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Multiple Equilibria of Elastic Strings under Central Forces: Highly Singular Nonlinear Boundary Value Problems of the Bernoullis

STUART S. ANTMAN

Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742

AND

PETER WOLFE

Department of Mathematics, University of Maryland, College Park, Maryland 20742

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1. INTRODUCTION. FORMULATION AND PRELIMINARY ANALYSIS OF THE GOVERNING EQUATIONS

In this paper we study the multiplicity of equilibrium states of nonlinearly elastic strings under central forces. This problem generalizes that studied by Joh. Bernoulli in 1728 for an inextensible string. (The theory of elastic strings was developed by Jas. Bernoulli in 1691–1704; cf. [11].) It is technically more difficult than the related problem of the elastic catenary studied in [2, 4, 8] because it does not have a convenient set of integrals and because the central force field may well be infinite at its center. The complications due to this singularity are magnified by the requirement that the compressive contact force in the string must become infinite where the local ratio of deformed to natural length becomes 0. We find that these problems have a multiplicity of both regular and singular solutions, with the analysis of the latter requiring a careful extension of the governing laws of mechanics to handle infinite forces.

In the rest of this section we formulate the governing equations. Here we pay special attention to questions of regularity, which cannot be handled routinely in virtue of the singularities in the equations. For a large class of problems we obtain detailed global information about the qualitative behavior of all regular solutions. In Section 2, we give a full characterization of singular solutions. In Section 3 we study purely radial solutions (by means

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of fixed point theorems). These problems have not only an intrinsic interest, but also play a central role in the multiplicity theory developed in Section 4 for arbitrary solutions. The analysis of Section 4 is based on a combination of shooting techniques with the use of the Leray–Schauder degree theory. In Section 5 we discuss existence and nonexistence of tensile solutions.

Let a nonlinearly elastic string have a natural length 1. We identify each material point of the string by its coordinate $s \in [0, 1]$. Let $\mathbf{r}(s)$ represent the deformed position in Euclidean 3-space $\mathbb{E}^3$ of the material point s. We require $\mathbf{r}$ to be absolutely continuous on $[0, 1]$. Let $r(s) = |\mathbf{r}(s)|$. We assume that the string is subjected to a central force field of intensity

$$\mathbf{f}(s) \equiv f(\mathbf{r}(s), s)\, \mathbf{r}(s)/r(s) \tag{1.1}$$

per unit reference length at s. We assume that $f(\cdot, s)$ is continuous on $\mathbb{E}^3\backslash\{0\}$, that $f(\mathbf{p}, \cdot)$ is integrable for each $\mathbf{p} \in E^3\backslash\{0\}$, and that f vanishes nowhere. The cases of most physical interest are those in which $\mathbf{f}$ describes a central electrostatic, magnetostatic, or gravitational force so that f is given by the inverse square law

$$f(\mathbf{r}, s) = \kappa(s)\, r^{-2} \tag{1.2}$$

and that in which $\mathbf{f}$ describes the centrifugal force due to the rotation of the string with constant angular velocity ω about an axis perpendicular to a plane in which the string is constrained to lie. In this case

$$f(\mathbf{r}, s) = \omega^2 (\rho A)(s) r, \tag{1.3}$$

where $(\rho A)(s)$ is the mass density per unit reference length at s.

Let $\mathbf{n}(s)$ denote the resultant contact force exerted by the material of $(s, 1]$ on the material of $[0, s]$. Then the equation of equilibrium for the string has the form

$$\mathbf{n}(s) - \mathbf{n}(a) + \int_a^s \mathbf{f}(\xi)\, d\xi = \mathbf{0} \tag{1.4}$$

for all intervals (a, s) in $[0, 1]$ on which $\mathbf{f}$ is integrable. The assumptions following (1.1) show that $\mathbf{f}$ is integrable on $P \equiv \{s: r(s) > 0\}$. Since $\mathbf{r}$ is continuous, P is the union $\bigcup I_m$ of a countable number of disjoint open subintervals I_m of $[0, 1]$. Equations (1.1) and (1.4) imply that

$$\mathbf{n}' + f(\mathbf{r}, s)\mathbf{r}/r = \mathbf{0} \qquad \text{a.e. on } P. \tag{1.5}$$

A string has the defining property that

$$\mathbf{n} = N\mathbf{e}, \qquad \text{where} \quad \mathbf{e} \equiv \mathbf{r}'/|\mathbf{r}'|. \tag{1.6}$$

$\mathbf{e}(s)$ is the unit tangent to the string at s and $N(s)$ is the *tension* at s. ($\mathbf{e}$ is defined a.e. on $[0, 1]$ since $\mathbf{r}$ is absolutely continuous.) We set

$$\nu(s) \equiv |\mathbf{r}'(s)| \tag{1.7}$$

and require that

$$\nu(s) > 0 \qquad \text{a.e. on } P. \tag{1.8}$$

We seek configurations $\mathbf{r}$ that satisfy (1.4) and (1.8) wherever $r > 0$ and that satisfy the boundary conditions

$$\mathbf{r}(0) = \mathbf{r}_0, \qquad \mathbf{r}(1) = \mathbf{r}_1, \tag{1.9}$$

where $\mathbf{r}_0$ and $\mathbf{r}_1$ are given vectors in $\mathbb{E}^3$.

If we substitute (1.1) and (1.6) into (1.4), we obtain

$$N(s)\,\mathbf{e}(s) = N(a)\,\mathbf{e}(a) - \int_a^s [f(\mathbf{r}(t), t)\,\mathbf{r}(t)/r(t)]\,dt \tag{1.10}$$

for (a, s) in a component open interval of P. It follows that if $\mathbf{r}$ and N satisfy (1.10), then $\mathbf{n} = N\mathbf{e}$ and, therefore, $|N| = |\mathbf{n}|$ are absolutely continuous on P. These facts do not ensure that N is itself absolutely continuous on P because N and $\mathbf{e}$ can have jumps that switch their signs at the same places. To exclude pathological configurations with such jumps we shall seek solutions for which N is continuous on P. Then (1.10) implies that N and $\mathbf{e}$ are absolutely continuous on P wherever N does not vanish.

We assume that the string is nonlinearly elastic so that the tension at s depends upon the stretch $\nu(s)$. Specifically we assume that there is a continuously differentiable function $(0, \infty) \times [0, 1] \ni (\nu, s) \mapsto \hat{N}(\nu, s) \in \mathbb{R}$ with

$$\hat{N}_\nu(\nu, s) > 0, \qquad \hat{N}(1, s) = 0, \tag{1.11}$$

$$\hat{N}(\nu, s) \to \infty \quad \text{as } \nu \to \infty, \qquad \hat{N}(\nu, s) \to -\infty \quad \text{as } \nu \to 0 \tag{1.12}$$

so that

$$N(s) = \hat{N}(\nu(s), s). \tag{1.13}$$

Properties (1.11) and (1.12) imply that $\hat{N}(\cdot, s)$ has an inverse $\hat{\nu}(\cdot, s)$ so that (1.13) is equivalent to

$$\nu(s) = \hat{\nu}(N(s), s). \tag{1.14}$$

Our boundary value problem is to find absolutely continuous functions $\mathbf{r}$ and functions N continuous on P that satisfy (1.4), (1.6), (1.13), and (1.1) subject to (1.8) and (1.9). We now obtain some useful properties of this system.

We operate on (1.5) with $\mathbf{r} \times$ to obtain

$$\mathbf{0} = \mathbf{r} \times \mathbf{n}' = (\mathbf{r} \times \mathbf{n})' \tag{1.15}$$

whenever (1.5) holds since $\mathbf{r}' \times \mathbf{n} = 0$ by (1.6). For each subinterval I_m of P Eq. (1.15) implies that there is a constant vector $\mathbf{c}_m$ such that

$$\mathbf{r} \times \mathbf{n} = \mathbf{c}_m, \tag{1.16a}$$

$$\mathbf{r} \cdot \mathbf{c}_m = 0, \tag{1.16b}$$

$$\mathbf{n} \cdot \mathbf{c}_m = 0. \tag{1.16c}$$

(Equation (1.16a) is statement of the balance of moments about $\mathbf{0}$ on any subinterval of I_m.)

We now limit our attention to the behavior of $\mathbf{r}$ and $\mathbf{n}$ on a representative subinterval I of $\{I_m\}$. (I would, of course, be the entire interval $[0, 1]$ if r does not vanish.) We denote the corresponding constant from $\{\mathbf{c}_m\}$ by $\mathbf{c}$. In the next section we examine the detailed structure of solutions over all of $[0, 1]$ when r vanishes at at least one point.

Equation (1.16a) implies that if $\mathbf{r} \times \mathbf{n}$ should vanish at at least one point s in I, it must vanish for all $s \in I$. Now (1.6) implies that $\mathbf{r}(\sigma) \times \mathbf{n}(\sigma) = \mathbf{0}$ only if either $N(\sigma) = 0$ or $\mathbf{r}(\sigma) \times \mathbf{r}'(\sigma) = \mathbf{0}$. If $\mathbf{n}$ and $\mathbf{r}$ were known to be continuously differentiable on I, then (1.5) would hold everywhere on I. Using (1.6) to write (1.5) as

$$N'\mathbf{e} + N\mathbf{e}' + f\mathbf{r}/r = \mathbf{0} \tag{1.17}$$

we see that the nonvanishing of f would then imply that $N'(\sigma) \neq 0$ if $N(\sigma) = 0$. Thus, N could vanish only at isolated points of I whence $\mathbf{r} \times \mathbf{r}'$ would have to vanish at the remaining points of I. Since $\mathbf{r} \times \mathbf{r}'$ is continuous, it would, therefore, have to vanish everywhere. Since $\nu > 0$ by (1.8) and $r > 0$ on I, this means that the string would have to lie on a ray from the center. (We give a formal proof of this statement below.) In summary, (1.16a) and the continuous differentiability of $\mathbf{n}$ and $\mathbf{r}$ imply that the material of I must either lie on a ray or be nowhere radial.

We now show that this same conclusion holds when $\mathbf{r}$ is merely absolutely continuous. We first show that the points of I where N vanishes must be isolated. Suppose that there were a σ in I such that $N(\sigma) = 0$ and σ is not an isolated zero of N. Thus, there would be a sequence $\{s_k\}$ of points of I converging to σ such that $N(s_k) = 0$ $(s_k \neq \sigma)$. Let us replace a and s in (1.4) by σ and s_k. Then this assumption would reduce (1.4) to

$$\int_\sigma^{s_k} [f(\mathbf{r}(s), s)\,\mathbf{r}(s)/r(s)]\,ds = \mathbf{0}, \qquad k = 1, 2, 3, \ldots . \tag{1.18}$$

Since $\mathbf{r}$ is absolutely continuous and nonzero on I,

$$s \mapsto \frac{\mathbf{r}(\sigma)}{r(\sigma)} \cdot \frac{\mathbf{r}(s)}{r(s)} \equiv R(s)$$

is absolutely continuous on I and equals 1 at σ. Thus, (1.18) implies that

$$\int_\sigma^{s_k} f(\mathbf{r}(s), s)\, R(s)\, ds = 0, \qquad k = 1, 2, 3, \ldots. \tag{1.19}$$

But R is positive on $[\sigma, s_k)$ if k is sufficiently large and f vanishes nowhere. Thus, the integrand of (1.19) has one sign for k sufficiently large so that (1.19) is impossible. Thus, the zeros of N on I must be isolated. (This conclusion does not prevent these zeros from accumulating at an end point of I.) Hence $\mathbf{r} \times \mathbf{r}'$, which is defined almost everywhere on I, must vanish almost everywhere on I. Since neither $\mathbf{r}$ nor $\mathbf{r}'$ can vanish on I because r and v are positive, $\mathbf{r}$ and $\mathbf{r}'$ must be parallel on I. Thus, there must be a nonvanishing locally integrable function g on I such that

$$\mathbf{r}'(s) = g(s)\, \mathbf{r}(s) \tag{1.20}$$

almost everywhere on I. The vector equation (1.20) is equivalent to an *uncoupled* system of three scalar, linear equations. Its solution is

$$\mathbf{r}(s) = \mathbf{r}(a) \exp \int_a^s g(t)\, dt, \tag{1.21}$$

where a is any point of I. This means that $\mathbf{r}$ lies on a ray.

Similarly if $\mathbf{r} \times \mathbf{r}'$ should vanish at a point in I, then (1.16a) would imply that $\mathbf{c} = \mathbf{0}$. By our preceding argument, $\mathbf{r}$ and $\mathbf{n}$ must again be radial. Thus, we find that the material of I must either lie on a ray or else be nowhere radial. The latter case occurs if and only if $\mathbf{c} \neq \mathbf{0}$. When this happens N cannot vanish on I (by (1.16a)) so that the configuration of the material of I can be characterized either as tensile ($N > 0$) or as compressive ($N < 0$). Moreover, if $\mathbf{c} \neq \mathbf{0}$, Eqs. (1.16b) and (1.16c) imply that $\mathbf{r}$ and, hence, $\mathbf{n} = N\mathbf{e}$ lie in the plane perpendicular to $\mathbf{c}$ for s in I. Since $\mathbf{e}$ is absolutely continuous on I where N does not vanish, it follows that if $\mathbf{c} \neq 0$, then $\mathbf{e}$ is absolutely continuous on all of I. We summarize these results:

1.22. PROPOSITION. *Let* $\mathbf{r}$ *be an absolutely continuous function, let* N *be continuous on* P, *and let* $\mathbf{r}$ *and* N *satisfy* (1.10) *on an open interval* I *on which* $r > 0$, $v > 0$. *Then* N, *which can vanish only at isolated points of* I, *is absolutely continuous on* I *and* $\mathbf{e}$ *is absolutely continuous on each open subinterval of* I *on which* N *does not vanish. If* N *vanishes anywhere on* I *or if the* *tangent to the string is radial at a point of I, then the configuration of I is radial. Otherwise the configuration is a plane curve that is nowhere radial and this configuration is everywhere in tension or else everywhere in compression.*

By introducing our constitutive assumptions (1.11)–(1.14) we can strengthen the regularity theory of Proposition 1.22:

1.23. PROPOSITION. *Suppose that* $\mathbf{r}$ *and* N *satisfy the hypotheses of Proposition* 1.22 *and satisfy* (1.13) *and* (1.14). *Then* $\mathbf{r}'$ *is absolutely continuous on every open subinterval of* I *on which* N *does not vanish. If* f *is continuous on* $(\mathbb{E}^3\backslash\{0\})\times I$, *then* $\mathbf{r}$ *is twice continuously differentiable on every subinterval of* I *on which* N *does not vanish.*

Proof. Since $\hat{\nu}$ is continuously differentiable the function $s\mapsto \nu(s)=\hat{\nu}(N(s),s)$ is absolutely continuous on I since N is. We multiply (1.10) by $\nu(s)/N(s)$ to obtain a representation for $\mathbf{r}'$. This representation yields the conclusion. ∎

The analysis leading to Proposition 1.22 shows that when $\mathbf{c}\neq\mathbf{0}$, the string is confined to a plane curve perpendicular to $\mathbf{c}$. Let us take the plane perpendicular to $\mathbf{c}$ to be spanned by the orthonormal pair $\{\mathbf{i},\mathbf{j}\}$. We can accordingly locate $\mathbf{r}(s)$ by polar coordinates r and φ:

$$r=r(\cos\varphi\mathbf{i}+\sin\varphi\mathbf{j}). \tag{1.24}$$

Thus,

$$\nu^2=(r')^2+(r\varphi')^2. \tag{1.25}$$

If we substitute (1.6) and (1.24) into (1.16a) and take the dot product of the resulting expression with $\mathbf{k}\equiv\mathbf{i}\times\mathbf{j}$, we find

$$r^2\varphi'=\mathbf{c}\cdot\mathbf{k}\nu/N. \tag{1.26}$$

Since r, ν, N, and $\mathbf{c}$ do not vanish, (1.26) implies that the sign of φ' is that of $\mathbf{c}\cdot\mathbf{k}/N$. Since the sign of $\mathbf{c}\cdot\mathbf{k}$ depends only on whether the parallel vectors $\mathbf{k}$ and $\mathbf{c}$ have the same or opposite sense, we can make φ' positive by choosing the basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ so that $\mathbf{c}\cdot\mathbf{k}$ has the same sign as N. Now $c\equiv|\mathbf{c}|=|\mathbf{c}\cdot\mathbf{k}|$. Thus, $\mathbf{c}\cdot\mathbf{k}=\pm c=\operatorname{sign}(N)c$ and (1.26) reduces to

$$r^2\varphi'=\pm c\nu/N=\operatorname{sign}(N)\,c\nu/N=c\nu/|N|. \tag{1.27}$$

When (1.13) holds, φ' is an absolutely continuous function on I by Proposition 1.22.

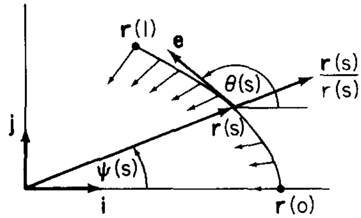

FIG. 1.30. Relationship of angles φ and θ. Here $I = [0, 1]$. The lightface arrows indicate an attractive central force. In this figure, $\theta' > 0$, $\varphi' > 0$.

To analyze the shape of the curves described by the restriction of $\mathbf{r}$ to I we represent $\mathbf{e}$ by

$$\mathbf{e}(s) = \cos\theta(s)\mathbf{i} + \sin\theta(s)\mathbf{j}, \tag{1.28}$$

whence

$$\mathbf{e}'(s) = [\mathbf{k} \times \mathbf{e}(s)]\,\theta'(s). \tag{1.29}$$

(See Fig. 1.30.) If we substitute these expressions into (1.17) and then take the dot product of (1.17) with $\mathbf{k} \times \mathbf{e}(s)$, we find that

$$N\theta' = \frac{f\mathbf{k} \cdot (\mathbf{r} \times \mathbf{e})}{r}. \tag{1.31}$$

From (1.24) and (1.28) we get

$$\mathbf{r} \times \mathbf{e} = r\sin(\theta - \varphi)\mathbf{k}. \tag{1.32}$$

If we differentiate (1.24), use (1.6) and (1.7), and equate the result with (1.28), we find

$$\cos\theta = \cos\varphi - \frac{r\varphi'}{v}\sin\varphi, \qquad \sin\theta = \sin\varphi + \frac{r\varphi'}{v}\cos\varphi, \tag{1.33}$$

whence

$$\frac{r\varphi'}{v} = \sin(\theta - \varphi). \tag{1.34}$$

Finally, if we combine (1.31), (1.32) and (1.34), we get

$$N\theta' = \frac{rf\varphi'}{v}. \tag{1.35}$$

If $\mathbf{c} \neq 0$, then φ' is positive on I and N does not vanish on I. Then θ', which does not vanish on I, has the same sign as Nf. θ' is not the curvature of $\mathbf{r}$ because the prime represents the derivative with respect to the reference length variable s and not the actual arc length parameter. The curvature is θ'/v and has the same sign as θ'. Note that this curvature is defined almost everywhere on I by virtue of Proposition 1.22 and Eq. (1.31). We say that the curve $\mathbf{r}$ is *bowed out* on I if $\theta' > 0$ on I as in Fig. 1.30 and *bowed in* if $\theta' < 0$ in I. Hence, we have

1.36. PROPOSITION. *Let the hypothesis of Proposition* 1.22 *hold. If* $c \neq 0$, *then the restriction of* $\mathbf{r}$ *to* I *is bowed-out if* $Nf > 0$ *and bowed-in if* $Nf < 0$. *(There are no other possibilities.) In particular, if* f *is attractive so that* $f < 0$, *then each bowed-out configuration of* I *is compressive and each bowed-in configuration of* I *is tensile, etc.*

Since φ' is positive on I when $c \neq 0$, the map $I \ni s \mapsto \varphi(s)$ has an inverse $\hat{s}$, which is continuously differentiable when (1.13) holds. We set

$$q(\varphi) = r(\hat{s}(\varphi)). \tag{1.37}$$

Let us denote derivatives with respect to φ by a superposed dot. We now substitute (1.24) and (1.37) into (1.17), use the chain rule, and substitute (1.27) into the radial component of the resulting equation to find that q and $u \equiv 1/q$ satisfy

$$(\dot{q}/q^2)^{\cdot} - 1/q + Nq^2 f/c^2 v = 0, \tag{1.38a}$$

$$\ddot{u} + u - Nf/c^2 v u^2 = 0 \tag{1.38b}$$

almost everywhere on I and everywhere on I that f and v are continuous.

Equation (1.38a) implies that if $Nf < 0$, then $\ddot{q} > 0$. This reinforces the conclusions of Proposition 1.36. Thus, a tensile configuration under an attracting force field has the expected U-shape.

We now obtain a specific ordinary differential equation for q or u when the string satisfies (1.13) or (1.14). The substitution of (1.27) and (1.37) into (1.25) yields

$$N^2 = c^2(q^2 + \dot{q}^2)/q^4 = c^2(u^2 + \dot{u}^2). \tag{1.39}$$

Then (1.14) yields

$$v(\hat{s}(\varphi)) = \hat{v}(\pm c\sqrt{u(\varphi)^2 + \dot{u}(\varphi)^2}, \hat{s}(\varphi)). \tag{1.40}$$

The substitution of (1.39) and (1.40) into (1.38b) and (1.27) yields the following coupled semilinear system of ordinary differential equations for u and $\hat{s}$:

$$\ddot{u}+u=\frac{\sqrt{u^2+\dot{u}^2}f((\cos\varphi\mathbf{i}+\sin\varphi\mathbf{j})/u,\hat{s})}{c\hat{\nu}(\pm c\sqrt{u^2+\dot{u}^2},\hat{s})\,u^2},\tag{1.41$^\pm$}$$

$$\hat{s}^{\cdot}=\frac{\sqrt{u^2+\dot{u}^2}}{\hat{\nu}(\pm c\sqrt{u^2+\dot{u}^2}),\hat{s})\,u^2}.\tag{1.42}$$

These equations are valid almost everywhere on I. If f is continuous, they are valid everywhere on I. In the important case that $I=[0,1]$ (by virtue of the positivity of r), these equations are to be supplemented by boundary conditions equivalent to (1.9), namely,

$$u(0)=u_1,\tag{1.43a}$$

$$u(\varphi_1)=u_1,\tag{1.43b}$$

$$\hat{s}(0)=0,\tag{1.43c}$$

$$\hat{s}(\varphi_1)=1.\tag{1.43d}$$

Here we have taken

$$\varphi(0)=0,\qquad \varphi(1)=\varphi_1\tag{1.44}$$

without loss of generality. $\varphi(1)$ is determined from (1.9). The four conditions of (1.43) correspond to the three constants of integration for (1.41) and (1.42) and the parameter c.

If f depends only on r and s, then (1.41) and (1.42) reduce to an autonomous system for u and $\hat{s}$. If, furthermore, the string and force field are uniform so that $\hat{\nu}$ is independent of s and if $f(\mathbf{r},s)=g(u)\,u^2$, then (1.41) and (1.42) uncouple with the former reducing to the autonomous equation

$$\ddot{u}+u=\pm\frac{\sqrt{u^2+\dot{u}^2}\,g(u)}{c\hat{\nu}(\pm c\sqrt{u^2+\dot{u}^2})}.\tag{1.45$^\pm$}$$

u must satisfy (1.43a) and (1.43b) and the auxiliary condition

$$\int_0^{\varphi_1}\frac{\sqrt{u^2+\dot{u}^2}}{\hat{\nu}(\pm c\sqrt{u^2+\dot{u}^2})}\,d\varphi=1\tag{1.46}$$

coming from (1.42), (1.43c), (1.43d), and (1.44). By multiplying (1.45) by $\dot{u}$ and then rearranging and integrating the resulting equation we find that (1.45) possesses the integral

$$J^\pm(u,\dot{u})\equiv H(\pm c\sqrt{u^2+\dot{u}^2})-G(u)=E(\text{const}),\tag{1.47$^\pm$}$$

where

$$H(N) \equiv \int_0^N \hat{\nu}(n)\,dn, \qquad G(u) \equiv \int_1^u g(v)\,dv. \tag{1.48}$$

Equation (1.47) is the equation of the phase-plane trajectories for (1.45). From the phase portraits further detailed information on the nature of solutions of (1.45) can be deduced.

1.49. *Remark.* If f satisfies the inverse square law (1.2) with κ constant, then $g(u) = \kappa$. In this case (1.45) and (1.47) become simpler. If, furthermore, the string is inextensible (so that $\hat{\nu} = 1$ and $H(N) = N$), then the integration of (1.47) is elementary. (This problem was treated by Joh. Bernoulli.)

Thus, the problem of determining the number of equilibrium states for elastic strings under central forces under the best of circumstances leads to the study of boundary value problems for the second-order semilinear equation (1.45) with the auxiliary condition (1.46). The technical difficulties connected with this problem are considerable. Moreover, we must contend with singular solutions for which r can vanish. We study the multiplicity of regular solutions in subsequent sections where we employ several techniques, both classical and modern, to handle this problem. We study the form of singular solutions in the next section.

2. Singular Solutions

We now study the nature of those solutions $\mathbf{r}$ that vanish somewhere and are thereby subjected to forces of possibly infinite intensity. We must not only resort to the underlying physics in order to define such solutions, which we term *singular*, but must also generalize the physical principles themselves (in an admittedly ad hoc way). To see the basic issues, we first study some simple physical situations.

We first ask whether it is possible to suspend the string from a point $\mathbf{r}_0$ (to be determined) so that the end $s = 1$ is free and has prescribed position. Thus, we would consider the problem of finding an $\mathbf{r}_0$ such that

$$\mathbf{r}(1) = \mathbf{r}_1, \tag{2.1a}$$

$$N(1) = 0, \tag{2.1b}$$

where $\mathbf{r}_1$ is given. Let us seek a straight (unfolded) configuration in which $|\mathbf{r}_0| > |\mathbf{r}_1|$. In this case (1.10) would reduce to

$$N(s) = N(0) + \int_0^s f(\mathbf{r}(t), t)\,dt. \tag{2.2}$$

Now suppose that

$$\int_0^s f(\mathbf{r},(t),t)\,dt \to -\infty \qquad \text{as} \quad s \to 1. \tag{2.3}$$

We now show that (2.1b) is inappropriate if $\mathbf{r}_1 = \mathbf{0}$. In this case (2.1b)–(2.3) would imply that $N(0) = \infty$. Then (2.2) would imply that $N(s) = \infty$ and, therefore, that $\nu(s) = \infty$ whenever $r(s) > 0$. This would mean that the deformed length of the string would be ∞, in contradiction to the fact that the deformed length is $|\mathbf{r}_0|$. The message in these remarks is simply this. The presence of infinitely large body forces may cause the distinction between the roles of body forces and contact forces to disappear. In the present example the body force is so dominant that we cannot afford the luxury of prescribing $N(1)$ to be anything other than $-\infty$ when $\mathbf{r}_1 = \mathbf{0}$. Indeed, the body force is so strong that it constrains the end $s = 1$ to remain at $\mathbf{0}$. Consequently, a more fitting boundary condition than (2.1) when $\mathbf{r}_1 = \mathbf{0}$ is just

$$\mathbf{r}(1) = \mathbf{0}. \tag{2.4}$$

The constraint force maintaining (2.4) is infinite and corresponds to the limit that $N \to -\infty$ as $s \to 1$ as given by (2.2) with $N(0)$ finite. These observations seem unremarkable. But their natural corollaries are critical in enabling us to avoid paradoxes in more complicated situations.

To clarify these issues we study the existence of singular radial solutions for a uniform string subject to a uniform attractive force field obeying the inverse square law (1.2) with $\kappa(s) = -\alpha$. We take as our boundary conditions

$$r(0) = r_0 > 0, \qquad r(s_1) = 0 \qquad \text{for some } s_1 \in (0, 1]. \tag{2.5}$$

We seek radial solutions with $r(s) < r_0$ for $s \in (0, s_1]$. (We ignore for the moment the behavior of the material of $(s_1, 1]$.) Then $\nu = -r'$, $\mathbf{e} = -\mathbf{r}/r$ so that (1.16) reduces to

$$\frac{d}{ds}\hat{N}(-r'(s)) + \frac{\alpha}{r(s)^2} = 0 \qquad \text{for} \quad s \in (0, s_1). \tag{2.6}$$

We multiply this equation by $r'(s)$ and integrate the resulting expression to obtain

$$\Psi(\nu) = h - \alpha/r, \tag{2.7}$$

where

$$\Psi(\nu) = \nu\hat{N}(\nu) - \int_1^\nu \hat{N}(\sigma)\,d\sigma = \hat{N}(\nu) + \int_1^\nu [\hat{N}(\nu) - \hat{N}(\sigma)]\,d\sigma \tag{2.8}$$

and

$$h = \Psi(v(0)) + \alpha/r_0. \tag{2.9}$$

The integral (2.7) implies that singular solutions occur only if

$$\Psi(v) \to -\infty \qquad \text{as} \quad v \to 0. \tag{2.10}$$

Now (1.11), (1.12), and (2.8) imply that $\Psi_v > 0$ and $\Psi(v) \to \infty$ as $v \to \infty$. If (2.10) holds, then Ψ has an inverse $v^{\#}: \mathbb{R} \to (0, \infty)$ with $v^{\#}_{\Psi}(\Psi) > 0$. Then (2.7) implies that (2.6) has a solution satisfying (2.5) if and only if there is a number h such that

$$s_1 = \int_0^{r_0} \frac{dr}{v^{\#}(h - \alpha/r)} \equiv \int_{-\infty}^{h-\alpha/r_0} \frac{\alpha\, dt}{v^{\#}(t)(h-t)^2} \equiv S(h; r_0). \tag{2.11}$$

It is clear that the existence of such an h devolves upon the behavior of $v^{\#}$ near $-\infty$ or the behavior of Ψ near 0. It is illuminating to study the example in which

$$\hat{N}(v) = A(v^p - v^{-q}) \qquad \text{with} \quad A > 0, p > 0, q > 0. \tag{2.12}$$

Then (2.10) does not hold for $q < 1$, (2.10) holds but $S(h; r_0)$ does not converge for $q \leqslant 2$ (so that (2.9) is not sufficient for existence), and (2.11) has a solution h if $q > 2$.

Now suppose that for a given r_0 and s_1 a string has a singular solution satisfying (2.5). We now show that this string has such a solution for all r_0's and s_1's. This means that one end of the string can be placed anywhere in space, any point s_1 of the string can be placed at the origin, and the string will have a straight equilibrium state joining these ends. (This means that a given string will have either no such equilibrium states or else all such states.)

2.13. Proposition. *Let* (2.6) *have a singular solution satisfying* (2.5). *Then given any* $r_0^* > 0$ *and any* $s_1^* \in (0, 1]$, *there is a solution of* (2.5), (2.6) *on* $[0, s_1^*)$ *with* $r(0) = r_0^*$ *and with* $r(s) \to 0$ *as* $s \searrow s_1^*$.

Proof. The second integral for $S(h; r_0)$ in (2.11) shows that if $S(h, r_0)$ converges for a given (h^*, r_0^*), then it converges for all (h, r_0). Since our hypotheses imply that there is an (h^*, r_0^*) satisfying (2.11), $S(h, r_0)$ converges for all (h, r_0). This fact and the properties of $v^{\#}$ ensure that $S(\cdot, r_0)$ is a continuous, decreasing map from $\mathbb{R}$ onto $(0, \infty)$, so that (2.11) must have a solution. ∎

This result depends critically upon hypothesis (1.12). Suppose that the string is exceedingly weak in tension with $\hat{N}$ bounded above; e.g., take

$$\hat{N}(v) = A(1 - v^{-q}). \tag{2.14}$$

(In this case $v^{\#}$ can be found explicitly.) Then $S(h, r_0)$ converges only for $q > 2$ (as for (2.12)). But $S(h, r_0)$ is bounded below. Thus, (2.11) has a solution if and only if r_0 is sufficiently small. In particular if $q = 3$, then $S(h, r_0)$ can be found explicitly and (2.11) has a solution if and only if

$$3\pi^2 a r_0 \leqslant 8As_1. \tag{2.15}$$

We now study the equilibrium of an elastic string satisfying (1.9) with $\mathbf{r}_0 \times \mathbf{r}_1 \neq \mathbf{0}$ under a force system that permits singular solutions. Suppose that a material point $\sigma \in (0, 1)$ is brought to the origin. Then it seems likely that there is a V-shaped equilibrium state (see Fig. 2.16) with $N(0)$ and $N(1)$ finite. If we routinely regard all the forces acting on the string as being accounted for by the contact forces $\mathbf{n}(0)$ and $\mathbf{n}(1)$ acting at the ends and by the body force $\int_0^1 [f(\mathbf{r}(s), s)\,\mathbf{r}(s)/r(s)]\,ds$ and if we assume that f satisfies conditions analogous to (2.3), then we would find that the resultant force on the string would not be zero. Thus, Fig. 2.16 could not represent an equilibrium state, in defiance of our intuition. If we insist on having equilibrium states like Fig. 2.16 we must call upon the interpretation made in the first example. We regard the force intensity as so strong that it effectively constrains $\mathbf{r}(\sigma)$ to remain at $\mathbf{0}$. The infinite reactions enforcing this constraint balance the forces already described. An effect of this interpretation is that there is no interaction or coupling between the equilibrium equations for the material segments $[0, \sigma)$ and $(\sigma, 1]$.

We can now contemplate situations in which a closed interval $[\sigma_1, \sigma_2]$ of material with $\sigma_1 < \sigma_2$ is pushed into the origin and kept there by the infinite forces that reduce the stretch v to 0. More generally, we could contemplate the situation in which any closed subset of $[0, 1]$ is constrained to remain at $\mathbf{0}$. By our interpretation of such configurations being maintained by induced constraint forces we need only show that the classical equilibrium equations are satisfied on each complementary open interval I.

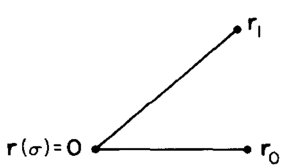

FIGURE 2.16

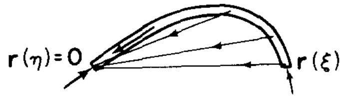

FIG. 2.17. Forces acting on curved segment of string passing through **0**.

We now inquire into the form of solutions on an interval I for which r vanishes at at least one end. So far we have only considered cases in which $\mathbf{r}$ describes a straight line. Must every solution be straight? To answer this question we first attempt a purely formal analysis. In Fig. 2.17 we show a free body diagram of a curved configuration of $I = (\xi, \eta)$ with $\mathbf{r}(\xi) \neq \mathbf{0}$, $\mathbf{r}(\eta) = \mathbf{0}$. The forces acting on this segment are the central body forces, the tangential contact force $-\mathbf{n}(\xi)$ and the possibly infinite reaction at $\mathbf{0}$ tangential to the string. Since all these forces except $-\mathbf{n}(\xi)$ pass through $\mathbf{0}$, the moment of this force system about $\mathbf{0}$ is $-\mathbf{r}(\xi) \times \mathbf{n}(\xi)$. Thus, the string cannot be in equilibrium unless $\mathbf{r}(\xi) \times \mathbf{n}(\xi) = \mathbf{0}$. Since $\mathbf{r}(\xi) \neq \mathbf{0}$, Proposition 1.22 would imply that $\mathbf{r}(I)$ must be radial. Unfortunately this argument, based on geometry and simple physical reasoning, is inadequate. This becomes apparent when we try to give an analytic justification of it. Now the summation of moments in the subinterval (ξ, s) of I is readily shown to yield

$$\mathbf{r}(\xi) \times \mathbf{n}(\xi) = \mathbf{r}(s) \times \mathbf{n}(s) \qquad \text{for} \quad s \in I. \tag{2.18}$$

This is equivalent to (1.16a). We recover the requirement that the configuration of $\mathbf{r}$ be straight from (2.18) whenever

$$\lim_{s \to \eta} \mathbf{r}(s) \times \mathbf{n}(s) = \lim_{s \to \eta} r(s)\, N(s) \left[\frac{\mathbf{r}(s)}{r(s)} \times \mathbf{e}(s) \right] = \mathbf{0}. \tag{2.19}$$

In the examples we have just considered, $r(s) \to 0$ but $N(s) \to -\infty$ as $s \to \eta$. For the segment shown in Fig. 2.16, $[\mathbf{r}(s)/r(s)] \times \mathbf{e}(s) \to \mathbf{0}$ as $s \to \eta$, but there is no a priori reason to exclude other possibilities such as that allowing the curve $\mathbf{r}$ to spiral into $\mathbf{0}$. Thus, we cannot conclude that (2.19) holds without further argument based upon the full boundary value problem. Indeed, the work of Devaney [6, 7] indicates that there can be nonradial singular solutions.

If, however, we know that N is bounded as $r(s) \to 0$ or, more generally, that rN is bounded as $r(s) \to 0$, then (2.19) holds and we can conclude that $\mathbf{r}(I)$ is radial. The properties of f themselves may enable us to show that rN is bounded as $r(s) \to 0$; e.g., (2.2) implies

2.20. PROPOSITION. *If* $\mathbf{r} \mapsto f(\mathbf{r}, s)$ *is bounded for* $\mathbf{r}$ *near* $\mathbf{0}$ *and if there is*

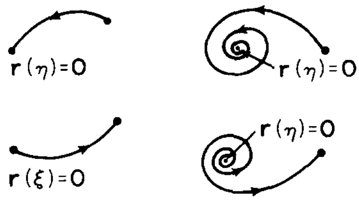

FIG. 2.21. Possible configurations of subintervals of I that have $\mathbf{0}$ as a limit point. The arrows indicate the direction of increasing s. Note that both φ' and θ' are positive.

a $\sigma \in [0, 1]$ such that $r(\sigma) = 0$, then N is bounded and $\mathbf{r}(I)$ is radial for every interval I on which $r > 0$.

Note that when the hypotheses of Proposition 2.20 hold, the ray on which $\mathbf{r}(I)$ lies may differ with I. Since $\{s \in [0, 1]: r(s) > 0\} = \bigcup I_m$, where $\{I_m\}$ is a countable collection of disjoint open intervals, $\mathbf{r}$ could represent a curve containing a countable number of radial pieces $\mathbf{r}(I_m)$, each of which may point in a different direction in $\mathbb{E}^3$. There are at most two intervals of $\{I_m\}$ that contain at least one of the points 0 and 1. On the other intervals I, $\mathbf{r}(I)$ is doubly-covered, i.e., $\mathbf{r}(I)$ has exactly one fold (provided the hypotheses of Proposition 2.20 hold). We discuss folded solutions in the next section.

We now develop some machinery to get further precise information. First we note that if the restriction of r to $I = (\xi, \eta)$ is positive but either $r(\xi)$ or $r(\eta)$ vanishes, then Proposition 1.36 and the positivity of φ' require that the configuration of I be bowed out, i.e., that θ' be positive on I. We sketch possible configurations of subintervals of I that have $\mathbf{0}$ as a limit point in Fig. 2.21. In the two cases on the left of Fig. 2.21 the positivity of θ' follows from geometric considerations. In the two cases on the right a free body diagram of a half-loop of the spiral shows that we must have $Nf > 0$ and, thus, $\theta' > 0$ by (1.35). If f is attractive and if $|\mathbf{r}(\eta) - \mathbf{r}(\xi)| \geqslant \eta - \xi$, then $\mathbf{r}(I)$ must be straight because otherwise N would be compressive and this would be incompatible with (1.11) and the hypothesis that the deformed length of I is not less than I.

Henceforth we restrict our attention to uniform strings acted on by uniform central force fields in which case (1.47) holds. Combining (1.39) and (1.47) yields

$$N(\hat{s}(\cdot)) = H^{-1}(E + G(u)). \tag{2.22}$$

This implies that $rN \to 0$ as $r(s) \to 0$ if and only if

$$\lim_{u \to \infty} \frac{H^{-1}(E + G(u))}{u} = \lim_{u \to \infty} \frac{g(u)}{\hat{\nu}(H^{-1}(E + G(u)))} = 0. \tag{2.23}$$

The first equality in (2.23) follows from l'Hôpital's rule and (1.48). Thus, we have

2.24. Proposition. *If* (2.23) *holds and if there is a* $\sigma \in [0, 1]$ *such that* $r(\sigma) = 0$, *then* $\mathbf{r}(I)$ *is radial for every interval* I *on which* $r > 0$.

We now seek further criteria ensuring that solutions are radial that can handle the case $r(s)\,N(s) \to 0$ as $r(s) \to 0$. We assume that (1.47) holds. If $c = 0$, the argument leading to Proposition 1.22 implies that the configuration of an open interval I on which $r > 0$ and $v > 0$ is radial. To study the possibility of the existence of curved states we accordingly take $c > 0$. Now the level curves of the graph of $J^{\pm}$, which is defined in (1.47), are the phase-plane trajectories for $(1.45^{\pm})$. If a nonradial configuration of I were to approach $\mathbf{0}$, then there would have to be a phase-plane trajectory in $(0, \infty) \times R$ on which $u \to \infty$. Since $J^{\pm}$ is continuous, we can demonstrate the nonexistence of such a trajectory by showing that $|J^{\pm}(u, \dot{u})| \to \infty$ as $u \to \infty$ for all $\dot{u} \in R$, for then there can be no level curve of the graph of $J^{\pm}$ on which $u \to \infty$. Since Nf must be positive for singular solutions, since H has the same sign as N, and since G has the same sign as $(u - 1)\,g(u)$, we readily obtain

2.25. Proposition. *If for each* $c > 0$ *and for each* $\dot{u} \in R$

$$|J^{\pm}(u, \dot{u})| \to \infty \qquad \text{as} \quad u \to \infty \tag{2.26}$$

and if there is a $\sigma \in [0, 1]$ *such that* $r(\sigma) = 0$, *then* $\mathbf{r}(I)$ *is radial for every interval* I *on which* $r > 0$. *The following results give readily verifiable sufficient conditions for* (2.26):

(i) $J^{+}(u, \dot{u}) \to \infty$ *as* $u \to \infty$ *if* $\lim_{u \to \infty} G(u) < \infty$ *(because* $H(c\sqrt{u^2 + \dot{u}^2}) > \infty$ *as* $u \to \infty$).

(ii) *If* $\hat{v}$ *is integrable on* $(-\infty, 0)$ *and if* $\lim_{u \to \infty} G(u) = -\infty$, *then* $J^{-}(u, \dot{u}) \to \infty$ *as* $u \to \infty$ *(because* $\lim_{u \to \infty} H(-c\sqrt{u^2 + \dot{u}^2}) > -\infty$ *in view of* (1.48)).

(iii) *If* $\hat{v}$ *is not integrable on* $(-\infty, 0)$ *and if* $-\infty \leqslant \lim_{u \to \infty} G(u) < \infty$, *then* $J^{-}(u, \dot{u}) \to -\infty$.

(iv) *If* $|H(\pm c\sqrt{u^2 + \dot{u}^2})| \to \infty$, *if* $|G(u)| \to \infty$ *as* $u \to \infty$, *and if*

$$\lim_{u \to \infty} \frac{G(u)}{H(\pm c\sqrt{u^2 + \dot{u}^2})} = \lim_{u \to \infty} \frac{g(u)}{\pm c\hat{v}(\pm c\sqrt{u^2 + \dot{u}^2})} = 0 \qquad \text{or} \qquad \infty, \tag{2.27}$$

then $|J^{\pm}(u, \dot{u})| \to \infty$ *as* $u \to \infty$. *(The first equality in* (2.27) *follows from l'Hôpital's rule.)*

2.28. COROLLARY. *If f satisfies the inverse square law $f(\mathbf{r}, s) = \kappa r^{-2}$ where $\kappa \in \mathbb{R}\backslash\{0\}$ and if there is a $\sigma \in [0, 1]$ such that $r(\sigma) = 0$, then $\mathbf{r}(I)$ is radial for every interval I on which $r > 0$.*

Proof. The case for J^+ is covered by (iv). To treat J^- we first observe that $\kappa > 0$ since $Nf > 0$. Then this case is covered by (i) or (ii) according to whether or not $\hat{\nu}$ is integrable. ∎

3. STRAIGHT AND FOLDED RADIAL CONFIGURATIONS

In this section we study the existence of straight and folded nonsingular radial configurations. Our results play a central role in the general treatment of existence in the next section. For simplicity of exposition we assume that the force is an attractive inverse square law force given by (1.2) with κ negative and continuous. For radial solutions, (1.17) can then be reduced to the system

$$|\mathbf{r}'| = \hat{\nu}(N, s), \qquad N' = -\kappa(s)/r^2. \tag{3.1}$$

We require that r satisfy the boundary conditions

$$r(0) = r_0, \qquad r(1) = r_1 \qquad \text{with} \quad 0 < r_0 < r_1. \tag{3.2}$$

We first seek *straight configurations* corresponding to solutions with $r' > 0$.

3.3. PROPOSITION. *For κ negative and continuous problem* (3.1), (3.2) *has a solution $r \in C^2([0, 1])$ with $r' > 0$.*

Proof. The initial value problem for (3.1) subject to $r(0) = r_0$ and $N(0) = A$ has a unique solution in a neighborhood of $s = 0$ that satisfies

$$r(s) = r_0 + \int_0^s \hat{\nu}(N(t), t)\, dt, \tag{3.4a}$$

$$N(s) = A - \int_0^s \kappa(t)\, r(t)^{-2}\, dt. \tag{3.4b}$$

From (3.4) we obtain

$$r_0 \leqslant r(s) \leqslant \int_0^1 \hat{\nu}(A + \max|\kappa|\, r_0^{-2}, t)\, dt, \qquad A \leqslant N(s) \leqslant A + \max|\kappa|\, r_0^{-2}. \tag{3.5}$$

Thus, the continuation theory of ordinary differential equations implies that the solution of the initial value problem is defined for all s in $[0, 1]$. This solution is a twice continuously differentiable solution of the boundary value problem (3.1), (3.2) if A can be chosen so that $r(1)=r_1$. Equations (3.4) imply that this can be done if A can be chosen such that

$$r_1 - r_0 = \int_0^1 \hat{v}\left(A - \int_0^s \kappa(t)\, r(t)^{-2}\, dt, s\right) ds \equiv R(A). \tag{3.6}$$

Now the properties of $\hat{v}$ imply that

$$\int_0^1 \hat{v}(A, s)\, ds \leqslant R(A) \leqslant \int_0^1 \hat{v}(A + \max|\kappa|\, r_0^{-2}, s)\, ds \tag{3.7}$$

so that R is a continuous function from $\mathbb{R}$ onto $(0, \infty)$. (Note that r itself depends continuously on A.) Thus, (3.6) has a solution. (This same conclusion could have been obtained by means of the Schauder Fixed Point Theorem.) ∎

By using (3.4) and the properties of $\hat{v}$ we could show that if (3.1) and (3.2) with κ negative and continuous has two distinct solutions r and $\tilde{r}$ with $\tilde{r}' > 0$, $r' > 0$ and with $\tilde{r}(0) > r(0)$, say, then $\tilde{r}(s) > r(s)$ for all s in $(0, 1)$. We can obtain a uniqueness theorem for solutions of Proposition 3.3 if we assume that the string is uniform so that

$$\kappa(s) = -\alpha, \qquad \alpha > 0 \qquad \text{and} \qquad \hat{v}(N, s) = \hat{v}(N). \tag{3.8}$$

We adhere to (3.8) in the rest of this section since this policy renders much of our ensuing development much more transparent, yields stronger results, and enables us to avoid a variety of technical difficulties.

3.9. Proposition. *Let* (3.8) *hold. Then* (3.1), (3.2) *has a unique solution* $r \in C^2([0, 1])$ *with* $r' > 0$.

Proof. Suppose that there were two solutions r and $\tilde{r}$ with $r'(0) < \tilde{r}'(0)$. Then (3.2) and the uniqueness theorem for initial value problems imply that there must be an $s^* \in (0, 1]$ such that $r(s^*) = \tilde{r}(s^*)$, $r'(s^*) > \tilde{r}'(s^*)$. Hence (2.7) implies that

$$\Psi(r'(0)) - \Psi(\tilde{r}'(0)) = \Psi(\tilde{r}'(s^*)) - \Psi(\tilde{r}'(s^*)). \tag{3.10}$$

The positivity of Ψ_v then ensures that the left side of (3.10) is negative while the right side is positive, which is absurd. ∎

We now study radial configurations that are folded at $s = \tau$, where τ is to be determined. We again take (3.2) as our boundary condition with $0 < r_0 \leqslant r_1$. Now the integral form (1.4) of the equilibrium equation implies that

$$\mathbf{n}(\tau-) = \mathbf{n}(\tau+) = \mathbf{n}(\tau). \tag{3.11}$$

But

$$\begin{aligned} \mathbf{n}(s) &= N(s)\,\mathbf{r}(s)/r(s) \qquad \text{for} \quad s \in [0, \tau) \\ &= -N(s)\,\mathbf{r}(s)/r(s) \qquad \text{for} \quad s \in (\tau, 1], \end{aligned} \tag{3.12}$$

so (3.11) can hold if and only if

$$N(\tau-) = -N(\tau+) = 0$$

or, equivalently,

$$\nu(\tau-) = \nu(\tau+) = 1. \tag{3.13}$$

Note that (3.13) implies that no configuration can have more than one fold because a segment of the string lying between two folds would be subject to zero contact force and to a nonzero resultant body force and so could not stay in equilibrium (i.e., (1.4) would not be satisfied).

We first study the problem in which $r(\tau) > r_0, r_1$ so that

$$\begin{aligned} \nu(s) &= r'(s) \qquad \text{for} \quad s \in [0, \tau) \\ &= -r'(s) \qquad \text{for} \quad s \in (\tau, 1]. \end{aligned} \tag{3.14}$$

(In this case (3.13) reduces to

$$r'(\tau-) = 1 = -r'(\tau+).) \tag{3.15}$$

We are, thus, led to study the two boundary value problems

$$[\hat{N}(r')]' = \alpha/r^2, \qquad r(0) = r_0, \qquad \hat{N}(r'(\tau)) = 0, \tag{3.16a}$$

$$-[\hat{N}(-r')]' = \alpha/r^2, \qquad r(1) = r_1, \qquad \hat{N}(-r'(\tau)) = 0. \tag{3.16b}$$

If (3.16a) has a solution $s \mapsto R_0(s, \tau)$ and if (3.16b) has a solution $s \mapsto R_1(s, \tau)$, then these two solutions describe a folded configuration of the string provided τ in $(0, 1)$ can be found so that

$$R_0(\tau, \tau) = R_1(\tau, \tau). \tag{3.17}$$

We now develop the existence and uniqueness theory for (3.16) and (3.17).

3.18. Lemma. *Problems* (3.16a) *and* (3.16b) *respectively have unique solutions* $R_0(\cdot, \tau)$ *and* $R_1(\cdot, \tau)$ *for* $r_0 > 0$, $r_1 > 0$.

Proof. We follow the derivation of (3.4) and (3.5) to show that (3.16a) and (3.16b) have solutions R_0 and R_1 if and only if

$$R_0(s, \tau) = r_0 + \int_0^s \hat{v}\left(-\int_t^\tau \alpha R_0(\sigma, \tau)^{-2}\, d\sigma\right) dt$$
$$\equiv (T_0 R_0)(s) \qquad \text{for} \quad s \in [0, \tau], \tag{3.19a}$$
$$R_1(s, \tau) = r_1 + \int_s^1 \hat{v}\left(-\int_\tau^t \alpha R_1(\sigma, \tau)^{-2}\, d\sigma\right) dt$$
$$\equiv (T_1 R_1)(s) \qquad \text{for} \quad s \in [\tau, 1]. \tag{3.19b}$$

The existence of solutions of (3.19) is ensured by the Schauder Fixed Point Theorem, which implies that T_1 and T_2 have fixed points in $K_0 \equiv \{r \in C^0([0, \tau]) : r_0 \leqslant r \leqslant r_0 + \tau\}$ and $K_1 \equiv \{r \in C^0([\tau, 1]) : r_1 \leqslant r \leqslant r_1 + 1 - \tau\}$, respectively. To prove the uniqueness of R_0 we suppose that (3.16a) or (3.19a) has two solutions R and $\tilde{R}$ with $\tilde{R}'(0) < R'(0)$. Then (2.7) implies that

$$\frac{\alpha}{R(\tau)} = \frac{\alpha}{r_0} + \Psi(R'(0)), \qquad \frac{\alpha}{\tilde{R}(\tau)} = \frac{\alpha}{r_0} + \Psi(\tilde{R}'(0)). \tag{3.20}$$

Since Ψ is strictly increasing, (3.20) implies that $\tilde{R}(\tau) > R(\tau)$. Thus, there must be an s^* in $(0, \tau)$ such that $R(s^*) = \tilde{R}(s^*)$ and $R'(s^*) < \tilde{R}'(s^*)$. Repeating this argument on $[s^*, \tau]$ we obtain a contradiction. The proof of the uniqueness of R_1 is analogous. ∎

We note that $R_0' > 0$ and $R_1' < 0$. (Here and below, the prime denotes the derivative with respect to the first argument of R_0 and R_1.) To obtain a folded configuration, we must show that (3.17) is satisfied.

3.21. Lemma. *$\tau \mapsto R_0(\tau, \tau)$ is strictly increasing and $\tau \mapsto R_1(\tau, \tau)$ is strictly decreasing.*

Proof. Integral (2.7) implies that

$$\frac{\alpha}{R_0(\tau, \tau)} = \frac{\alpha}{r_0} + \Psi(R_0'(0, \tau)). \tag{3.22}$$

Let $\tau_1 < \tau_2$. If $R_0'(0, \tau_1) > R_0'(0, \tau_2)$, then (3.22) implies that $R_0(\tau_1, \tau_1) < R_0(\tau_2, \tau_2)$. If $R_0'(0, \tau_1) = R_0'(0, \tau_2)$, then the uniqueness theory for initial value problems implies that $R_0(\cdot, \tau_1) = R_0(\cdot, \tau_2)$. Since $R_0(\cdot, \tau)$ is increasing, we, therefore, have that $R_0(\tau_1, \tau_1) < R_0(\tau_2, \tau_1) = R_0(\tau_2, \tau_2)$. If $R_0'(0, \tau_1) < R_0'(0, \tau_2)$, then $R_0(s, \tau_1) < R_0(s, \tau_2)$ for s in some neighborhood of 0. If this last inequality holds for s in $[0, \tau_1]$, then the monotonicity of $R_0(\cdot, \tau_2)$

implies that $R_0(\tau_1, \tau_1) < R_0(\tau_2, \tau_2)$. The same conclusion follows if $R_0(\tau_1, \tau_1) \leqslant R_0(\tau_1, \tau_2)$. If, however, there is a first s^* in $[0, \tau_1)$ such that $R_0(s^*, \tau_1) = R_0(s^*, \tau_2)$, then either $R_0'(s^*, \tau_1) = R_0'(s^*, \tau_2)$, again implying that $R_0(\cdot, \tau_1) = R_0(\cdot, \tau_2)$ and thence the conclusion, or else $R_0'(s^*, \tau_1) > R_0'(s^*, \tau_2)$. In this case (2.7) implies that

$$\frac{\alpha}{R_0(\tau_1, \tau_1)} - \Psi(R_0'(s^*, \tau_1)) = \frac{\alpha}{R_0(\tau_2, \tau_2)} - \Psi(R_0'(s^*, \tau_2)), \tag{3.23}$$

whence it follows that $R_0(\tau_1, \tau_1) < R_0(\tau_2, \tau_2)$. The proof for R_1 is analogous. ∎

In view of Lemma 3.21 we see that (3.17) holds for $\tau \in (0, 1)$ if and only if the curves $\tau \mapsto R_0(\tau, \tau)$ and $\tau \mapsto R_1(\tau, \tau)$ cross in (0, 1). This occurs if and only if

$$r_0 \equiv R_0(0, 0) > R_1(0, 0) \tag{3.24a}$$

and

$$R_0(1, 1) > R_1(1, 1) \equiv r_1. \tag{3.24b}$$

Since $R_1(0, 0) > r_1 \geqslant r_0$ by (3.19b), inequality (3.24a) is automatically satisfied. Now (3.19a) implies that

$$\begin{aligned} r_0 + \hat{v}(-\alpha r_0^{-2}) &< r_0 + \int_0^1 \hat{v}(-\alpha r_0^{-2}(1-t))\,dt < R_0(1, 1) \\ &< r_0 + \int_0^1 \hat{v}(-\alpha R_0(1, 1)^{-2}(1-t))\,dt < r_0 + 1. \end{aligned} \tag{3.25}$$

Now $\rho \mapsto \rho - \int_0^1 \hat{v}(-\alpha \rho^{-2}(1-t))\,dt$ is a strictly increasing function from $[0, \infty)$ onto $[0, \infty)$. Let g be its inverse. Then the third inequality of (3.25) is equivalent to

$$R_0(1, 1) < g(r_0). \tag{3.26}$$

These results together with the consequences of Lemmas 3.18 and 3.21 yield

3.27. THEOREM. *Let* (3.8) *hold and let* $0 < r_0 \leqslant r_1$. *Then there is at most one folded configuration with the fold more remote from the origin than the supports. There is such a configuration, which is compressive, if and only if* $R_0(1, 1) > r_1$. *This condition is satisfied if* $r_0 + \int_0^1 \hat{v}(-\alpha r_0^{-2}(1-t))\,dt \geqslant r_1$ *and a fortiori if* $r_0 + \hat{v}(-\alpha r_0^{-2}) \geqslant r_1$. *In particular, there is always such a folded configuration if* $r_0 = r_1$ *(in which case the fold is at* $s = 1/2$*). There is no such configuration if* $g(r_0) \leqslant r_1$ *and a fortiori if* $r_0 + 1 \leqslant r_1$.

We now study the existence of regular folded configurations satisfying (3.2) with the fold "below" the supports. By following the development centered on (3.12)–(3.19) we show that there is such a solution folded at $s = \tau$ if and only if there are functions P_0 and P_1 satisfying

$$P_0(s, \tau) = r_0 - \int_0^s \hat{v}\left(\int_t^\tau \alpha P_0(\sigma, \tau)^{-2}\, d\sigma\right) dt$$
$$\equiv (T_0 P_0)(s) \qquad \text{for} \quad s \in [0, \tau], \tag{3.28a}$$
$$P_1(s, \tau) = r_1 - \int_s^1 \hat{v}\left(\int_\tau^t \alpha P_1(\sigma, \tau)^{-2}\, d\sigma\right) dt$$
$$\equiv (T_1 P_1)(s) \qquad \text{for} \quad s \in [\tau, 1], \tag{3.28b}$$

and (3.17). Let m be the minimum of $(0, \infty) \ni b \mapsto b + \int_0^1 \hat{v}(t\alpha b^{-2})\, dt$.

3.29. Lemma. *Let $m \leqslant r_0 \leqslant r_1$. Then for each $\tau \in [0, 1]$, Eqs.* (3.28a) *and* (3.28b) *have continuous solutions.*

Proof. The Schauder Fixed Point Theorem implies that T_0 and T_1 have fixed points on $\{r \in C^0([0, \tau]) \colon m \leqslant r \leqslant r_0\}$ and $\{r \in C^0([\tau, 1]) \colon m \leqslant r \leqslant r_1\}$ respectively. ∎

Note that these solutions are tensile because the arguments of $\hat{v}$ in (3.28) are positive except at τ. Thus, there cannot be a regular folded configuration with the fold below the supports if $r_0 + r_1 \leqslant 1$. From (3.28) we find that $\tau \mapsto P_0(\tau, \tau)$ is decreasing and $\tau \mapsto P_1(\tau, \tau)$ is increasing on $[0, 1]$. Following the arguments for strings folded above the supports we find that (3.17) is satisfied for τ in (0, 1) if and only if

$$P_1(0, 0) < r_0. \tag{3.30}$$

We, thus, obtain

3.31. Theorem. *Let* (3.8) *hold and let $m \leqslant r_0 \leqslant r_1$. Then there is a folded configuration, which is tensile, with the fold closer to the origin than the supports if and only if* (3.30) *holds. This condition is satisfied if $r_1 - \int_0^1 \hat{v}(\alpha t r_1^{-2})\, dt \leqslant r_0$ and a fortiori if $r_1 - r_0 \leqslant 1$.*

Note that we do not obtain uniqueness and nonexistence results because there is no evidence suggesting that there should be any. The function playing the role of g^{-1} (cf. (3.26)) is just the nonmonotone function $b \mapsto b + \int_0^1 \hat{v}(t\alpha b^{-2})\, dt$. (It is easy to choose reasonable $\hat{v}$'s for which this integral can be evaluated.)

4. Existence and Connectivity of Nonradial Compressive States

In this section we study the existence and connectivity of nonradial compressive configurations under an attractive central force field. For simplicity we shall ultimately restrict our attention to the behavior of uniform strings under a field satisfying the inverse square law. We show how these solutions evolve from the radial solutions treated in the preceding section as the supports are moved apart.

We begin our analysis by studying compressive solutions of (1.4), (1.6)–(1.8), and (1.13) on [0, 1] subject to the initial conditions

$$\mathbf{r}(0)=\mathbf{r}_0, \qquad \mathbf{n}(0)=\mathbf{n}_0, \tag{4.1}$$

where $\mathbf{r}_0 \neq \mathbf{0}$ and $\mathbf{n}_0$ are prescribed. (We do not yet take the string to be uniform and the force to satisfy the inverse square law.)

Now (1.4), (1.6), and (4.1) imply that

$$\mathbf{n}(s)=N(s)\,\mathbf{e}(s)=\mathbf{n}_0-\int_0^s \mathbf{f}(\mathbf{r}(t),t)\,dt \equiv \mathbf{n}^*(\mathbf{r},\mathbf{n}_0)(s). \tag{4.2}$$

We seek compressive solutions for which $N \leqslant 0$, whence

$$N(s)=-|\mathbf{n}(s)|, \qquad \mathbf{e}(s)=-\mathbf{n}(s)/|\mathbf{n}(s)|. \tag{4.3}$$

In this case (1.7), (1.14), and (4.1) imply that

$$\mathbf{r}(s)=\mathbf{r}_0-\int_0^s \hat{\nu}(-|\mathbf{n}(t)|,t)[\mathbf{n}(t)/|\mathbf{n}(t)|]\,dt. \tag{4.4}$$

We study the existence of solutions $\mathbf{n}$ and $\mathbf{r}$ of (4.2) and (4.4) on [0, 1] that depend continuously on $\mathbf{n}_0$ and $\mathbf{r}_0$. That $\mathbf{n}/|\mathbf{n}|$ need not be continuous means that (4.4) does not satisfy the Carathéodory conditions (described in standard texts on ordinary differential equations) so that its treatment is not routine.

4.5. Proposition. *Let* $\mathbf{f}\colon (\mathbb{E}^3\backslash\{0\})\times[0,1]\to\mathbb{E}^3\backslash\{0\}$ *be continuous and let* $\mathbf{f}(\cdot,t)$ *be Lipschitz continuous on compact subsets of* $\mathbb{E}^3\backslash\{0\}$ *uniformly for* $t\in[0,1]$. ($\mathbf{f}$ *need not be a central force.*) *Then for any* $\mathbf{n}_0\in\mathbb{E}^3$ *and for any* $\mathbf{r}_0\in\mathbb{E}^3$ *for which* $r_0\equiv|\mathbf{r}_0|>1$, *there exists a unique, continuous solution* $\mathbf{n}, \mathbf{r}$ *of* (4.2) *and* (4.4) *on* [0, 1] *that depends continuously on* $\mathbf{n}_0$ *and* $\mathbf{r}_0$.

Proof. Let

$$\mathbf{e}^*(\mathbf{r}, \mathbf{n}_0) = -\mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)/|\mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)|. \tag{4.6}$$

Then (4.2) and (4.4) are equivalent to

$$\mathbf{r}(s) = \mathbf{r}_0 + \int_0^s \hat{v}(-|\mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)(t)|, t)\, \mathbf{e}^*(\mathbf{r}, \mathbf{n}_0)(t)\, dt \equiv \mathbf{p}(\mathbf{r}_1, \mathbf{r}_0, \mathbf{n}_0)(s). \tag{4.7}$$

Now for $r > 0$, $\mathbf{f}(\mathbf{r}(\cdot), \cdot)$ never vanishes. Then the zeros of $\mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)$ must be isolated (cf. Section 1). Thus, $\mathbf{e}^*$ is defined for continuous nonvanishing $\mathbf{r}$ everywhere except possibly at a finite set of s's. Since $r_0 > 1$, (4.7) implies that $|\mathbf{p}(\mathbf{r}, \mathbf{r}_0, \mathbf{n}_0)(s) - \mathbf{r}_0| \leqslant 1$ for such $\mathbf{r}$. Thus, $\mathbf{p}(\cdot, \mathbf{r}_0, \mathbf{n}_0)$ is readily shown to be a compact and continuous mapping of the closed convex set $\{\mathbf{r} \in C^0([0, 1]) : |\mathbf{r}(s) - \mathbf{r}_0| \leqslant 1,\ s \in [0, 1]\}$ of the Banach space $C^0([0, 1])$ into itself. The Schauder Fixed Point Theorem, thus, implies that $\mathbf{p}(\cdot, \mathbf{r}_0, \mathbf{n}_0)$ has a fixed point, which is a solution of (4.7). This solution generates a solution of (4.2) and (4.4), which we denote by $\mathbf{r}(\cdot, \mathbf{r}_0, \mathbf{n}_0)$, $\mathbf{n}(\cdot, \mathbf{r}_0, \mathbf{n}_0)$. We now show that this solution is unique. (Uniqueness would follow from the continuous dependence of solutions on $\mathbf{r}_0$ and $\mathbf{n}_0$. The possible lack of continuity of $\mathbf{e}$ makes it convenient to prove uniqueness first.) Suppose that (4.2) and (4.4) have two continuous solutions $(\mathbf{r}^1, \mathbf{n}^1)$ and $(\mathbf{r}^2, \mathbf{n}^2)$ with $|\mathbf{r}^1 - \mathbf{r}_0| \leqslant 1$, $|\mathbf{r}^2 - \mathbf{r}_0| \leqslant 1$. Set $\delta\mathbf{r} = \mathbf{r}^1 - \mathbf{r}^2$, $\delta\mathbf{n} = \mathbf{n}^1 - \mathbf{n}^2$. Then the Lipschitz continuity of $\mathbf{f}(\cdot, s)$ and $\hat{v}(\cdot, s)$ implies that there is a positive constant C (depending on the two solutions) such that

$$|\delta\mathbf{r}(s)| \leqslant C \int_0^s |\delta\mathbf{n}(t)|\, dt + C \int_0^s \left| \frac{\mathbf{n}^1(t)}{|\mathbf{n}^1(t)|} - \frac{\mathbf{n}^2(t)}{|\mathbf{n}^2(t)|} \right| dt, \tag{4.8}$$

$$|\delta\mathbf{n}(s)| \leqslant C \int_0^s |\delta\mathbf{r}(t)|\, dt. \tag{4.9}$$

Let us first suppose that $\mathbf{n}_0 = 0$. Then

$$\mathbf{e}^*(\mathbf{r}^i, \mathbf{0})(s) \equiv \frac{\int_0^s \mathbf{f}(\mathbf{r}^i(t), t)\, dt}{|\int_0^s \mathbf{f}(\mathbf{r}^i(t), t)\, dt|} \equiv \frac{\int_0^1 \mathbf{f}(\mathbf{r}^i(s\tau), s\tau)\, d\tau}{|\int_0^1 \mathbf{f}(\mathbf{r}^i(s\tau), s\tau)\, d\tau|} \to \frac{\mathbf{f}(\mathbf{r}_0, 0)}{|\mathbf{f}(\mathbf{r}_0, 0)|}$$
$$\text{as } s \searrow 0 \text{ for } i = 1, 2. \tag{4.10}$$

Let s_1 be greater than 0 and smaller than the first positive zero ζ_1 of $\mathbf{n}^1$. Then there is a number C_1 such that $|\int_0^1 \mathbf{f}(\mathbf{r}^1(s\tau), s\tau)\, d\tau|^{-1} \leqslant C_1$ for $s \in [0, s_1]$. Let us set

$$\Delta_r(s) \equiv \max_{t \in [0,s]} |\delta\mathbf{r}(t)|, \qquad \Delta_n(s) \equiv \max_{t \in [0,s]} |\delta\mathbf{n}(t)|. \tag{4.11}$$

Then the Lipschitz continuity of $\mathbf{f}$ implies that there is a constant C_2 such that

$$\left|\frac{\mathbf{n}^1(s)}{|\mathbf{n}^1(s)|}-\frac{\mathbf{n}^2(s)}{|\mathbf{n}^2(s)|}\right| \leqslant C_2 \frac{\int_0^1 |\mathbf{f}(\mathbf{r}^1(s\tau), s\tau)-\mathbf{f}(\mathbf{r}^2(s\tau), s\tau)|\, d\tau}{|\int_0^1 \mathbf{f}(\mathbf{r}^1(s\tau), s\tau)\, d\tau|}$$
$$\leqslant C_2 \Delta_r(s) \tag{4.12}$$

for $s \in [0, s_1]$. Thus (4.8), (4.9), (4.11), and (4.12) imply that there is a $C > 0$ such that

$$\Delta_r(s) \leqslant C \int_0^s \Delta_n(t)\, dt + C \int_0^s \Delta_r(t)\, dt, \quad \Delta_n(s) \leqslant C \int_0^s \Delta_r(t)\, dt \qquad \text{for} \quad s \in [0, s_1]. \tag{4.13}$$

A standard application of the Gronwall inequality to a certain scalar equation obtained from (4.13) shows that $\Delta_r(s) = 0$, $\Delta_n(s) = 0$ for $s \in [0, s_1]$ and, thus, on $[0, \zeta_1]$. Since $\mathbf{n}^1$ has only a finite number of zeros, the same argument can be repeated to show that $\Delta_r = 0$, $\Delta_n = 0$ on $[0, 1]$ and, thus, that the solution is unique.

If $\mathbf{n}_0 \neq \mathbf{0}$, then there is an interval about $s = 0$ on which $\mathbf{n}^1$ does not vanish. A simpler version of the preceding argument yields uniqueness on this interval. If this interval is not $[0, 1]$, then the preceding argument gives the uniqueness on the remainder of $[0, 1]$. Thus, in every case the solutions are unique.

To prove the continuous dependence of solutions on $\mathbf{r}_0$ and $\mathbf{n}_0$ we let $\{(r_j, n_j)\}$ be a sequence of vectors converging to $(\mathbf{r}_0, \mathbf{n}_0)$. Then (4.7) implies that $\{\mathbf{r}(\cdot, \mathbf{r}_j, \mathbf{n}_j)\}$ is uniformly bounded and equicontinuous and, therefore, by the Arzelà–Ascoli theorem, possesses a subsequence converging uniformly to a limit, which we readily show to be the unique solution $\mathbf{r}(\cdot, \mathbf{r}_0, \mathbf{n}_0)$ of (4.7). The uniqueness of this solution shows that the whole sequence $\{\mathbf{r}(\cdot, \mathbf{r}_j, \mathbf{n}_j)\}$ converges to $\mathbf{r}(\cdot, \mathbf{r}_0, \mathbf{n}_0)$. ∎

A useful variant of this proposition holds when $\mathbf{f}$ is an attractive central force, which for simplicity we take to be uniform.

4.14. PROPOSITION. *Let* $(0, \infty) \ni r \mapsto f(r) \in (-\infty, 0)$ *be continuously differentiable and let* $\mathbf{f}$ *of* (4.2) *have the form* $\mathbf{f}(\mathbf{r}, s) = f(\mathbf{r})\, \mathbf{r} r^{-1}$. *Let the string be uniform and let* $|J^-(u, \dot{u})| \to \infty$ *as* $u \to \infty$. (J^- *is defined in* (1.47).) *Then for every* $\mathbf{r}_0, \mathbf{n}_0$ *for which* $\mathbf{r}_0 \times \mathbf{n}_0 \neq \mathbf{0}$, *there exists a unique continuously differentiable solution* $(\mathbf{n}, \mathbf{r})$ *of* (4.2) *and* (4.4) *on* $[0, 1]$, *which is a continuously differentiable function of* $\mathbf{r}_0$ *and* $\mathbf{n}_0$ *for* $\mathbf{r}_0 \times \mathbf{n}_0 \neq \mathbf{0}$. *For every* $\mathbf{r}_0, \mathbf{n}_0$ *with* $\mathbf{r}_0 \neq \mathbf{0}$, *there exists a unique continuous solution* $(\mathbf{n}, \mathbf{r})$ *of* (4.2) *and* (4.4) *on* $(0, \sup\{s\colon r(s) > 0\})$, *which depends continuously on* $\mathbf{r}_0$ *and* $\mathbf{n}_0$.

Proof. For $\mathbf{r}_0 \times \mathbf{n}_0 \neq \mathbf{0}$, we obtain the existence and continuous dependence of solutions for small s by the standard arguments based on the Contraction Mapping Principle. This approach is justified by Proposition 1.22, which implies that $\mathbf{e}$ must be absolutely continuous for small s because the solution is not radial at $s = 0$. (The technical difficulties in the proof of Proposition 4.5 were all caused by the possible discontinuity of $\mathbf{e}$.) Then the continuation theory for ordinary differential equations ensures that solutions exist for all s in $[0, 1]$ because (1.47^-) yields an a priori estimate embodied in Proposition 2.25 ensuring that solutions of the initial value problem cannot "blow up" for finite s. The last statement of this proposition is proved as in Proposition 4.5. ∎

Note that the results of Sections 2 and 3 yield sufficient conditions on radial solutions ensuring that $\sup\{s: r(s) > 0\} > 0$. Below we shall obtain a simple condition yielding this inequality.

Now we restrict our attention to the case in which the string is uniform and the force field obeys the inverse square law with $f(r) = -\alpha r^{-2}$. The discussion of Section 1 shows that every regular configuration lies in a plane spanned by the orthonormal pair $\{\mathbf{i}, \mathbf{j}\}$ of vectors. Without loss of generality we set

$$\mathbf{r}_0 = r_0\mathbf{j}, \qquad \mathbf{r}_1 = a\mathbf{i} + b\mathbf{j}, \qquad \mathbf{n}_0 = \lambda\mathbf{i} + \mu\mathbf{j}. \tag{4.15}$$

Since we seek regular solutions we take $r_0 > 0$ and $a^2 + b^2 > 0$. Then $\mathbf{r}(\cdot, \mathbf{r}_0, \mathbf{n}_0)$, which we henceforth write as $\mathbf{r}(\cdot, \mathbf{r}_0, \lambda, \mu)$, is a compressive

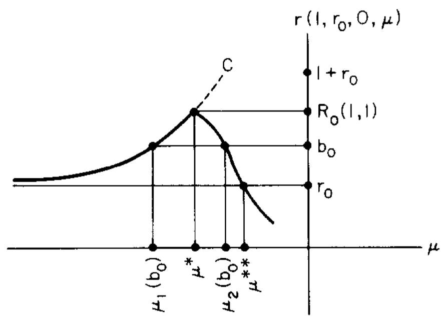

FIG. 4.18. The function $r(1, r_0, 0, \cdot)$ increases strictly on $(-\infty, \mu^*)$ from r_0 to $R_0(1, 1)$, which is defined in Proposition 3.18. On $[\mu^*, 0]$ it decreases from $R_0(1, 1)$ to a value between 0 and r_0. Its slope has a jump at μ^*. For $\mu < \mu^*$, $r(\cdot, r_0, 0, \mu)$ describes straight compressive radial configurations and for $\mu^* < \mu < 0$, $r(\cdot, r_0, 0, \mu)$ describes folded radial configurations. The smooth continuation C of the left piece of the curve represents (4.19) for $\mu > \mu^*$. This corresponds to a straight configuration that is compressive at the low end and tensile at the high end. These solutions which are of physical interest play no role in the construction of compressive states (cf. Theorem 3.27).

solution of our boundary value problem (1.4), (1.6)–(1.9), (1.13) if and only if $\mathbf{r}(\cdot, \mathbf{r}_0, \lambda, \mu)$ is nonzero on $[0, 1]$ and there are numbers λ, μ satisfying

$$\mathbf{r}(1, r_0, \lambda, \mu) = a\mathbf{i} + b\mathbf{j}. \tag{4.16}$$

Since regular solutions are either everywhere radial or nowhere radial we find that (4.16) has a solution for $a = 0$ if and only if

$$r(1, r_0, 0, \mu) \equiv \mathbf{r}(1, r_0, 0, \mu) \cdot \mathbf{j} = b \equiv r_1. \tag{4.17}$$

Let us suppose that $b \geqslant r_0$. Then a careful study of (4.7) when $\lambda = 0$, which is essentially carried out in the proofs of Section 3, shows that $r(1, r_0, 0, \cdot)$ has the form shown in Fig. 4.18. In particular, (3.4) implies that $r(1, r_0, 0, \mu)$ satisfies

$$r(1, r_0, 0, \mu) = r_0 + \int_0^1 \hat{v}\left(\mu + \int_0^s ar(t, r_0, 0, \mu)^{-2}\, dt\right) ds \tag{4.19}$$

provided μ is such that $N(s)$ is everywhere negative; then (4.19) is valid on the interval $(-\infty, \mu^*)$ where μ^* satisfies

$$\mu^* + \int_0^1 ar(t, r_0, 0, \mu^*)^{-2}\, dt = 0. \tag{4.20}$$

By comparing (4.19) with (3.19a) we find that

$$r(1, r_0, 0, \mu^*) = R_0(1, 1). \tag{4.21}$$

From (4.7) we find that

$$\begin{aligned} r(1, r_0, 0, \mu) = r_0 &+ \int_0^{\tau(\mu)} \hat{v}\left(\mu + \int_0^s ar(t, r_0, 0, \mu)^{-2}\, dt\right) ds \\ &- \int_{\tau(\mu)}^1 \hat{v}\left(-\int_{\tau(\mu)}^s ar(t, r_0, 0, \mu)^{-2}\, dt\right) ds \end{aligned} \tag{4.22}$$

where the fold point $\tau(\mu)$ satisfies

$$\mu + \int_0^{\tau(\mu)} ar(t, r_0, 0, \mu)^{-2}\, dt = 0 \tag{4.23}$$

provided $\mu^* < \mu \leqslant 0$. Now (4.23) implies that $\tau(\mu) \searrow 0$ as $\mu \nearrow 0$ so that $r(1, r_0, 0, \mu)$ falls below r_0 as μ nears 0. Let μ^{**} be the smallest value of μ for which $r(1, r_0, 0, \mu^{**}) = r_0$.

For the ensuing analysis we need to determine the behavior of $r(1, r_0, 0, \cdot)$ on $(\mu^{**}, 0]$. The low point on this curve occurs at $\mu = 0$. We ensure that the solution is regular for $\mu \leqslant 0$ by making r_0 so large that $r(1, r_0, 0, 0) > 0$. We know that taking $r_0 \geqslant 1$ suffices for this purpose. We seek a sharper bound for r_0. Now (4.22) and the properties of $\tau(\cdot)$ say that $r(1, r_0, 0, 0) > 0$ if and only if

$$r_0 > \int_0^1 \hat{\nu}\left(-\int_0^s ar(t, r_0, 0, 0)^{-2}\, dt\right) ds. \tag{4.24}$$

Since $\tau(0) = 0$ we know that $r(\cdot, r_0, 0, 0)$ is an unfolded state that is the limit of a sequence of states with folds at $\tau(\mu)$. Consequently, $r(s, r_0, 0, 0) < r_0$ for $s > 0$. Thus, a sufficient condition for (4.24) is

$$r_0 \geqslant \int_0^1 \hat{\nu}\left(-\int_0^s ar_0^{-2}\, dt\right) ds \equiv \int_0^1 \hat{\nu}(-asr_0^{-2})\, ds > 0. \tag{4.25}$$

Let ρ_0 denote the infimum of r_0's that satisfy this condition. We know that $\rho_0 \in [0, 1]$. (Note that the integral on the right can be explicitly evaluated if $\hat{\nu}(N) = (1 - AN)^{-\gamma}$ for $N \leqslant 0$ where $A > 0$, $\gamma > 0$. Thus, ρ_0 can be found in terms of aA and γ. In some cases $\rho_0 = 0$.)

Now let us fix a value b_0 of b to lie in $(r_0, R_0(1, 1))$. Then Fig. 4.18 implies that (4.17) has exactly two solutions, $\mu_1(b_0)$ corresponding to a straight compressive state and $\mu_2(b_0)$ to a folded compressive state. We wish to prove that (4.16) has a family of solutions (λ, μ) "parametrized" by (a, b) that connect $(0, \mu_1(b_0))$ to $(0, \mu_2(b_0))$. Note that we cannot use the classical Implicit Function Theorem to prove the existence of solutions near these special solutions because $\mathbf{r}(1, r_0, \cdot, \cdot)$ is not differentiable for $\lambda = 0$. Instead, we obtain the global existence of solutions by using the homotopy invariance

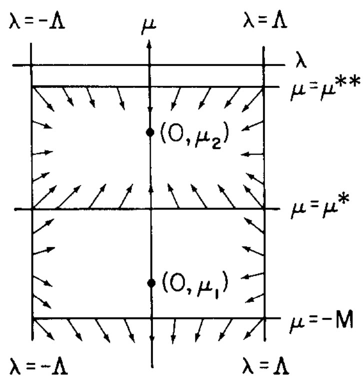

FIG. 4.26. Schematic diagram of the vector field

$$\frac{\mathbf{r}(1, r_0, \cdot, \cdot) - b_0\mathbf{j}}{|\mathbf{r}(1, r_0, \ ,\ ) - b_0\mathbf{j}|}.$$

of (Brouwer) degree. We now get some preliminary results, which will enable us to carry out this theory.

We first observe that (4.7) implies that $\mathbf{r}_s(0, r_0, \lambda, \mu) \cdot \mathbf{i}$ has the sign opposite to that of λ. Equation (4.7) then implies that the sign of $\mathbf{r}(s, r_0, \lambda, \mu) \cdot \mathbf{i}$ is opposite that of λ for all $s \in (0, 1]$ and, moreover, $\mathbf{r}(s, r_0, \cdot, \mu) \cdot \mathbf{i}$ is odd. Thus, $\lambda \mathbf{r}(1, r_0, \lambda, \mu) \cdot \mathbf{i} < 0$ and $\mathbf{r}(1, r_0, \cdot, \mu) \cdot \mathbf{i}$ is odd. We use this fact, the results of Fig. 4.18, and the continuity of $\mathbf{r}(1, r_0, \cdot, \cdot)$ to sketch the continuous vector field $\lambda \mathbf{i} + \mu \mathbf{j} \mapsto \mathbf{r}(1, r_0, \lambda, \mu) - b_0 \mathbf{i}$ on the lines $\lambda = \pm \Lambda$, $\mu = -M$, $\mu = \mu^*$, $\mu = \mu^{**}$ where Λ is small and positive and where $-M < \mu_1(b_0)$. See Fig. 4.26. Since $(0, \mu_1(b_0))$ and $(0, \mu_2(b_0))$ are the only singular points of this vector field, we can compute the rotation of these vector fields on the rectangles enclosing these singular points to obtain

$$\begin{aligned} &\operatorname{ind}(\mathbf{r}(1, r_0, \cdot, \cdot) - b_0 \mathbf{j}, (0, \mu_1(b))) \\ &\qquad \equiv \deg(\mathbf{r}(1, r_0, \cdot, \cdot) - b_0 \mathbf{j}, \{(\lambda, \mu): |\lambda| \leqslant \Lambda, -M \leqslant \mu \leqslant \mu^*\}) = 1, \\ &\operatorname{ind}(\mathbf{r}(1, r_0, \cdot, \cdot) - b_0 \mathbf{j}, (0, \mu_2(b))) \\ &\qquad \equiv \deg(\mathbf{r}(1, r_0, \cdot, \cdot) - b_0 \mathbf{j}, \{(\lambda, \mu): |\lambda| \leqslant \Lambda, \mu^* \leqslant \mu \leqslant 0\}) = -1. \end{aligned} \tag{4.27}$$

Here "deg" stands for (Brouwer) degree and "ind" stands for (Brouwer) index. Compare [9] for a discussion of these notions and the computation yielding (4.27).

Now let $\mathbf{r}(\cdot) = \mathbf{r}(\cdot, r_0, \lambda, \mu)$ represent a compressive solution of our boundary value problem (1.4), (1.6)–(1.9), (1.13) so that (4.16) holds. Then

$$\begin{aligned} |\mathbf{r}_1 - \mathbf{r}_0| &= \sqrt{a^2 + (b - r_0)^2} = \left| \int_0^1 \mathbf{r}'(s)\, ds \right| \\ &\leqslant \int_0^1 \hat{v} \left(- \left| \mathbf{n}_0 + \int_0^s \alpha \mathbf{r}(t)\, r(t)^{-3}\, dt \right| \right) ds \\ &\leqslant \hat{v} \left(-\sqrt{\lambda^2 + \mu^2} + \int_0^1 \alpha r(t)^{-2}\, dt \right) \leqslant \hat{v}(-\sqrt{\lambda^2 + \mu^2} + \alpha r_{01}^{-2}), \end{aligned} \tag{4.28}$$

where r_{01} is the distance from $\mathbf{0}$ to the line joining $\mathbf{r}_0$ to $\mathbf{r}_1$. The last inequality of (4.28) is a consequence of Proposition 1.36. If $|\mathbf{r}_1 - \mathbf{r}_0| > 0$, then (4.28) implies that there is a decreasing function M on $(0, \infty)$ such that $\lambda^2 + \mu^2 < M(|\mathbf{r}_1 - \mathbf{r}_0|)$ if (λ, μ) satisfies (4.16). Moreover, (4.7) and (4.16) imply that

$$\begin{aligned} b - r_0 = -\int_0^1 \hat{v}(-|n^*(\mathbf{r}, \mathbf{n}_0)(s)|) \left[\mu + \int_0^s \alpha \mathbf{r}(t) \cdot \mathbf{j} r(t)^{-3}\, dt \right] \\ \times |n^*(\mathbf{r}, \mathbf{n}_0)(s)|^{-1}\, ds. \end{aligned} \tag{4.29}$$

Then Proposition 1.36 implies that $\mu < 0$.

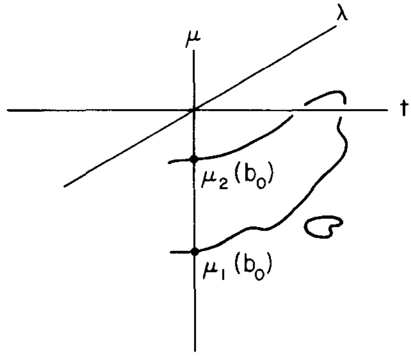

FIG. 4.31. Schematic diagram of solutions pairs. A plane perpendicular to the t-axis through a small value of t intersects the set of solutions at an even number of points. Note that there might be disconnected collections of solutions as well.

Now let $\mathbb{R} \ni t \mapsto (a(t), b(t)) \in \mathbb{R}^2$ be a continuous curve with $(a(0), b(0)) = (0, b_0)$, with $a(t) = 0$ if and only if $t = 0$, and with $a(t)^2 + b(t)^2 \to \infty$ as $t \to \pm\infty$. Suppose further that there are numbers $\varepsilon > 0$, $\delta > 0$ such that $b(t) > \rho_0$ for $|t| < \varepsilon$ and $|a(t)| \geqslant \delta$ for $|t| \geqslant \varepsilon$. We say that $(t, (\lambda, \mu))$ is a *solution pair* for (4.16) if (4.16) is satisfied when a and b are replaced by $a(t)$ and $b(t)$. We can now state our basic result.

4.30. THEOREM. *Let $\hat{v}$ be independent of s and let $f(\mathbf{r}) = -ar^{-2}$. Let $r_0 > \rho_0$, let $b_0 \in (r_0, R_0(1, 1))$, and let $(a(\cdot), b(\cdot))$ have the properties described above. Then the set of solution pairs for* (4.16) *contains a connected set joining* $(0, (0, \mu_1(b_0)))$ *to* $(0, (0, \mu_2(b_0)))$. (*See Fig.* 4.31.)

Proof. Equations (4.7) and (4.16) imply that

$$a(t) = -\int_0^1 \hat{v}(-|\mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)(s)|) \left[\lambda + \int_0^s a\mathbf{r}(\sigma) \cdot \mathbf{i}r(\sigma)^{-3}\, d\sigma\right] \times |n^*(\mathbf{r}, \mathbf{n}_0)(s)|^{-1}\, ds. \tag{4.32}$$

Proposition 4.14 implies that the right side of (4.32) depends continuously on λ for $\lambda \neq 0$. Moreover, in the development associated with Fig. 4.26 we showed that the right side of (4.32) is an odd function of λ that does not vanish for $\lambda \neq 0$. Thus, there is a number $\gamma > 0$ such that $|\lambda| \geqslant \gamma$ when $|a(t)| \geqslant \delta$. Let

$$U \equiv \{(t, \lambda, \mu) \colon \lambda^2 + \mu^2 < M(|\mathbf{r}_1 - \mathbf{r}_0|), |a(t)| < 1\} \backslash \{(t, \lambda, \mu) \colon |t| \leqslant \varepsilon, \mu \geqslant 0, |\lambda| \leqslant \gamma\}. \tag{4.33}$$

Proposition 4.14 implies that $(t, \lambda, \mu) \mapsto \mathbf{r}(1, r_0, \lambda, \mu) - a(t)\mathbf{i} + b(t)\mathbf{j}$ is continuous on the closure of U and that (4.16) has no solutions on the complement of U.

Suppose that $(0, (0, \mu_1(b_0)))$ and $(0, (0, \mu_2(b_0)))$ were not joined by a connected set of solution pairs. Let $\mathscr{C}$ be the maximal connected set of solution pairs containing $(0, (0, \mu_1(b_0)))$. The set of all solution pairs is closed because it is the inverse image of $\mathbf{0}$ under the continuous function $(t, \lambda, \mu) \mapsto \mathbf{r}(1, r_0, \lambda, \mu) - a(t)\mathbf{i} - b(t)\mathbf{j}$ (cf. (4.16)). It is bounded because it lies in U. It follows that $\mathscr{C}$ is also closed and bounded. Moreover, $\mathscr{C}$ can be enclosed in an open set $\mathscr{O}$ whose closure lies in U and contains no solution pairs other than those of $\mathscr{C}$. Let $\mathscr{O}_\omega \equiv \{(t, \lambda, \mu) \in \mathscr{O}: t = \omega\}$. Then (4.27) implies that $\deg(\mathbf{r}(1, r_0, \cdot, \cdot) - b_0\mathbf{j}, \mathscr{O}_0) = 1$. Now the properties of $(a(\cdot), b(\cdot))$ imply that there is a number $\xi < 0$ and a number $\eta > 0$ such that $\mathscr{O}_\xi$ and $\mathscr{O}_\eta$ contain no solution pairs, whence

$$\begin{aligned}\deg(\mathbf{r}(1, r_0, \cdot, \cdot) &- a(\xi)\mathbf{i} - b(\xi)\mathbf{j}, \mathscr{O}_\xi) \\ &= 0 = \deg(\mathbf{r}(1, r_0, \cdot, \cdot) - a(\eta)\mathbf{i} - b(\eta)\mathbf{j}, \mathscr{O}_\eta).\end{aligned}$$

But this equation is incompatible with the homotopy invariance of degree (cf. [9, 10], e.g.), which requires these degrees on $\mathscr{O}_\xi$ and $\mathscr{O}_\eta$ to equal the degree on $\mathscr{O}_0$. ∎

This result tells us that for $|t|$ small there are at least two compressive solutions of our boundary value problem and that such compressive solutions come in pairs. We briefly discuss generalizations of this result in Section 6.

5. Tensile Solutions

In this section we comment briefly on tensile configurations. For simplicity we continue to assume that the string is uniform and that $f(r) = -ar^{-2}$. The initial value problem (1.4), (1.6)–(1.8), (1.13), (4.1) describes a tensile configuration if and only if r satisfies

$$\mathbf{r}(s) = \mathbf{r}_0 + \int_0^s \hat{v}(|\mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)(s)|)\, \mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)(s)\, |\mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)(s)|^{-1}\, ds, \tag{5.1a}$$

$$\mathbf{n}^*(\mathbf{r}, \mathbf{n}_0)(s) \equiv \mathbf{n}_0 + \int_0^s a\mathbf{r}(t)\, r^{-3}(t)\, dt. \tag{5.1b}$$

(This equation is analogous to (4.7).) Now (2.7) and (2.8) imply that tensile solutions cannot be singular. Hence, we can imitate the proof of Proposition 4.14 to obtain

5.2. PROPOSITION. *For every* $\mathbf{r}_0, \mathbf{n}_0$ *for which* $\mathbf{r}_0 \times \mathbf{n}_0 \neq \mathbf{0}$, *there exists a unique continuously differentiable solution* $\mathbf{r}$ *of* (5.1) *on* $[0, 1]$, *which is a continuously differentiable function of* $\mathbf{r}_0$ *and* $\mathbf{n}_0$ *for* $\mathbf{r}_0 \times \mathbf{n}_0 \neq \mathbf{0}$. *For every* $\mathbf{r}_0, \mathbf{n}_0$ *with* $r_0 > 0$, *there exists a unique continuous solution* $\mathbf{r}$ *of* (5.1) *on* $[0, 1]$, *which depends continuously on* $\mathbf{r}_0$ *and* $\mathbf{n}_0$.

We adhere to the notation of (4.15). We then denote solutions of (5.1) by $\mathbf{r}(\cdot, r_0, \lambda, \mu)$. Such a solution generates a solution of our boundary value problem if and only if (4.16) holds.

Now we can compute the index of $\mathbf{r}(1, r_0, \cdot, \cdot) - b_0\mathbf{j}$ at $(0, \mu(b_0))$ where $(0, \mu(b_0))$ is a solution of (4.16) with $b = b_0 \geqslant r_0$ and show that this index is not zero for nearly all b_0's. (For purely radial configurations it can be shown that $[0, \infty) \ni \mu \mapsto |r(1, r_0, 0, \mu)|$ is strictly increasing in keeping with Proposition 3.9.) Let us again introduce a curve $\mathbb{R} \ni t \mapsto (a(t), b(t)) \in \mathbb{R}^2$ of terminal points with $(a(0), b(0)) = (0, b_0)$, with $a(t) = 0$ if and only if $t = 0$, and with $a(t)^2 + b(t)^2 \to \infty$ as $t \to \pm\infty$. Then a simpler version of the proof of Theorem 4.30 yields

5.3. THEOREM. *Let* $b_0 \geqslant r_0 > 0$ *and let* $\operatorname{ind}(\mathbf{r}(1, r_0, \cdot, \cdot) - b_0\mathbf{j}, (0, \mu(b_0))) = 0$. *Then* (4.16) *has a connected family* $\mathscr{C}$ *of solution pairs* $(t, (\lambda, \mu))$ *containing* $(0, (0, \mu(b_0)))$ *that has at least one of the following two properties*: (i) $\mathscr{C}$ *is unbounded*, (ii) $\mathscr{C}$ *contains another tensile radial solution* $(0, (0, \mu^{\#}))$.

(The difficulty in the proof of Theorem 4.30 was in showing that $\mathscr{C}$ could not satisfy a third possibility, namely, that it approach the boundary of the set on which $\mathbf{r}(1, r_0, \cdot, \cdot)$ is defined.)

We complement this theorem with a nonexistence theorem generalizing a remark following the proof of Lemma 3.29.

5.4. THEOREM. *Let* $\mathbf{f}$ *be an attractive central force* (*so that it has the form* (1.1) *with* $f(r, s) < 0$). *If* $r_0 + r_1 \equiv |\mathbf{r}_0| + |\mathbf{r}_1| < 1$, *then the boundary value problem* (1.4), (1.6)–(1.9), (1.13) *has no tensile solution.*

Proof. The proof for radial solutions is a consequence of Lemma 3.29. We therefore assume that $\mathbf{r}_0 \times \mathbf{r}_1 \neq 0$. Since Propositions 1.22 and 1.36 imply that the configuration of the string is bowed-in and nowhere radial, it follows that $r_0 + r_1$ exceeds the sum L of the lengths of the tangent lines to the curve $\mathbf{r}$ at $\mathbf{r}_0$ and $\mathbf{r}_1$ from their points of tangency $\mathbf{r}_0$ and $\mathbf{r}_1$ to their intersection point. The convexity of $\mathbf{r}$, proved in the discussion preceding the statement of Proposition 1.36, implies that L exceeds the length of $\mathbf{r}$. But Proposition 1.22 then implies that the length of r must exceed 1. ∎

6. Conclusion

Most of our results in Sections 3–5 were stated for uniform strings subjected to uniform attractive forces satisfying the inverse square law. All these results can be immediately generalized to uniform strings subject to uniform attractive forces with $f(r) \searrow -\infty$ as $r \searrow 0$. At the cost of much harder estimates, many of these results can be extended to nonuniform strings under nonuniform attractive forces provided the nonuniformity is controlled. The resulting theorems have weaker conclusions, however. In particular, various uniqueness properties can be lost in the process of generalization. A complete theory for repulsive central forces along the lines of Sections 3–5 could be easily constructed. The results would differ in minor respects from those we obtained and could have obtained for attractive forces. A complete theory for central forces that are bounded near $\mathbf{r} = \mathbf{0}$ could also be obtained. The richness of the theory of radial states under such forces can be seen in the analysis of [5] on the effects of centrifugal force in a related problem.

In Sections 4 and 5 we restricted the parameters a and b to lie on a curve $t \mapsto (a(t), b(t))$ in $\mathbb{R}^2$. This enabled us to treat a one-parameter problem to which we could apply the theory of homotopy invariance of degree. This restriction of a and b to a curve is somewhat artificial. We could consider the union of solution pairs generated by all such curves, but the geometric properties of such unions cannot be deduced from degree theory. For this purpose the more refined topological methods of [3] are appropriate. When a and b are allowed to vary freely the results of [3] (together with some refinements of [1]) can be used to generalize the results of Section 4 to show that the solution pairs $((a, b), (\lambda, \mu)) = ((0, b_0), (0, \mu_1(b_0)))$, $((0, b_0), (0, \mu_2(b_0))$ of (4.16) are connected by a family of solution pairs each point of which has Lebesgue dimension at least 2. If the number of parameters is increased (by including r_0 and α, say), then an analogous result holds with the connecting family of solution pairs having dimension at least equal to the number of parameters. The problems we study in Sections 4 and 5 actually have infinite-dimensional parameters $(a, b, \alpha, \hat{v})$. Indeed, by replacing the force satisfying the inverse square law with an arbitrary central force field f, we can take the parameters to be $(a, b, f, \hat{v})$. A theory capable of handling this case is developed in [2]. This theory shows among other things that the connecting family of solution pairs has infinite dimension at each point. Fortunately, the use of these more exotic theories depends exactly upon the developments we carried out in Sections 4 and 5.

We have used methods based upon degree theory rather than those based on variational methods because the former yield sharp results on connectivity. Variational methods (cf. [6, 7]) yield different kinds of multiplicity results; these are related to questions of stability.

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