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COMMUNICATION

DETERMINING THE TOTAL COLOURING NUMBER IS NP-HARD*

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In this paper it is proved that the problem of determining the total chromatic number of an arbitrary graph is NP-hard. The problem remains NP-hard even for cubic bipartite graphs.

1. Introduction

The *total chromatic number* $X_T(G)$ of a graph G is the minimum number of colours required to colour the edges and vertices of G in such a way that no two adjacent or incident elements of G have the same colour. Clearly $X_T(G) \geqslant d(G)+1$, where $d(G)$ is the maximum degree of G. It is a long standing conjecture that $X_T(G) \leqslant d(G)+2$. For example it is easily seen to be true if G is bipartite.

Consider the following problem:

Total colouring

Instance: A graph G.
Question: Is G $(d(G)+1)$-total colourable?

In this paper we prove that Total Colouring is NP-complete (see [1] for terminology and definitions). In fact we prove a stronger result, i.e. that Total Colouring is NP-complete even for cubic bipartite graphs. Thus the problem has no polynomial time algorithm unless $P=\text{NP}$.

It is clear that Total Colouring is in the class NP. To prove that the problem is NP-complete we exhibit a polynomial reduction from a known NP-complete problem concerning edge-colourings in regular graphs, which is defined as follows:

Regular edge-colouring

Instance: A r-regular graph G.
Question: Is G r-edge colourable?

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This problem was shown to be NP-complete by Holyer [2] for $r=3$, and by Leven and Galil [3] for $r \geq 4$. We will transform Regular Edge-Colouring for $r=4$ to Total Colouring restricted to cubic bipartite graphs.

2. The component used in the construction

Given a 4-regular graph G of the 4-edge colouring problem we will show how to construct a cubic bipartite graph G' which is 4-total colourable if and only if G is 4-edge colourable. The graph G' will be constructed by replacing each vertex, v, of G with a replacement graph R, having 4 pendant edges. These edges will be associated with the edges of G incident to v.

The replacement graph will be constructed by putting together several copies of the basic graph S of *Fig. 1.* The following are basic results concerning the graph S.

2.1. (a) *The graph S is 4-total colourable.*
(b) *In any 4-total colouring of S the edges e_1, e_2 and e_3 are coloured the same.*

Proof. The 4-total colouring in *Fig. 1* proves (a). In order to prove (b) assume that there is a 4-total colouring of S, in which two edges, say e_1 and e_2, are coloured differently. Thus the graph S' constructed from S by deleting e_3 and identifying c_1 with c_2 is 4-total colourable. However, S' is isomorphic to K_{33}\edge which is not 4-total colourable. This is the required contradiction. □

A *partial k-total colouring* is a colouring of some of the vertices and edges so that no two adjacent or incident elements have the same colour.

2.2. *Consider a partial 4-total colouring of S in which the pendant edges have the same colour (say colour 4) and c_1 and c_2 are also coloured (and nothing else is*

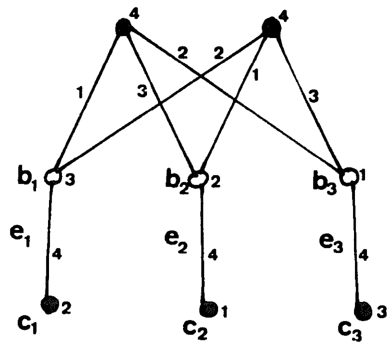

Fig. 1. The graph S.

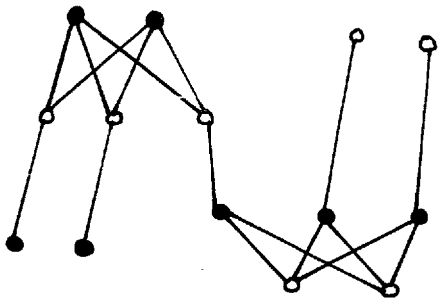

Fig. 2. The graph H.

coloured). Then

(a) *this extends to a* 4-*total colouring of* S, *and*

(b) *if* c_1 *and* c_2 *have different colours then the vertices* b_3 *and* c_3 *can be coloured with any colour except colour* 4.

Proof. If c_1 and c_2 have the same colour, say colour 3, then colour c_3 with colour 1 or 2. Colour b_i with colour i for $i = 1, 2, 3$. Finally use the colour sets of 2.1(a) to colour the remaining part of S. This proves part (a) and part (b) follows immediately from 2.1(a). □

Now consider the "replacement" graph R, constructed as follows (see *Fig. 3*). First replace each vertex of the graph K_4 with a copy of the graph H shown in *Fig. 2*. Insert a new vertex in every edge of the original K_4. Finally add three 'cross' edges. Observe that the replacement graph is cubic and bipartite.

2.3. (a) *Consider a partial* 4-*total colouring of* R *in which the four pendant edges*

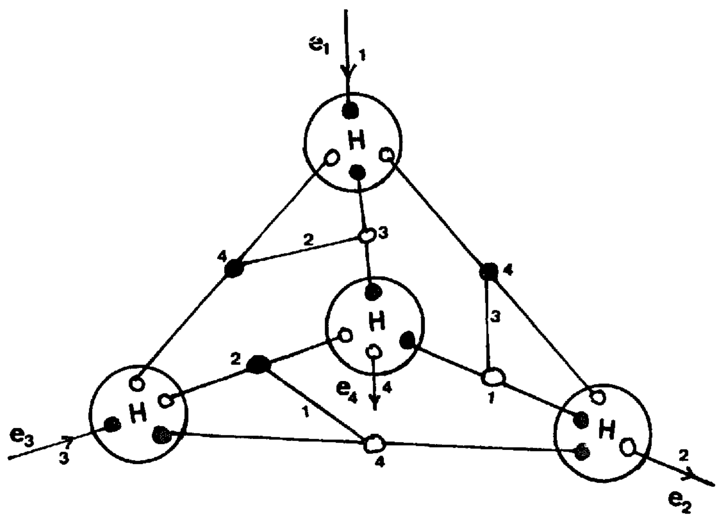

Fig. 3. The replacement graph R.

have different colours and the pendant vertices are also coloured (and nothing else is coloured). Then this extends to a 4-total colouring of R.

(b) *In every 4-total colouring of R the four pendant edges must all have different colours.*

Proof. In order to prove part (a), first extend the given partial 4-total colouring of R as indicated in *Fig. 3,* then apply (2.2) twice on each graph H. By (2.2)b it follows that this partial colouring can always be extended to a complete 4-total colouring of R. Part (b) follows easily from (2.1)b. □

3. The main theorem

We are now in position to prove the following theorem.

Theorem 3.1. *Total Colouring restricted to cubic bipartite graphs is NP-complete.*

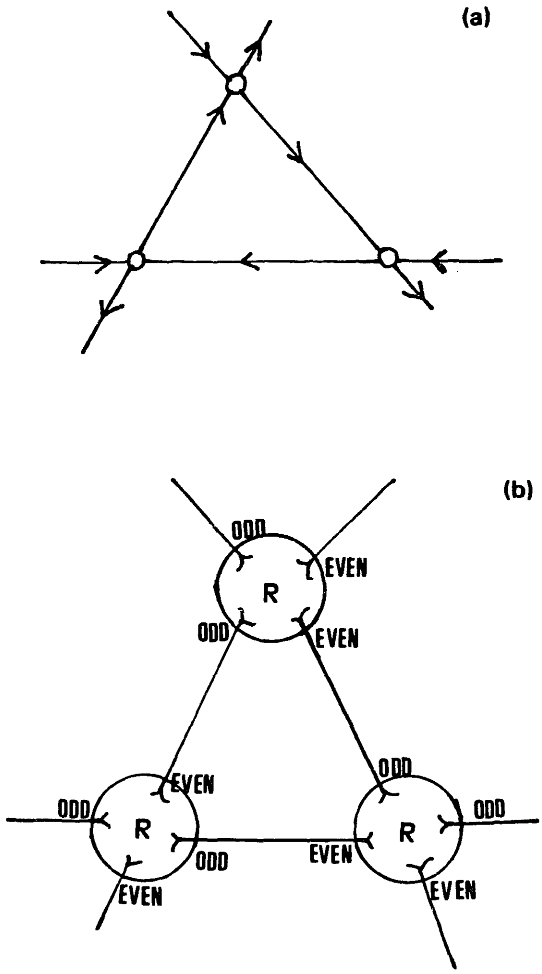

Fig. 4. The transformation. (a) Typical part of G. (b) Typical part of G'.

Proof. The problem is clearly in the class NP. We exhibit a polynomial transformation from the 4-edge colouring problem. Consider a 4-regular graph G and construct from it a graph G' as follows:

(i) Orient the edges of G so that each vertex has two arrows coming in and two arrows going out of it (see *Fig. 4a*). We can do this by finding eulerian cycles.

(ii) Replace each vertex v of G by a replacement graph R, in which the incoming edges are associated with the odd terminal edges e_1, e_3 and the outgoing edges with the even terminal edges e_2, e_4 (see *Fig. 4b*).

Clearly the resulting graph G' is cubic and bipartite, and the transformation can be carried out in polynomial time. We must show that G is 4-edge colourable if and only if G' is 4-total colourable.

One part is easy by (2.3)b if G' is 4-total colourable then clearly G is 4-edge colourable.

Suppose now we have a 4-edge colouring of G. This yields a colouring of the 'link' edges of G'. By (2.3)a we can extend this colouring to a 4-total colouring of G' by extending this partial colouring to each replacement graph in turn. This finishes the proof of theorem 3.1. □

Acknowledgement

I would like to thank Colin McDiarmid for suggesting a simplification of the original replacement graph.

References

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[2] I.J. Holyer, The NP-completeness of edge colourings, SIAM J. Computing 10(1981) 718–720.
[3] D. Leven and Z. Galil, NP-completeness of finding the chromatic index of regular graphs, J. Algorithms 4 (1983) 35–44.