset $M$ in $\mathbb{N}$ there are $N = N(h)$ in $\mathbb{N}$, and for an infinite subset $\tilde{M} \subseteq M$, two numbers $i_0 = i_0(h), j_0 = j_0(h)$ such that for $m \in \tilde{M}$, $k > N$, $t > N$,

1. $\tilde{h} = \tilde{h}_i \tilde{h}_j \tilde{h}_i \tilde{h}_j$ is a proximal element.
2. $A(\tilde{h}) \in P \setminus A'(h_i)$ for $1 \leq j \leq t$.
3. $A(h_i) \in P \setminus A'(\tilde{h})$ for $1 \leq j \leq t$.

Proof. Let $B = A'(h)$; then since $B \neq P$ and because of (iv), there is $i_0, 1 \leq i \leq t$, such that $A(h_{i_0}) \notin B$. Now by [7, Lemma 3.9], there is an infinite subset, $\widetilde{M}_1 \subseteq M$, such that the sequence $(\tilde{h}_i^m A(h_{i_0}))_{m \in \tilde{M}_1}$ converges to some point $p \in P$ when $m$ goes to infinity. Then, by (3), one can find $j_0, 1 \leq j_0 \leq t$, such that $p \notin A'(h_{i_0})$. For all $r$ and $S, 1 \leq r \leq t, r \neq i_0, 1 \leq s \leq t$, take a compact subset $K_r$ of $P$ such that $A(h_i) \in K_r \subseteq K_r \subseteq P \setminus A'(h_{i_0})$.

Let $W$ be an open subset containing a point $p$ such that $\tilde{W} \subseteq P \setminus A'(h_{i_0})$. Then there is a compact subset $K_{i_0}$ in $P$ and number $M = M(K_{i_0}, W)$ such that

$A(h_{i_0}) \subseteq \tilde{K}_{i_0} \subseteq K_{i_0} \subseteq P \setminus A'(h_{i_0})$

and

$\tilde{h}_i^m K_{i_0} \subseteq W$ for all $m \in \tilde{M}_1, m > M$.

There exists $\tilde{N}, \tilde{N} \in \mathbb{N}$, such that

$\tilde{h}_i^n (K_r) \subseteq \tilde{K}_{i_0}$ for all $n > \tilde{N}$ and $r, 1 \leq r \leq t$,

and

$\tilde{h}_i^n (\tilde{W}) \subseteq \tilde{K}_{j_0}$ for all $n > \tilde{N}$.

Now, we put $\tilde{M} = \{m \in \tilde{M}_1, m > M(K_{i_0}, W)\}$.

Then for all $k, n, \ell > \tilde{N}, m \in \tilde{M}$, and $K = \bigcup_{r=1}^t K_r$,

$\tilde{h}_i^k \tilde{h}_i^n \tilde{h}_i^\ell K \subseteq \tilde{K}_i \subseteq K$. 


Now, by [7, Lemma 3.7], there are two integers \( N_{i_0} \) and \( N_{j_0} \) such that if \( n \geq N_{j_0} \), then
\[
\| \hat{h}_{j_0}^n K \| < 1, \quad \text{because} \quad K \subseteq P \setminus A'(h_{j_0}),
\]
and if \( n \geq N_{i_0} \), then
\[
\| \hat{h}_{i_0}^n K \| < 1, \quad \text{because} \quad K \subseteq P \setminus A'(h_{i_0}).
\]

For an element \( h \), we have
\[
\| \hat{h} K_{\hat{h}_{i_0}} \| \leq 1, \quad \text{for all} \quad m \in \hat{M}.
\]

Thus, if \( \bar{N} = \max\{N_{i_0}, N_{j_0}\}, k, n, \ell \geq \bar{N} \) and \( m \in \hat{M} \),
\[
\| \hat{h}_{i_0}^k \hat{h}_{j_0}^n \hat{h}_{i_0}^m \hat{h}^\ell K \| < 1.
\]

Then, by Tits’ criterion, for \( k, n, \ell \geq \max\{\bar{N}, \bar{N}\}, m \in \hat{M} \),
\begin{enumerate}
\item \( \bar{h} = h_i^k h_j^n h^\ell h_{i_0}^m \) is proximal;
\item \( A(\bar{h}) \in K \subseteq P \setminus \bigcup_{\ell=1}^P A'(h_i) \);
\item \( K \cap A'(\bar{h}) = \emptyset \), so \( \bigcup_{\ell=1}^P A(h_i) \subseteq P \setminus A'(\bar{h}) \);
\end{enumerate}
and the statement of the lemma is proved.

4. DISCONTINUOUSLY ACTING SEMIGROUPS

Let \( g \in \text{Isom} \mathbb{R}^n \) and let \( V_0(g) = \{v \in \mathbb{R}^n : \ell(g)v = v\} \). There is a maximal \( g \)-invariant affine subspace \( A^0(g) \) of \( \mathbb{R}^n \) such that \( g \) induces a translation on it. This translation can be zero. In that case all points on \( A^0(g) \) are fixed. Assume first that \( g \) has no fixed points. For example, if \( g \) is an element of \( S \) of infinite order and \( S \) acts discontinuously on \( \mathbb{R}^n \), then \( g \) has no fixed points. So for such elements \( g \), we will define the vector \( v^0(g) \) as follows:
\[
v^0(g) = \frac{g x - x}{\|g x - x\|}, \quad x \in A^0(g).
\]

It is easy to see that the vector \( v^0(g) \) does not depend on \( x \). If the element \( g \) has a fixed point we will set \( v^0(g) = 0 \).

The subset \( C \) of \( \mathbb{R}^n \) is called a convex cone if
\begin{enumerate}
\item for every \( \alpha \in \mathbb{R}, \alpha > 0, v \in C \Rightarrow \alpha v \in C \);
\item for every \( v, w \in C \Rightarrow v + w \in C \).
\end{enumerate}
The intersection of any set of convex cones is a convex cone. So for every subset $X$ in $\mathbb{R}^n$ there is a smallest convex cone $C_X$ such that $X \subseteq C_X$.

For any subsemigroup $S, S \subseteq \text{Isom } \mathbb{R}^n$ let us now define the convex cone $C(S)$ as $C_X$, where $X = \{v^\ell(g), g \in S, \ell(g) \text{ is a regular element of the group } \overline{\ell(S)}\}$.

Let us recall the following well-known fact that a compact closed subgroup of $\text{Isom } \mathbb{R}^n$ is Zariski closed. So for every group $M$ let us recall that an element $g$ of an algebraic linear group $G$ is called regular iff

$$\dim V_0(g) = \min_{h \in G} V_0(h).$$

The following simple statement is probably well known.

**Lemma 4.0.** Let $C$ be a convex cone in $\mathbb{R}^n$ such that the closure $\overline{C}$ in the Euclidean topology is a vector subspace of $\mathbb{R}^n$. Then $\overline{C} = C$.

**Proof.** We assume that $\overline{C} = \mathbb{R}^n$, and we will prove that $\mathbb{R}^n = C$. First, there are $n$ linear independent vectors $v_1, \ldots, v_n$ which belong to $C$. Let $\bar{v} = -v_1 - \cdots - v_n$. As $\bar{v} \in \overline{C}$, we can find $v \in C$, such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ and $\alpha_i < -1/2$ for all $i, 1 \leq i \leq n$. Then

$$C \supseteq \text{span}^\perp \langle v, v_1, \ldots, v_n \rangle = \{\alpha v + \alpha_1 v_1 + \cdots + \alpha_n v_n, \alpha > 0, \alpha_i > 0, i = 1, \ldots, n \} \supseteq \text{span}(v_1, \ldots, v_n) = \mathbb{R}^n.$$

**Remark.** Assume that the closure $\overline{C(S)} = \mathbb{R}^n$, where $S$ is a subsemigroup of $\text{Isom } \mathbb{R}^n$. Then, by Lemma 4.0, there is a finite set of elements $g_1, \ldots, g_r$ of $S$ such that

1. $\text{span}(v^\ell(g_1), \ldots, v^\ell(g_r)) = \mathbb{R}^n$.
2. There are positive scalars $\alpha_1, \ldots, \alpha_r$ such that $\alpha_1 v^\ell(g_1) + \cdots + \alpha_r v^\ell(g_r) = 0$.

Let us recall the following well-known fact that a compact closed subgroup of $O(n)$ in Euclidean topology is Zariski closed in $O(n)$, since the group $O(n)$ is compact, the Euclidean closure of $\ell(S)$ is Zariski closed. So in the sequel we will just use the word “closure” and will not specify the topology and write $\overline{\ell(S)}$.

**Lemma 4.1.** The closure $\overline{C(S)}$ is $\ell(S)$-invariant.

**Proof.** Let $g$ be a regular element of $S$, i.e., $\ell(g)$ is regular in $\overline{\ell(S)}$, and let $s$ be any non-trivial element in $S$. There are three subsets $M_1, M_2,$ and $M_3$ of $\mathbb{N}$ such that

1. $(\ell(g)^m)_{m \in M_1}$ converges to $\ell(g)$ when $m \to \infty, m \in M_1$.
2. $(\ell(s)^m)_{m \in M_2}$ converges to $\ell(s)$ when $m \to \infty, m \in M_2$.
3. $(\ell(s)^m)_{m \in M_3}$ converges to $\ell(s)^{-1}$ when $m \to \infty, m \in M_3$.
(4) \( s^{m_1}; s^{m_2}; s^{m_3} \) is a regular element of \( S \) for all \( m_1 \in M_1, m_2 \in M_2, m_3 \in M_3 \).

Let use briefly explain the idea of the proof. Let \( m_i(n) \in M_i, i = 1, 2, 3; g(n) = s^{m_1(n)} s^{m_2(n)} s^{m_3(n)} \); and \( m_1(n) \to \infty \) if \( n \to \infty \). Then \( \ell(g(n)) \to \ell(s) \ell(g) \ell(s)^{-1} \), and as a first step we will show that the sequence of subspaces \( V_0(g(n)) \) converges to \( \ell(s) V_0(g) \). Then, assuming additionally that \( m_1(n) \) converges to infinity “much faster” than \( m_2(n) \) and \( m_3(n) \) and looking at the projection of a vector \( g(n) x - x \) onto \( V_0(g(n)) \), we will show that the corresponding norm one vector converges to \( \ell(s) v_0(g) \). This means that \( \ell(s) v_0(g) \in C(3) \) for every \( s \in S \).

Let \( V_0(g) \perp \) be the orthogonal complement to \( V_0(g) \). Then there is a positive constant \( c = c(g) \), such that

\[
\| \ell(g) w - w \| \geq c \| w \| \quad \text{for all } w \in V_0(g) \perp.
\]

So we will assume, because of (1), that

\[
\| \ell(g)^m w - w \| \geq c \| w \| \quad \text{for all } w \in V_0(g) \perp \quad \text{and} \quad m \in M_1.
\]

Consider three functions \( m_i : \mathbb{N} \to M_i, i = 1, 2, 3 \), such that \( m_i(n) \to \infty \) when \( n \to \infty \) and put \( g(n) = s^{m_1(n)} s^{m_2(n)} s^{m_3(n)} \).

Let \( V_n = V_0(g(n)) \). We will show now that the sequence \( (V_n)_{n \in \mathbb{N}} \) converges to \( \ell(s) V_0(g) \) when \( n \to \infty \). If instead of \( g(n) \), we consider \( \bar{g}(n) = s^{-1} g(n) \), then since \( V_0(g(n)) = \ell(s)^{-1} V_0(g(n)) \), we have to show that the sequence \( V_0(\bar{g}(n)) \) converges to \( V_0(g) \) if \( n \to \infty \). We do this now.

It is clear that \( g(n) = s^{m_1(n)} s^{m_2(n)} s^{m_3(n)} \) where \( m_2(n) = m_2(n) - 1, m_3(n) = m_3(n) + 1 \), and

1. \( \ell(s)^{m_1(n)} \to e \), when \( n \to \infty \);
2. \( \ell(s)^{m_2(n)} \to e \), when \( n \to \infty \);
3. \( \ell(s)^{m_3(n)} \to e \), when \( n \to \infty \);

for all \( n, n \in \mathbb{N} \), \( \bar{g}(n) \) is a regular element.

Taking a subsequence, we can assume that the sequence \( (\bar{V}_n)_{n \in \mathbb{N}}, \bar{V}_n = V_0(\bar{g}(n)) \), converges to the subspace \( \bar{V}_0 \) and \( \bar{V}_0 \neq V_0(g) \). So we can assume, since it is true for big \( n \), that for every \( n \) there is a vector \( v_n, v_0(n) \in \bar{V}_n \), such that if

\[
v_n = v_0(n) + v_0(n),
\]

where \( v_0(n) \in V_0(g), v_0(n) \in V_0(g) \perp \), then

\[
\| v_0(n) \| \geq c_1 \| v_n \|
\]

for some constant \( c_1 \), which does not depend on \( n \).
Let $\bar{C}$ be the projection of $\mathbb{R}^n$ onto $V_0(g)$ parallel to $V_0(g)^\perp$. Then for every point $p$, $v_0(g) = \pi_g(gp - p)/\|\pi_g(gp - p)\|$. Take $p \in A^0(g)$, $q_n = s^{-m_3(n)}p$; then

$$w_n = s^{m_3(n)}g^{m_3(n)}s^{m_3(n)}q_n - q_n = (s^{m_3(n)}g^{m_3(n)}p - p) + (s^{m_3(n)}q_n - q_n)$$

$$= (s^{m_3(n)}p - p) + \ell(s)^{m_3(n)}(g^{m_3(n)}p - p) + (p - s^{m_3(n)}p).$$

As above, let $\pi_n$ be the projection of $\mathbb{R}^n$ onto $V_0(g(n))$ parallel to $V_0(g(n))^{\perp}$. Now because of (i), if $\tilde{w}_n = \ell(s)^{m_3(n)}(g^{m_3(n)}p - p)$, the sequence $(\pi_n(\tilde{w}_n)/\|\pi_n(\tilde{w}_n)\|)_{n \in \mathbb{N}}$ has the same limit as $(\pi_n(w_n)/\|\pi_n(w_n)\|)_{n \in \mathbb{N}}$. On the other hand, it is easy to see that

(i) $m_1(n) \geq m_2^2(n) + m_3^2(n)$.

(ii) $\tilde{w}_n = \ell(s)^{m_3(n)}(m_1(n) \cdot v_0(g))$;

(iii) $(\pi_n)_{n \in \mathbb{N}}$ converges to $\pi_g$.

Then, $(\pi_1(w_n)/\|\pi_1(w_n)\|)$ converges to $\ell(s) \cdot v_0(g)$, which means that for every $s \in S$ and a regular element $g$ from $S$, $\ell(s) \cdot v_0(g) \in \bar{C}(S)$, which proves the statement.

**Lemma 4.2.** Assume that the restriction of $\ell(S)$ to every $\ell(S)$-invariant irreducible subspace is non-trivial; then $C(S)$ is a vector subspace of $\mathbb{R}^n$.

**Proof.** Because of Lemma 4.0, it is enough to show that $\bar{C}(S)$ is a subspace of $\mathbb{R}^n$. Assume that this is not true. Then there is a hyperplane $W$ of $\mathbb{R}^n$ and a vector $v_0$ such that $\bar{C}(S) \subseteq \{v \in \mathbb{R}^n : \langle v, v_0 \rangle \geq 0\}$ and $W = \{v \in \mathbb{R}^n : \langle v, v_0 \rangle = 0\}$. Let $T = \bar{C}(S)$ and let $\mu$ be the Haar measure on the compact group $T$. Let $v_0$ be a vector from $C(S)$ such that $\langle v_0, v_0 \rangle > 0$. Then the vector $v_1 = \int_{g \in T} g v_0 d\mu_g$ is $\neq 0$, because $\langle v_1, v_0 \rangle > 0$ and $\ell(g)v_1 = v_1$ for all $g \in S$. This is impossible, because of our assumption.
Corollary 4.3. Assume that the semigroup $S$ is as in Lemma 4.2. Then there are regular elements $g_1, \ldots, g_r$ from $S$ such that

1. $\text{span}(v^0(g_1), \ldots, v^0(g_r)) = \mathbb{R}^n$.
2. There are positive scalars $\alpha_1 + \cdots + \alpha_r \in \mathbb{R}^n$ such that
   \[ \alpha_1 v^0(g_1) + \cdots + \alpha_r v^0(g_r) = 0. \]

Proof. The proof follows immediately from Lemmas 4.1 and 4.2 and Remark 1.

Lemma 4.4. Suppose a Euclidean semigroup $S$ contains elements $s_1, \ldots, s_m$ with the following properties:

1. $\text{span}(v^0(s_1), \ldots, v^0(s_m)) = \mathbb{R}^n$;
2. there are $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^n$, $\alpha_i > 0$, $i = 1, \ldots, m$, such that
   \[ \alpha_1 v^0(s_1) + \cdots + \alpha_m v^0(s_m) = 0. \]

Then there is a compact subset $K$ of $\mathbb{R}^n$, such that for every point $p \in \mathbb{R}^n$ the intersection $K \cap S \bar{p}$ is non-empty, where $S$ is the subsemigroup of $S$ generated by $s_1, \ldots, s_m$.

Proof. Let us first show that there is a constant $\alpha$, $\alpha \neq 0$, $\alpha = \alpha(s_1, \ldots, s_m)$, such that for every non-zero vector $v$, $v \in \mathbb{R}^n$, there is a number $i$, $1 \leq i \leq m$, such that $(v, v^0(s_i)) \leq -\alpha^2 \|v\|$. For any vector $v$ in $\mathbb{R}^n$, $\|v\| = 1$, we will set
\[ \alpha_v = \max_{1 \leq i \leq m} \{-(v, v^0(i)), 0\}. \]
From (2) it follows that $\alpha_v > 0$ for every $v$ with $\|v\| = 1$. Now let
\[ \alpha^2 = \inf_{\|v\| = 1} \alpha_v. \]
Compactness of the sphere $\{v \in \mathbb{R}^n; \|v\| = 1\}$ implies that $\alpha > 0$.

Let us fix a point $p_0$, $p_0 \in \mathbb{R}^n$ and show that there exist two constants $R_0 = R_0(\alpha) \in \mathbb{R}$, $R_0 > 0$, and $B = B(\alpha)$, $0 < B < 1$, $B \in \mathbb{R}$, such that if $d(p, p_0) > R_0$; then there is an $s_{i_0}$ such that
\[ d(s_{i_0}^m p, p_0) < B d(p, p_0) \quad \text{for some } m \in \mathbb{Z}, m > 0. \]
Namely, let
\[ a = \max_{1 \leq i \leq m} d(p_0, A^0(s_i)), \]
\[ b = \max_{1 \leq i \leq m} \|s_i x - x\|. \]
Let $v$ be the vector $p - p_0$ (i.e., started in $p_0$ with the endpoint $p$). Then there is an index $i_0$, such that $(v^0(s_{i_0}), v) \leq -\alpha^2 \|v\|$. Let $\pi$ be an affine subspace, orthogonal to $A^0(s_{i_0})$ and containing the point $p_0$. There
is $m, m \in \mathbb{Z}, m > 0$, such that

\[
(1) \quad d(p_0, s_i^n p) \leq \sqrt{(1 - \alpha^2)d(p_0, q) + b + a}.
\]

Let $\beta$ be a real number, $\sqrt{1 - \alpha^2} < \beta < 1$. Let us set $R_0 = (b + a)/(\beta - \sqrt{1 - \alpha^2})$ and assume that $d(p_0, p) > R_0$. Then $b + a < (\beta - \sqrt{1 - \alpha^2})d(p_0, p)$ and therefore $d(p_0, s_i^n p) \leq \beta d(p_0, p).

Let $K = U(p_0, 2R_0) = \{x \in \mathbb{R}^n : d(p_0, x) \leq 2R_0\}$. We will show now that for every point $p$ on $\mathbb{R}^n$ the intersection $K \cap \bar{S}P \neq \emptyset$. Let $O_p$ be the closure of the orbit $\bar{S}P$ and let $q$ be a point on $O_p$ such that $d(p_0, q) = d(p_0, O_p)$. Assume that $q \notin K$, then $d(p_0, q) > 2R_0$. Hence, there is $m, m > 0, m \in \mathbb{Z}$, and $s_{i_0}$ such that $d(p_0, s_i^n q) \leq \beta d(p_0, q) < d(p_0, q)$. It is clear that $s_i^n q \in O_p$, which contradicts our assumption that $d(p_0, q) = d(p_0, O_p)$.

**Proposition 4.5.** Let $S$ be a subsemigroup of $\text{Isom}(\mathbb{R}^n)$. Assume that $\overline{\text{Isom}(\mathbb{R}^n)}$ is a connected non-solvable group. Then for every finite set $g_1, \ldots, g_m \subseteq S$, such that $v^\beta(g_i)$ non-zero for all $i, 1 \leq i \leq m$, and every positive $e, e < 1$, $\in \mathbb{R}$, there are elements $g_1^*, \ldots, g_m^* \subseteq S$ such that

\[
(1) \quad g_1^*, \ldots, g_m^* \text{ freely generate a free semigroup};
\]

\[
(2) \quad (v^\beta(g_i^*), v^\beta(g_i)) \geq 1 - e \text{ for all } i, 1 \leq i \leq m.
\]

**Proof.** There is [7] an irreducible representation $\rho: S \rightarrow \text{GL}(V)$ of $S$ where $V$ is a finite-dimensional space over a local field, such that there is a proximal element in $\rho(S)$. Then, because of Lemma 3.2, there is a finite set $h_1, \ldots, h_t$ of elements in $S$ such that

\[
(i) \quad \rho(h_i) \text{ is proximal for all } i, 1 \leq i \leq t.
\]

\[
(ii) \quad A(\rho(h_i)) \subseteq P \setminus A(\rho(h_j)), 1 \leq i, j \leq t.
\]

\[
(iii) \quad \bigcap_{1 \leq j \leq t} A(\rho(h_j)) = \emptyset.
\]

\[
(iv) \quad \text{The lines corresponding to } A(h_i), 1 \leq i \leq t, \text{ span } V.
\]

Now let $M$ be an infinite set, $M \subseteq \mathbb{N}$, such that the sequence $\ell(g_i)^m, m \in M$, converges to $\ell(g_i)$ when $m$ converges to infinity. By Lemma 3.3, there is a number $N, N \in \mathbb{N}$, and an infinite subset $\tilde{M}, \tilde{M} \subseteq M, i$ and $j$, such that for $k > N, n > N, r > N,$ and $m \in \tilde{M},$

\[
(1) \quad \text{The element } \rho(h_i^m h_j^k h_i^k h_j^m) \text{ is proximal.}
\]

\[
(2) \quad A(\rho(h_i^m h_j^k h_i^k h_j^m)) \subseteq P \setminus A(\rho(h_j)), 1 \leq j \leq t.
\]

\[
(3) \quad A(\rho(h_j)) \subseteq P \setminus A(\rho(h_i^m h_j^k h_i^k h_j^m)), 1 \leq j \leq t.
\]
Now we will consider subsequences \((n_i)_{i \in \mathbb{N}}, (k_i)_{i \in \mathbb{N}}, (r_i)_{i \in \mathbb{N}}, (m_i)_{i \in \mathbb{N}}\) such that
\begin{equation}
\begin{aligned}
& (6) \text{ The sequences } (\ell(h_i)^n)_{i \in \mathbb{N}}, (\ell(h_j)^{k_j})_{i \in \mathbb{N}}, (\ell(h_j)^{r_j})_{i \in \mathbb{N}} \text{ converge to the identity in the Euclidean topology when } t \to \infty. \\
& (7) \quad n_i^2 + k_i^2 + r_i^2 \leq m_i, \quad t \in \mathbb{N}, \\
& (8) \quad N \leq \min_{i \in \mathbb{N}} (n_i, k_i, r_i) \geq N.
\end{aligned}
\end{equation}

Then the subspaces \(A^0(h_i^{n_i} h_j^{k_j} g_1^m h_i^{r_i})\) converge to \(A^0(g_1)\) by (6). If \(p\) is any point in \(A^0(g_1)\), then because of (7), the direction of the vector \(h_i^{-r_i} h_i^{k_i} h_j^{n_i} h_i^{r_i} p - g_1^{-m_i} p\) converges to \(v^0(g_1)\). Hence the sequence of vectors \(v^0(h_i^{n_i} h_j^{k_j} g_1^m h_i^{r_i})\) converges to \(v^0(g_1)\). This means that for any \(1 \geq \varepsilon > 0, \varepsilon \in \mathbb{R}\), there is \(T, T \in \mathbb{N}, T = T(\varepsilon)\), such that
\begin{equation}
(\varepsilon(\varepsilon(\varepsilon(\varepsilon(v^0(h_i^{n_i} h_j^{k_j} g_1^m h_i^{r_i})), v^0(g_1))) \geq 1 - \varepsilon \quad \text{ if } t > T(\varepsilon).
\end{equation}

So it is enough to set
\begin{equation}
\tilde{g}_1 = h_i^{n_i} h_j^{k_j} g_1^m h_i^{r_i} \quad \text{ where } t > T(\varepsilon).
\end{equation}

As far as the set \(h_1, \ldots, h_n, \tilde{g}_1\) is concerned, it satisfies the same conditions (i)–(iv) as above, and we can continue the process, and finally we will find elements \(\tilde{g}_1, \ldots, \tilde{g}_m\) with the following properties:
\begin{enumerate}
\item The element \(\rho(\tilde{g}_1)\) is proximal for every \(i, 1 \leq i \leq m\).
\item \(A(\rho(\tilde{g}_i)) \subseteq P \backslash A'(\rho(\tilde{g}_i)), 1 \leq i \leq m,\)
\item \((\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(v^0(\tilde{g}_i)), v^0(g_i)))) \geq 1 - \varepsilon, 1 \leq i \leq m.\)
\end{enumerate}

Now, by Lemma 2.2, there is \(\bar{N}, N \in \mathbb{N}\), such that \(\tilde{g}_1^n, \ldots, \tilde{g}_m^n\) is a freely generated free semigroup if \(n \geq \bar{N}\), so it is enough for all \(i, 1 \leq i \leq m\), to set \(g_i^n = \tilde{g}_1^n\), where \(n_0 \geq \bar{N}\), because \(v^0(\tilde{g}_1^n) = v^0(\tilde{g}_i)\) for every \(m, m > 0\).

**Theorem.** If \(S\) is a Euclidean semigroup which acts discontinuously on \(\mathbb{R}^n\), then the connected component \(\ell(S)\) of the Zariski closure of the linear part \(\ell(S)\) of \(S\) is a reducible group.

**Proof.** Assume that the semigroup \(\ell(S)\) is irreducible, then the closure \(\ell(S)\) is an irreducible group. The subspace \(\text{span}\{v^0(s), s \in S\}\) is \(\ell(S)\)-invariant; then span \(\{v^0(s), s \in S\} = \mathbb{R}^n\). By Corollary 4.3 there are elements \(g_1, \ldots, g_m\) on \(S\) and positive numbers \(\alpha_1, \ldots, \alpha_m, \alpha, \in \mathbb{R}\), such that
\begin{enumerate}
\item \(\alpha_1 v^0(g_1) + \cdots + \alpha_m v^0(g_m) = 0.\)
\item \(\text{span}\{v^0(g_1), \ldots, v^0(g_m)\} = \mathbb{R}^n.\)
\end{enumerate}

Let \(\varepsilon > 0\) be a real number such that if \(v_1, \ldots, v_m\) are norm 1 vectors and \((v_i, v^0(g_i)) \geq 1 - \varepsilon\) for all \(i = 1, \ldots, m\), then the convex cone spanned by \(v_1, \ldots, v_m\) also contains zero.
Using Proposition 4.5, we can find a set $g_1^*, \ldots, g_m^*$ and $g^*$ such that

1. $g_1^*, \ldots, g_m^*$ freely generate a free subsemigroup in Isom $\mathbb{R}^n$;
2. $(v^0(\tilde{g}_i^*), v^0(g_i)) \geq 1 - \epsilon$, $i = 1, \ldots, m$;
3. $\text{span}(v^0(g_1^*), \ldots, v^0(g_m^*)) = \mathbb{R}^n$.

The zero belongs to the convex cone spanned by $v^0(\tilde{g}_1^*), \ldots, v^0(g_m^*)$. Then, by Lemma 4.4, the subsemigroup generated by $g_1^*, \ldots, g_m^*$ is crystallographic. On the other hand, it is easy to extend the set $g_1^*, \ldots, g_m^*$ to the set $g_1^*, \ldots, g_m^*$ such that these elements freely generate a free semigroup. Now, using the fact that a semigroup generated by $g_1^*, \ldots, g_m^*$ is crystallographic, one can arrive at a contradiction by using the following arguments.

Let $K$ be a compact subset in $\mathbb{R}^n$ and let $SK = \mathbb{R}^n$, where $S$ is a subsemigroup generated by $g_1^*, \ldots, g_m^*$. Then for every $n, m \in \mathbb{Z}$, $n > 0$, we can find positive integers $k_1(n), \ldots, k_m(n)$ such that

$$(g^*)^{-n}K \cap (g_1^*)^{k_1(n)} \cdots (g_m^*)^{k_m(n)} K \neq \emptyset.$$ 

Thus,

$$(g^*)^n(g_1^*)^{k_1(n)} \cdots (g_m^*)^{k_m(n)} K \cap K \neq \emptyset.$$ 

The subsemigroup $\tilde{S}$ generated by $g^*, g_1^*, \ldots, g_m^*$ is free and acts discontinuously on $\mathbb{R}^n$; then by Lemma 2.2 from [6] we will immediately get a contradiction.

The group $O(n)$ is irreducible; hence we have

**Corollary.** Let $S$ be a Euclidean semigroup acting discontinuously on $\mathbb{R}^n$; then the linear part $\ell(S)$ of $S$ is not Zariski dense in $O(n)$.

Now let us show how by using these results one can prove Margulis’ conjecture for a crystallographic semigroup acting on a three-dimensional space. Namely, if $S$ is a crystallographic semigroup, $S \subseteq \text{Isom}(\mathbb{R}^3)$, then $S$ is a group.

These results were first proved in [6], but we used a different technique.

So by the theorem, $\overline{\ell(S)}$ is almost abelian. Assume first that the connected component $\overline{\ell(S)^0}$ is trivial. Then $\ell(S)$ is a finite group. Let $s_1, \ldots, s_m$ be a subset in $S$ such that for every $s \in S$ there is an element $s_i$, $1 \leq i \leq m$, with $\ell(s) = \ell(s_i)$. We can write any element $s \in S$ as a product $s = v_i \ell(s)$. There is $i = i(s)$, $1 \leq i \leq m$, such that $\ell(ss_i) = 1$ and so, $v_i + \ell(s_i)^{-1}v_i = ss_i \in S \cap \mathbb{R}^n$. Thus, $\tilde{S}$ is crystallographic semigroup, because $\tilde{S} \cdot \tilde{K} = \mathbb{R}^n$, and, therefore, $\tilde{S}$ is a group.
is group. Then, $S$ is a group. Let us now assume that \( \overline{\ell(S)} \neq \{e\} \) and \( V_0 = \text{span}(\ell^0(s), s \in S, \ell(s) - \text{regular}) \). Assume that \( \dim V_0 = 2 \). Using almost the same arguments as in the proof of Lemma 4.1, one can show that \( V_0 \) is an \( \overline{\ell(S)} \)-invariant subspace. So, the orthogonal complement \( V_0^\perp \) is also \( \overline{\ell(S)} \) invariant, and the restriction \( \overline{\ell(S)}_{|V_0^\perp} \) is trivial. Then for every regular element \( s \) in \( S \), \( \dim V_s^0 = 2 \), where \( V_s^0 = \{v \in \mathbb{R}^n, \ell(s)v = v\} \).

Hence, \( \dim V_s^0 = 3 \) and \( \overline{\ell(S)}_{|V_s^0} = \mathbb{R}^3 \), or \( \overline{\ell(S)} = 1 \). So, \( \dim V_0 = 1 \).

Fix a vector \( \nu_0, \nu_0 \in V_0 \). The semigroup \( S \) is crystallographic; then we can show that there are two regular elements \( s_1, s_2 \) from \( S \) such that

1. \( v_1 = \{v : \ell(s_1)v = v\} = V_2 = \{v : \ell(s_2)v = v\} \) for all \( m \in \mathbb{Z} \).

The easiest way to find such an element is the following: let \( K \) be a compact subset in \( \mathbb{R}^3 \) such that \( SK = \mathbb{R}^3 \) and let \( L \) be a line parallel to \( v_0 \) such that the intersection \( K \cap L \) is a non-empty set. Let \( x_0 \in K \cap L \) and let \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) be subsets of points of \( L \) with the following properties:

1. \( x_n - x_0, y_n - x_0, v_0 \) (for all \( n, n \in \mathbb{Z} \))
2. \( d(x_n, x_0) \to \infty \) and \( d(y_n, x_0) \to \infty \) when \( n \to \infty \).

Then there are two subsets of elements in \( S: \{\overline{\ell_s}\}_{s \in \mathbb{N}} \) and \( \{\overline{\ell_n}\}_{n \in \mathbb{N}} \) such that

1. \( \overline{\ell_s}^{-1}x_n \in K, \overline{\ell_n}^{-1}y_n \in K \) for all \( n, n \in \mathbb{N} \);
2. \( \overline{\ell}(\overline{\ell_n}) \) and \( \overline{\ell}(\overline{\ell_n}) \) are regular for all \( n, n \in \mathbb{N} \).

Now, using (2) and (3), we can see that for a large enough number \( N, \)

1. \( v_0 = \text{span}(\ell^0(s_1), s \in S, \ell(s) - \text{regular}) \). Assume that \( \dim V_0 = 2 \). Using \n
Now it is easy to see that there are two infinite sets \( N_1, N_1 \subseteq \mathbb{N} \) and \( N_2, N_2 \subseteq \mathbb{N} \) such that if \( p_0 \in A^0(s_1) \) and \( d_0 = d(A^0(s_1), A^0(s_2)) \), the distance between lines \( A^0(s_1) \) and \( A^0(s_2) \), then \( d(p_0, s_2^{n_1} s_1^{n_2} p_0) \leq 2d_0 \) for all \( n_1 \in N_1, n_2 \in N_2 \). But this is impossible since \( S \) acts discontinuously on \( \mathbb{R}^3 \).

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