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Explicit irrationality measures for continued fractions

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ABSTRACT

Let $\tau = [a_0; a_1, a_2, \ldots]$, $a_0 \in \mathbb{N}$, $a_n \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$, be a simple continued fraction determined by an infinite integer sequence (a_n). We are interested in finding an effective irrationality measure as explicit as possible for the irrational number τ. In particular, our interest is focused on sequences (a_n) with an upper bound at most (a^{n^k}), where $a > 1$ and $k > 0$. In addition to our main target, arithmetic of continued fractions, we shall pay special attention to studying the nature of the inverse function $z(y)$ of $y(z) = z \log z$.

1. Introduction

Let $\tau = [a_0; a_1, a_2, \ldots]$, $a_0 \in \mathbb{N}$, $a_n \in \mathbb{Z}^+$, $n \in \mathbb{Z}^+$, be a simple continued fraction determined by an infinite integer sequence (a_n). For example, set $\tau = [f_0; f_1, f_2, \ldots]$, where $(f_n) = (0, 1, 1, 2, \ldots)$ is the Fibonacci sequence. We prove an explicit lower bound

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(\frac{1+\sqrt{5}}{2} N^{\frac{D}{\sqrt{\log N}}} + 3)}, \quad D = \frac{\log \frac{3+\sqrt{5}}{2}}{\sqrt{\log 2}},$$

valid for every $M, N \in \mathbb{Z}$ with $N \geqslant 2$.

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In Theorem 2.1 we give a general method which is applied through the rest of the work. First we consider the case where (a_n) is bounded by linear functions. Here the applications include the explicit irrationality measure for the Siegel's constant $[0; 1, 2, 3, \dots]$ and for the Napier's constant $e = [2; 1, 2, 1, 1, 4, 1, \dots]$. Then we deal with exponential bounds. In this case we obtain explicit irrationality measures for continued fractions determined by the Fibonacci sequence or the sequence $(\operatorname{lcm}(1, 2, \dots, n))$. Finally we have the case where the sequence $(\log a_n)$ is bounded by polynomials, say $a_n = n!$. If the sequence $(\log a_n)$ grows faster than (K^n), $K > 1$, then the irrationality measure exponent is at least $K + 1 > 2$. This case is covered in Hančl, Matala-aho and Pulcerová [9].

By an explicit irrationality measure of an irrational number τ we mean any positive lower bound

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2 I(N)} = \frac{1}{N^{2+\delta(N)}}, \quad \delta(N) \geqslant 0, \tag{1}$$

satisfied by all $M, N \in \mathbb{Z}$, $N \geqslant N_0$, where the dependence $I(N)$ on N is explicit and N_0 is a given or at least an effectively computable number. Usually the infimum of $2 + \delta(N)$ is called irrationality measure of τ.

In the following we would like to give a survey on results considering Diophantine properties of e. In 1971 Bundschuh [4] proved that formula (1) is valid with $I(N) = 18\frac{\log 4N}{\log\log 4N}$ and $N_0 = 1$. Later Davis [6] found an asymptotical result that for every $\alpha > 1$ there exists N_0 such that $I(N) = 2\alpha\frac{\log 4N}{\log\log 4N}$ is valid for all $N > N_0$. For references on similar results see Borwein and Borwein [3], Fel'dman and Nesterenko [7] and Shiokawa [14]. In 1976 Galochkin [8] showed that we may take $I(N) = 0.001\frac{\log(N+2)}{\log\log(N+2)}$ and $N_0 = 1$. Later Alzer [2] improved the results of Bundschuh [4] and Galochkin [8] by proving the estimate $\mid e - \frac{M}{N} \mid > \frac{C\log\log N}{N^2 \log N}$ for all $M, N \in \mathbb{Z}$ with $N \geqslant 2$ if and only if $C < 0.386249\dots$. In 2000 Tasoev [17] proved similar result for continued fractions generalizing the expansion of e. Recently Sondow [16] received a new type of explicit irrationality measure for e, namely that for all $M, N \in \mathbb{Z}$ with $N \geqslant 2$ we have $\mid e - \frac{M}{N} \mid > \frac{1}{(S(N)+1)!}$, where $S(N)$ is the smallest positive integer such that $S(N)!$ is the multiple of N.

In [17] Tasoev considers also continued fractions of the kind $\tau = [a_0, \overline{a^\lambda, \dots, a^\lambda}]_{\lambda=0}^{\infty}$, $a_0, a, m \in \mathbb{Z}^+$ with $a \geqslant 2$ and $m \geqslant 2$. In [17] it is proved that for every $\alpha > 1$ there exists N_0 such that $I(N) = \sqrt{a}\alpha N^{\sqrt{\frac{2\log a}{m\log N}}}$ is valid for every $N > N_0$. However, this does not cover our example $\tau = [f_0; f_1, f_2, \dots]$.

In addition to our main target, arithmetic of continued fractions, we shall pay special attention to studying the nature of the inverse function $z(y)$ of $y(z) = z\log z$. We obtain a presentation for $z(y)$ by nested logarithms which besides its own interest is our main tool considering the polynomial growth cases. Namely, nested logarithm approximations of $z(y)$ give a natural framework considering e.g. the arithmetic nature of e thus allowing certain generalizations and improvements of the results of Alzer [2], Bundschuh [4] and Davis [6] for e.

2. Results

Throughout the whole paper we use the notations $\mathbb{N}$ and $\mathbb{Z}^+$ for the sets of non-negative and positive integers, respectively. In the following

$$\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$$

denotes the nth convergent of $[a_0; a_1, a_2, \dots]$.

Theorem 2.1. *Let $g_1(x), g_2(x)\colon \mathbb{R}^+ \to \mathbb{R}^+$ be increasing functions such that*

$$a_n \leqslant g_1(n), \qquad g_2(n) \leqslant q_n$$

for every $n \in \mathbb{Z}^+$ and let $g_2^{-1}(x)$ denote the inverse function of $g_2(x)$. Then

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(g_1(g_2^{-1}(N)+1)+3)} \tag{2}$$

for all $M \in \mathbb{Z}$, $N \in \mathbb{Z}^+$.

Define $z(y)$ to be the inverse function of the function $y(z) = z\log z$ when $z \geqslant 1/e$.

Theorem 2.2. *The inverse function $z(y)$ of the function*

$$y(z) = z\log z, \quad z \geqslant \frac{1}{e}, \tag{3}$$

is strictly increasing. When $y > e$ the inverse function may be given by the infinite nested logarithm fraction

$$z(y) = \frac{y}{\log \frac{y}{\log \frac{y}{\log \cdots}}}. \tag{4}$$

Let $z_0(y) = y$ and $z_n(y) = \frac{y}{\log z_{n-1}}$ for all $n \in \mathbb{Z}^+$. Then we also have

$$z_1 < z_3 < \cdots < z < \cdots < z_2 < z_0.$$

Theorem 2.3. *Let $a, B, c > 0$ be given and suppose*

$$a_n \leqslant an, \qquad (Bn)^{cn} \leqslant q_n$$

for all $n \in \mathbb{Z}^+$. Then

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(\frac{a}{B}z(\frac{B}{c}\log N) + a + 3)} \tag{5}$$

for all $M, N \in \mathbb{Z}$ with $N \geqslant 1$.

When $N > e^{\frac{ce}{B}}$, then we have a nested logarithm representation (4) of $z(\frac{B}{c}\log N)$. By Theorem 2.2 we get

$$z\left(\frac{B}{c}\log N\right) < z_2\left(\frac{B}{c}\log N\right) = \frac{\frac{B}{c}\log N}{E(N)}, \tag{6}$$

where

$$E(N) = \log \frac{\frac{B}{c}\log N}{\log \frac{B}{c}\log N}.$$

By using estimates (5) and (6) we get

$$\left|\tau - \frac{M}{N}\right| > \left(\frac{c}{a} - \epsilon(N)\right)\frac{E(N)}{N^2\log N}, \tag{7}$$

where

$$\epsilon(N) = \frac{c^2(a+3)}{a(\frac{a\log N}{E(N)} + c(a+3))} \underset{N\to\infty}{\to} 0. \tag{8}$$

From our result (7) we could deduce

$$\left|\tau - \frac{M}{N}\right| > \frac{C\log\log N}{N^2\log N} \tag{9}$$

with an explicit $C = C(N)$. However, bound (9) would not be as sharp as (7). To obtain a sharper bound let $B, c > 0$ be given. Define w as the solution (larger solution if $B > c$) of the equation

$$\left(\frac{w}{de}\right)^{\frac{w}{de}} = e^{\frac{-\log d}{de}}, \tag{10}$$

where $d = B/c$.

Corollary 2.4. *Let $a, B, c > 0$ be given and suppose*

$$a_n \leqslant an, \qquad (Bn)^{cn} \leqslant q_n$$

for all $n \in \mathbb{Z}^+$. Then

$$\left|\tau - \frac{M}{N}\right| > \left(\frac{c}{a} - \epsilon(N)\right)\left(1 - \frac{1}{w}\right)\frac{\log\log N}{N^2\log N}, \tag{11}$$

where $\epsilon(N)$ is defined as in (8) and w as in (10), for all $M, N \in \mathbb{Z}$ with $N > e^{\frac{ce}{B}}$.

Note that w in bound (11) is absolute for all N. If we want a better dependence, then $C(N)$ will be rather complicated function of N and we prefer to use bound (7).

Corollary 2.5. *Let $a, b > 0$ be given and suppose*

$$bn \leqslant a_n \leqslant an$$

for all $n \in \mathbb{Z}^+$. Then

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(\frac{ae}{b}z(\frac{b}{e}\log N) + a + 3)}$$

for all $M, N \in \mathbb{Z}$ with $N \geqslant 1$.

Theorem 2.3 can be generalized as follows.

Theorem 2.6. *Let $a, b, l > 0$ be given and suppose*

$$a_n \leqslant an^l, \qquad (Bn)^{cn} \leqslant q_n$$

for all $n \in \mathbb{Z}^+$. *Then*

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(a(\frac{1}{B}z(\frac{B}{c}\log N)+1)^l+3)}$$

for all $M, N \in \mathbb{Z}$ *with* $N \geqslant 1$.

Corollary 2.7. *Let* $a, b, h, l > 0$ *be given and suppose*

$$bn^h \leqslant a_n \leqslant an^l$$

for all $n \in \mathbb{Z}^+$. *Then*

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(a(\frac{e}{\sqrt[h]{b}}z(\frac{\sqrt[h]{b}}{he}\log N)+1)^l+3)}$$

for all $M, N \in \mathbb{Z}$ *with* $N \geqslant 1$.

Theorem 2.8. *Let* $a, B > 1$, $h, l > 0$ *be given. Suppose*

$$a_n \leqslant a^{n^l}, \qquad B^{n^{h+1}} \leqslant q_n$$

for all $n \in \mathbb{Z}^+$. *Then*

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(N^{\frac{\log a}{\log N}(\sqrt[h+1]{\frac{\log N}{\log B}}+1)^l}+3)}$$

for all $M, N \in \mathbb{Z}$ *with* $N \geqslant 2$.

Corollary 2.9. *Let* $a, b > 1$, $h, l > 0$ *be given and suppose*

$$b^{n^h} \leqslant a_n \leqslant a^{n^l}$$

for all $n \in \mathbb{Z}^+$. *Then*

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(N^{\frac{\log a}{\log N}(\sqrt[h+1]{\frac{(h+1)\log N}{\log b}}+1)^l}+3)}$$

for all $M, N \in \mathbb{Z}$ *with* $N \geqslant 2$.

3. Proofs

Let (a_n) be an integer sequence with $a_n \geqslant 1$, when $n \geqslant 1$. Consider the irrational value τ of the simple continued fraction

$$\tau = [a_0; a_1, a_2, \ldots].$$

From the theory of continued fractions we know the recurrence formulas

$$p_{n+2} = a_{n+2}p_{n+1} + p_n, \qquad q_{n+2} = a_{n+2}q_{n+1} + q_n \tag{12}$$

for all $n \in \mathbb{N}$ with the initial values

$$p_0 = a_0, \qquad q_0 = 1, \qquad p_1 = a_0 a_1 + 1, \qquad q_1 = a_1,$$

and the estimates

$$\frac{a_{n+2}}{q_n q_{n+2}} < \left|\tau - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \tag{13}$$

for all $n \in \mathbb{N}$, see e.g. Hardy and Wright [10]. By denoting $r_n = q_n \tau - p_n$ we have

$$\frac{1}{q_{n+2}} < |r_n| < \frac{1}{q_{n+1}}. \tag{14}$$

Let $M \in \mathbb{Z}$ and $N \in \mathbb{Z}^+$ be given. Write

$$\Lambda = N\tau - M, \qquad \Delta = p_n N - q_n M,$$

which gives

$$\Delta = q_n \Lambda - r_n N.$$

1. Suppose that

$$\frac{M}{N} \neq \frac{p_n}{q_n}$$

for all $n \in \mathbb{N}$. Thus

$$1 \leqslant |\Delta| = |p_n N - q_n M| \leqslant q_n |\Lambda| + |r_n| N. \tag{15}$$

By (14) there exists a positive integer h such that

$$|r_h| < \frac{1}{q_{h+1}} < \frac{1}{2N} < \frac{1}{q_h},$$

which gives

$$|r_h| N < \frac{1}{2}, \qquad q_h < 2N. \tag{16}$$

Now (15) and (16) imply

$$\frac{1}{2} < q_h |\Lambda| < 2N |N\tau - M|$$

or

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{4N^2}.$$

2. Suppose that

$$\frac{M}{N} = \frac{p_n}{q_n}$$

for some $n \in \mathbb{N}$. By (13) we get

$$\left|\tau - \frac{M}{N}\right| = \left|\tau - \frac{p_n}{q_n}\right| > \frac{a_{n+2}}{q_n q_{n+2}} > \frac{1}{(a_{n+1}+2)q_n^2} > \frac{1}{(a_{n+1}+3)N^2}. \tag{17}$$

The constant 2 is increased to 3 in (17) to ensure

$$\frac{1}{4N^2} \geqslant \frac{1}{(a_{n+1}+3)N^2}.$$

Proof of Theorem 2.1. All we need to do is to estimate a_{n+1} in (17). Denoting $q_n = N$ we have

$$a_{n+1} \leqslant g_1(n+1) \leqslant g_1\big(g_2^{-1}(N)+1\big)$$

and (2) follows. □

Proof of Theorem 2.2. In the following we suppose $y > e$. Note that

$$z = \frac{y}{\log z} = \frac{y}{\log \frac{y}{\log z}} = \cdots.$$

Eq. (3) has a unique solution z for a fixed y and thus the equation

$$z = \frac{y}{\log \frac{y}{\log z}} \tag{18}$$

has the same unique solution $z = z(y)$, too.

Let us define a function

$$l(x) = \frac{y}{\log x}, \quad x > 1,$$

and set $l^n(y) = z_n(y)$ for all $n \in \mathbb{N}$. Note that $l^{2k}(x)$ is increasing and $l^{2k+1}(x)$ is decreasing for all $k \in \mathbb{N}$. From $y > e$ we get $l(y) > e$. Since $l(e) = y$ we have

$$z_{2k+1} = l^{2k+1}(y) = l^{2k+2}(e) < l^{2k+2}\big(l(y)\big) = l^{2k+3}(y) = z_{2k+3},$$
$$z_{2k} = l^{2k}(y) = l^{2k+1}(e) > l^{2k+1}\big(l(y)\big) = l^{2k+2}(y) = z_{2k+2}$$

and

$$z_{2k} = l^{2k}(y) = l^{2k+1}(e) > l^{2k+1}(y) = z_{2k+1}$$

for all $k \in \mathbb{N}$. Therefore

$$z_1 < z_3 < \cdots < z_2 < z_0.$$

Hence the limits

$$\lim_{k\to\infty} z_{2k+1} = A \quad \text{and} \quad \lim_{k\to\infty} z_{2k} = B$$

exist and by applying the function $l^2(x)$ we see that they satisfy

$$A = \frac{y}{\log \frac{y}{\log A}} \quad \text{and} \quad B = \frac{y}{\log \frac{y}{\log B}}.$$

By the uniqueness of the solution of (18) we get $A = B = z(y)$ and thus

$$z(y) = l^{\infty}(y) = \frac{y}{\log \frac{y}{\log \frac{y}{\log\cdots}}}. \qquad \square$$

Proof of Theorem 2.3. We shall estimate a_{n+1} in (17). Denoting $q_n = N$ and using Theorem 2.2 we have

$$y(Bn) = Bn \log Bn \leqslant \frac{B}{c} \log N,$$

so

$$Bn \leqslant z\left(\frac{B}{c} \log N\right)$$

because $z(y)$ is strictly increasing. Note that if $Bn < 1/e$ then we have $Bn \leqslant z(y(Bn))$ too. Now

$$a_{n+1} \leqslant a(n+1) \leqslant \frac{a}{B} z\left(\frac{B}{c} \log N\right) + a. \qquad \square$$

Proof of Corollary 2.4. Denote $d = B/c$ and $x = \log\log N$. The function

$$f(x) = \frac{\log(\log d + x) - \log d}{x} \tag{19}$$

will obtain its maximum $1/w$ on the positive real axis at $w - \log d$, where w is the solution of Eq. (10) (larger solution if $d > 1$). Now the result follows from (7) as

$$E(N) = \log \frac{d \log N}{\log d \log N} = \log d + \log\log N - \log(\log d + \log\log N) \geqslant \left(1 - \frac{1}{w}\right) \log\log N. \qquad \square$$

Proof of Corollary 2.5. Recurrence (12) implies

$$b^n n! \leqslant q_n. \tag{20}$$

Using Stirling's formula (see e.g. [1])

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{\theta(n)}{12n}}, \quad 0 < \theta(n) < 1,$$

with (20) we get

$$\left(\frac{bn}{e}\right)^n \leqslant q_n.$$

Now we use Theorem 2.3 with $B = b/e$ and $c = 1$. □

Proof of Theorem 2.6. From the proof of Theorem 2.3 we get

$$Bn \leqslant z\left(\frac{B}{c}\log N\right)$$

and so

$$a_{n+1} \leqslant a(n+1)^l \leqslant a\left(\frac{1}{B}z\left(\frac{B}{c}\log N\right)+1\right)^l. \qquad \square$$

Proof of Corollary 2.7. Similarly to (20) we have

$$b^n (n!)^h \leqslant q_n.$$

Again, Stirling's formula implies

$$\left(\frac{\sqrt[h]{b}n}{e}\right)^{hn} \leqslant q_n.$$

Now we use Theorem 2.6 with $B = \sqrt[h]{b}/e$ and $c = h$. □

Proof of Theorem 2.8. Here

$$n \leqslant \sqrt[h+1]{\frac{\log N}{\log B}},$$

and thus

$$a_{n+1} \leqslant a^{(n+1)^l} \leqslant a^{\left(\sqrt[h+1]{\frac{\log N}{\log B}}+1\right)^l}. \qquad \square$$

Proof of Corollary 2.9. By recurrence (12) we get

$$b^{1^h+2^h+\cdots+n^h} \leqslant q_n.$$

Now we use Theorem 2.8 with $B = \sqrt[h+1]{b}$. □

4. Applications

First we consider Siegel's continued fraction

$$\tau = [0; 1, 2, \ldots],$$

which is transcendental by Siegel's theory of E-functions [15]. Now $a_n = n$ for all $n \in \mathbb{Z}^+$. Using Theorem 2.3 with $a = 1$, $B = 1/e$ and $c = 1$ (see the proof of Corollary 2.5) we obtain

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(ez(\frac{1}{e}\log N) + 4)}$$

for all $M, N \in \mathbb{Z}$ with $N \geqslant 1$.

Now Eq. (10) has the form

$$w^w = e,$$

which has a unique solution w, where $1/w = 0.567143\ldots$. Using (11) we can write the lower bound as

$$\left|\tau - \frac{M}{N}\right| > \bigl(1 - \epsilon(N)\bigr)\left(1 - \frac{1}{w}\right)\frac{\log\log N}{N^2 \log N}$$

for all $M, N \in \mathbb{Z}$ with $N \geqslant 1619$, where

$$\epsilon(N) = \frac{4}{\frac{\log N}{\log \frac{\log N}{\log\log N - 1} - 1} + 4}.$$

In the next example we consider the Napier's constant

$$e = [2; 1, 2, 1, 1, 4, 1, \ldots].$$

For the proof see e.g. Perron [12] or Cohn [5]. Now we are looking for the lower bound of q_n. We have $a_0 = 2$ and

$$a_n = \begin{cases} \frac{2n+2}{3}, & n \equiv 2 \pmod 3, \\ 1, & n \not\equiv 2 \pmod 3) \end{cases}$$

when $n \in \mathbb{Z}^+$. From recurrence formula (12) we obtain

$$q_{n+2} = (a_n a_{n+1} a_{n+2} + a_n + a_{n+2})q_{n-1} + (a_{n+1}a_{n+2} + 1)q_{n-2} \tag{21}$$

for all $n \geqslant 2$.

When $n \equiv 0 \pmod 3$ and $n \geqslant 3$ then we have $a_n = 1$, $a_{n+1} = 1$ and $a_{n+2} = (2n+6)/3$. By recurrence (21)

$$q_{n+2} = \frac{4n+15}{3}q_{n-1} + \frac{2n+9}{3}q_{n-2}. \tag{22}$$

When $n \equiv 2 \pmod 3$ then $a_n = (2n+2)/3$, $a_{n+1} = 1$, $a_{n+2} = 1$ and

$$q_{n+2} = \frac{4n+7}{3}q_{n-1} + 2q_{n-2} \geqslant \frac{4n+7}{3}q_{n-1}. \tag{23}$$

In the case $n \equiv 1 \pmod 3$ and $n \geqslant 4$ we have $a_{n+2} = a_{n-1} = 1$. So (22) and (23) give

$$q_{n+2} = q_{n+1} + q_n \geqslant \frac{4n+11}{3} q_{n-2} + (2n+2) q_{n-3} \geqslant \frac{4n+4}{3}(q_{n-2} + q_{n-3}) = \frac{4n+4}{3} q_{n-1}.$$

Therefore for all $n \in \mathbb{Z}^+$ we have a lower bound

$$q_{n+2} \geqslant \frac{4n+4}{3} q_{n-1}.$$

Hence

$$\sqrt[3]{\left(\frac{4}{3}\right)^n n!} \leqslant q_n$$

and by Stirling's formula [1] we get

$$\left(\frac{4}{3e} n\right)^{\frac{1}{3}n} \leqslant q_n$$

for all $n \geqslant 1$. Using Theorem 2.3 with $a = 1$, $B = 4/3e$ and $c = 1/3$ we obtain

$$\left| e - \frac{M}{N} \right| > \frac{1}{N^2(\frac{3e}{4} z(\frac{4}{e} \log N) + 4)}$$

for all $M, N \in \mathbb{Z}$ with $N \geqslant 1$ (naturally any choice of $a > 2/3$ holds for N big enough, but this one holds for all $N \geqslant 1$). When $N \geqslant 7$ we can take $a = 4/5$. So we get $1/w = 0.278383\ldots$ and thus

$$\left| e - \frac{M}{N} \right| > \left(\frac{5}{12} - \epsilon(N) \right) \left(1 - \frac{1}{w} \right) \frac{\log\log N}{N^2 \log N}$$

for all $M, N \in \mathbb{Z}$ with $N \geqslant 7$, where

$$\epsilon(N) = \frac{95}{\frac{144 \log N}{\log \frac{4 \log N}{\log 4 \log N - 1} - 1} + 228}.$$

For example if $N \geqslant 7$ then $C = 0.165684\ldots$ and for $N \geqslant 32$ we have $C = 0.197639\ldots$. For $N \geqslant 39$ we can take $a = 3/4$ to get larger C and so on. With any constant $C < 1/2$ the bound can be improved to $C \log\log N / N^2 \log N$ for big enough N, as $a \to 2/3$, $\epsilon(N) \to 0$ and $f(x) \to 0$, $f(x)$ decreasing, see (19).

In the following set $a_n = f_n$. We choose $h = 1$, $l = 1$, $B = \sqrt[4]{2}$ and $a = \frac{1+\sqrt{5}}{2}$. Then Theorem 2.8 gives

$$\left| \tau - \frac{M}{N} \right| > \frac{1}{N^2(\frac{1+\sqrt{5}}{2} N^{\frac{D}{\sqrt{\log N}}} + 3)}$$

where $M, N \in \mathbb{Z}$ with $N \geqslant 2$ and $D = \frac{\log \frac{3+\sqrt{5}}{2}}{\sqrt{\log 2}}$. This improves considerably the result of Matala-aho and Merilä [11].

When $a_n = \mathrm{lcm}(1, 2, \ldots, n)$ we choose $h = 1$, $l = 1$, $b = \sqrt{2}$ and $a = e^{1.030883}$ (see e.g. Rosser and Schoenfeld [13]). Then Corollary 2.9 gives

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(e^{1.030883} N^{\frac{D}{\sqrt{\log N}}} + 3)}$$

where $M, N \in \mathbb{Z}$ with $N \geqslant 2$ and $D = \frac{2.061766}{\sqrt{\log 2}}$.

In the case $a_n = n!$ we may choose $a = e$, $b = \sqrt{2}$, $l = 1.280678$ and $h = 1$. Then Corollary 2.9 gives

$$\left|\tau - \frac{M}{N}\right| > \frac{1}{N^2(N^{\frac{1}{\log N}(2\sqrt{\frac{\log N}{\log 2}}+1)^{1.280678}} + 3)}$$

for all $M, N \in \mathbb{Z}$ with $N \geqslant 2$.

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