Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 285 (2005) 856–867
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The upper bound of Frobenius related length functions

Jinjia Li

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

Received 20 September 2004
Available online 22 January 2005
Communicated by Paul Roberts

Abstract

In this paper, we study the asymptotic behavior of lengths of Tor modules of homologies of complexes under the iterations of the Frobenius functor in positive characteristic. We first give upper bounds to this type of length functions in lower dimensional cases and then construct a counterexample to the general situation. The motivation of studying such length functions arose initially from an asymptotic length criterion given in [S.P. Dutta, Intersection multiplicity of modules in the positive characteristics, J. Algebra 280 (2004) 394–411] which is a sufficient condition to a special case of nonnegativity of χ_∞. We also provide an example to show that this sufficient condition does not hold in general.

Keywords: Complex; Homology; Frobenius; Intersection multiplicity

Introduction and notations

In this paper, (A, m, k) will be a complete local ring of characteristic $p > 0$, m its maximal ideal, $k = A/m$ and k is perfect. By a free complex we mean a complex $F_\bullet = (F_i, d_i)_{i \geqslant 0}$ $(\cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$ of finitely generated free A-modules. We define codimension of M to be $\dim A - \dim M$ (denoted by $\operatorname{codim} M$) for any A-module M. The

E-mail address: jinjiali@math.uiuc.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.11.015

Frobenius endomorphism $f_A : A \to A$ is defined by $f_A(r) = r^p$ for $r \in A$. Each iteration f_A^n defines a new A-module structure on A, denoted by ${}^{f^n}A$ for which $a \cdot b = a^{p^n} b$. Write $F_A^n(M)$ for $M \otimes_A {}^{f^n}A$ and $F_A^n(F_\bullet)$ for $F_\bullet \otimes_A {}^{f^n}A$. We drop the subscript A when there is no ambiguity.

In [D1], Dutta introduced the following definition of χ_∞.

Definition. Let R be a local ring in characteristic $p > 0$. Let M and N be two finitely generated modules such that $\ell(M \otimes_R N) < \infty$ and $\operatorname{proj\,dim} M < \infty$. Define

$$\chi_\infty(M, N) = \lim_{n\to\infty} \chi\big(F^n(M), N\big)/p^{n \operatorname{codim} M}.$$

χ_∞ plays an important role in the study of intersection multiplicity χ defined by Serre [S], especially in the nonsmooth situation. For example, over complete intersections, $\chi_\infty(M, N) = \chi(M, N)$ when both M and N are of finite projective dimension [D4, Corollary to Theorem 1.2]. Thus the positivity (or nonnegativity) of χ_∞ settles the positivity (respectively nonnegativity) conjecture of χ over complete intersections.

Our main object is to examine the following sufficient condition for the nonnegativity of χ_∞ [D4, Corollary 1 to Theorem 2.2].

Theorem (Dutta). *Let R be a local Gorenstein ring in characteristic $p > 0$. Let M and N be finitely generated modules of finite projective dimensions such that $\ell(M \otimes N) < \infty$. Suppose $\dim M + \dim N = \dim R$, $\dim N = \operatorname{depth} N + 1 = s$ and $\dim M = \operatorname{depth} M + 1 = 2$. Then $\chi_\infty(M, N) \geqslant 0$, if*

$$\lim_{n\to\infty} \ell\big(\operatorname{Ext}^3(N, R) \otimes H_m^0\big(F^n\big(\operatorname{Ext}^{s+1}(M, R)\big)\big)^\vee\big)/p^{ns} = 0. \tag{1}$$

Note here $\operatorname{proj\,dim} M = s + 1$ and $\operatorname{proj\,dim} N = 3$ by Auslander–Buchsbaum formula, and these Exts are the natural duals under the generalized "Matlis" duality.

This study leds us to investigate the asymptotic behavior of $\ell(\operatorname{Tor}_j^A(H_i(F^n(F_\bullet)), N))$, where $F_\bullet$ is a free complex with homologies of finite length. ($F_\bullet$ is not necessarily a bounded complex here!)

In [D3], Dutta established that

$$\ell\big(\operatorname{Tor}_j^A\big(H_i\big(F^n(F_\bullet)\big), N\big)\big) \leqslant C_{ij}\, p^{n \dim N}$$

when $\operatorname{codim} N = 1$ [D3, Proposition 1.3]. Naturally, one can ask whether this inequality is still valid when N has higher codimension. Investigation of the length condition (1) raises the same question. The expectation was that the same inequality should hold in general for any N, namely, $\ell(\operatorname{Tor}_j^A(H_i(F^n(F_\bullet)), N)) \leqslant C_{ij}\, p^{n \dim N}$. A positive answer to this question in codimension 3 would yield an affirmative answer for (1). However, our investigation revealed that one can only extend this for $\operatorname{codim} N \leqslant 2$.

The following result in Section 1 shows that one can extend this inequality for $\operatorname{codim} N \leqslant 2$.

Theorem (Corollary 1.3 in Section 1). *Let $F_\bullet$ be a free complex with homologies of finite length over a Cohen–Macaulay local ring A. Let N be a finitely generated A-module such that* $\operatorname{codim} N \leqslant 2$. *Then there exist constants C_{ij}'s, such that*

$$\ell\big(\operatorname{Tor}_j^A\big(H_i\big(F^n(F_\bullet)\big), N\big)\big) \leqslant C_{ij}\, p^{n \dim N}$$

for all $i, j \geqslant 0$.

When $\operatorname{codim} N = 3$, we provide a counterexample in Section 2. This counterexample in turn leads to us our main theorem in Section 3.

Main Theorem (Theorem 3.2 in Section 3). *Let $R = K[[X, Y, U, V]]/(XY - UV)$ where K is a field of characteristic $p > 0$ and X, Y, U, V are indeterminates. There exist finitely generated modules M, N over R as in the above theorem with $s = 1$, such that the sufficient condition* (1) *for nonnegativity of χ_∞ fails to hold.*

Nevertheless, this counterexample does not give a negative χ_∞.

1.

We first state a proposition due to Seibert [Se, Proposition 1, Section 3] which plays a crucial role in our proof.

Proposition 1.1 (Seibert). *Let $F_\bullet$ be a free complex over A with homologies of finite length and N be any finitely generated A-module. Then there exist constants C_i's such that*

$$\ell\big(H_i\big(F^n(F_\bullet) \otimes_A N\big)\big) \leqslant C_i\, p^{n \dim N}.$$

The following is our first result which generalizes a result due to Dutta [D3, Proposition 1.3].

Proposition 1.2. *Let $F_\bullet$ be a free complex with homologies of finite length over A. Let N be A/xA or $A/(x, y)$ where $\{x\}$ or $\{x,y\}$, respectively, forms a regular sequence. Then there exist constants C_{ij}'s, such that*

$$\ell\big(\operatorname{Tor}_j^A\big(H_i\big(F^n(F_\bullet)\big), N\big)\big) \leqslant C_{ij}\, p^{n \dim N} \tag{2}$$

for all $i, j \geqslant 0$.

The following special lemma has been used repeatedly in the proof of Proposition 1.2. We leave the proof as an exercise for the reader.

Special Lemma. *Let A be a local ring and M be a module over A such that $\ell(M) < \infty$. Suppose x is A-regular. Then*

$$\ell\big(\mathrm{Tor}_1^A(M, A/xA)\big) = \ell\big(M \otimes_A (A/xA)\big).$$

Proof of Proposition 1.2. We write $\bar{A} = A/xA$ and $\bar{F}_\bullet = F_\bullet \otimes_A \bar{A}$.

Case 1. $N = A/xA$. This case has already been demonstrated in [D3] in a more general set up. (See the proof of Proposition 1.3 in [D3], although the official statement there is in the form of limit.) We give a simple proof of this case anyway for completeness.

Since $\operatorname{proj\,dim} N = 1$,

$$\mathrm{Tor}_j^A\big(H_i\big(F^n(F_\bullet)\big), N\big) = 0$$

for $j \geqslant 2$ and by the special lemma

$$\ell\big(\mathrm{Tor}_1^A\big(H_i\big(F^n(F_\bullet)\big), N\big)\big) = \ell\big(H_i\big(F^n(F_\bullet)\big) \otimes_A N\big).$$

Thus it suffices to prove the result for $j = 0$.

If $i = 0$, since $H_0(F^n(F_\bullet)) \otimes N = H_0(F^n(F_\bullet) \otimes N)$, we get the desired inequality by Proposition 1.1.

If $i \geqslant 1$, since $F_A^n(F_\bullet) \otimes_A \bar{A} = F_{\bar{A}}^n(\bar{F}_\bullet)$, there is a short exact sequence of complexes

$$0 \to F^n(F_\bullet) \xrightarrow{x} F^n(F_\bullet) \to F_{\bar{A}}^n(\bar{F}_\bullet) \to 0.$$

Taking the associated long exact sequence of homologies, we get

$$\cdots \to H_i\big(F^n(F_\bullet)\big) \xrightarrow{x} H_i\big(F^n(F_\bullet)\big) \to H_i\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big) \to H_{i-1}\big(F^n(F_\bullet)\big) \to \cdots.$$

It yields the following short exact sequence:

$$0 \to H_i\big(F^n(F_\bullet)\big) \otimes A/xA \to H_i\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big) \to (0:x)_{H_{i-1}(F^n(F_\bullet))} \to 0 \tag{3}$$

for $i \geqslant 1$. So,

$$\ell\big(H_i\big(F^n(F_\bullet)\big) \otimes A/xA\big) \leqslant \ell\big(H_i\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big)\big)$$

and again, the desired inequality follows from Proposition 1.1 with $N = \bar{A}$.

Case 2. $N = A/(x, y)$. In this case, since $\operatorname{proj\,dim} N = 2$,

$$\mathrm{Tor}_j^A\big(H_i\big(F^n(F_\bullet)\big), N\big) = 0$$

for $j \geqslant 3$. By a result due to Serre [S, Theorem 1, Chapter IV],

$$\sum_{j=0}^{2}(-1)^j \ell\big(\mathrm{Tor}_j^A\big(H_i\big(F^n(F_\bullet)\big), N\big)\big) = \chi\big(H_i\big(F^n(F_\bullet)\big), N\big) = 0.$$

Hence, it is enough to prove the result for $j = 0$ and 1.

Tensor (3) with $A/(x, y)$ ($\simeq \bar{A}/y\bar{A}$) over $\bar{A}$. We obtain

$$\begin{aligned} \cdots &\to \mathrm{Tor}_1^{\bar{A}}\big((0:x)_{H_{i-1}(F^n(F_\bullet))}, \bar{A}/y\bar{A}\big) \to H_i\big(F^n(F_\bullet)\big) \otimes_A A/(x, y) \\ &\to H_i\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big) \otimes_{\bar{A}} \bar{A}/y\bar{A} \to (0:x)_{H_{i-1}(F^n(F_\bullet))} \otimes_{\bar{A}} \bar{A}/y\bar{A} \to 0 \end{aligned} \tag{4}$$

for $i \geqslant 1$. It follows that

$$\begin{aligned} &\ell\big(H_i\big(F^n(F_\bullet)\big) \otimes_A A/(x, y)\big) \\ &\quad\leqslant \ell\big(\mathrm{Tor}_1^{\bar{A}}\big((0:x)_{H_{i-1}(F^n(F_\bullet))}, \bar{A}/y\bar{A}\big)\big) + \ell\big(H_i\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big) \otimes_{\bar{A}} \bar{A}/y\bar{A}\big). \end{aligned}$$

Notice that by the special lemma,

$$\ell\big(\mathrm{Tor}_1^{\bar{A}}\big((0:x)_{H_{i-1}(F^n(F_\bullet))}, \bar{A}/y\bar{A}\big)\big) = \ell\big((0:x)_{H_{i-1}(F^n(F_\bullet))} \otimes_{\bar{A}} \bar{A}/y\bar{A}\big)$$

and from the above long exact sequence (4),

$$\ell\big((0:x)_{H_{i-1}(F^n(F_\bullet))} \otimes_{\bar{A}} \bar{A}/y\bar{A}\big) \leqslant \ell\big(H_i\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big) \otimes_{\bar{A}} \bar{A}/y\bar{A}\big).$$

Hence

$$\ell\big(H_i\big(F^n(F_\bullet)\big) \otimes_A A/(x, y)\big) \leqslant 2\ell\big(H_i\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big) \otimes_{\bar{A}} \bar{A}/y\bar{A}\big).$$

Therefore by Case 1, we are done for $j = 0$.

Finally, for $j = 1$, we use the following spectral sequence obtained by base change:

$$\mathrm{Tor}_p^{\bar{A}}\big(\mathrm{Tor}_q^A\big(H_i\big(F^n(F_\bullet)\big), \bar{A}\big), \bar{A}/y\bar{A}\big) \quad \underset{p}{\Longrightarrow} \quad \mathrm{Tor}_{p+q}^A\big(H_i\big(F^n(F_\bullet)\big), A/(x, y)\big).$$

It follows that

$$\begin{aligned} &\ell\big(\mathrm{Tor}_1^A\big(H_i\big(F^n(F_\bullet)\big), A/(x, y)\big)\big) \\ &\quad\leqslant \ell\big(\mathrm{Tor}_1^{\bar{A}}\big(\mathrm{Tor}_0^A\big(H_i\big(F^n(F_\bullet)\big), \bar{A}\big), \bar{A}/y\bar{A}\big)\big) + \ell\big(\mathrm{Tor}_0^{\bar{A}}\big(\mathrm{Tor}_1^A\big(H_i\big(F^n(F_\bullet)\big), \bar{A}\big), \bar{A}/y\bar{A}\big)\big) \\ &\quad= \ell\big(H_i\big(F^n(F_\bullet)\big) \otimes A/(x, y)\big) + \ell\big(\mathrm{Tor}_1^A\big(H_i\big(F^n(F_\bullet)\big), \bar{A}\big) \otimes A/(x, y)\big). \end{aligned}$$

The last equality here is by the special lemma again.

Since x is A-regular, $\operatorname{Tor}_1^A(H_i(F^n(F_\bullet)), \bar{A}) \simeq (0:x)_{H_i(F^n(F_\bullet))}$. Therefore by (3), we have a surjection

$$H_{i+1}\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big) \otimes A/(x,y) \twoheadrightarrow \operatorname{Tor}_1^A\big(H_i\big(F^n(F_\bullet)\big), \bar{A}\big) \otimes A/(x,y).$$

Thus

$$\begin{aligned}&\ell\big(\operatorname{Tor}_1^A\big(H_i\big(F^n(F_\bullet)\big), A/(x,y)\big)\big)\\&\quad\leqslant \ell\big(H_i\big(F^n(F_\bullet)\big)\otimes A/(x,y)\big) + \ell\big(H_{i+1}\big(F_{\bar{A}}^n(\bar{F}_\bullet)\big)\otimes A/(x,y)\big).\end{aligned}$$

Both of the terms on the right-hand side of the above inequality are bounded by a constant times $p^{n\dim N}$ by the $j=0$ case, and so we are done for $j=1$ which finishes our proof. □

Corollary 1.3. *Let A be a Cohen–Macaulay local ring and let $F_\bullet$ be as in Proposition* 1.2*. Let N be a finitely generated A-module such that* $\operatorname{codim} N \leqslant 2$*. Then there exist constants C_{ij}'s, such that*

$$\ell\big(\operatorname{Tor}_j^A\big(H_i\big(F^n(F_\bullet)\big), N\big)\big) \leqslant C_{ij}\, p^{n\dim N}$$

for all $i, j \geqslant 0$.

Proof. Suppose $\operatorname{codim} N = h$, $h = 1$ or 2. Then $\operatorname{Ann}_A N$ contains an A-regular sequence $\{x_1, \ldots, x_h\}$. We have the following short exact sequence:

$$0 \to Q \to \big(A/(x_1,\ldots,x_h)\big)^t \to N \to 0.$$

Tensoring the above short exact sequence with $H_i(F^n(F_\bullet))$, we get a long exact sequence

$$\begin{aligned}&\cdots \to \operatorname{Tor}_1^A\big(\big(A/(x_1,\ldots,x_h)\big)^t, H_i\big(F^n(F_\bullet)\big)\big)\\&\quad\to \operatorname{Tor}_1^A\big(N, H_i\big(F^n(F_\bullet)\big)\big) \to Q \otimes H_i\big(F^n(F_\bullet)\big)\\&\quad\to \big(A/(x_1,\ldots,x_h)\big)^t \otimes H_i\big(F^n(F_\bullet)\big) \to N \otimes H_i\big(F^n(F_\bullet)\big) \to 0.\end{aligned}$$

By Proposition 1.2 and induction on j, we obtain the desired inequality. □

Remark 1.4. If A is a regular local ring, since the functor $F^n(-)$ is exact [K, Theorem 3.3], $\operatorname{Tor}_j^A(H_i(F^n(F_\bullet)), N) \simeq \operatorname{Tor}_j^A(F^n(H_i(F_\bullet)), N)$. Thus by Proposition 1.1, the inequality in Proposition 1.2 holds for any finitely generated A-module N.

2.

Now, we demonstrate an example to show that the inequality (2) in Proposition 1.2, as well as the one in Corollary 1.3, can fail when $\operatorname{codim} N = 3$.

We first state two standard facts in commutative algebra which will be used in the proof of Proposition 2.4.

Fact 2.1. *Let R be a finitely generated algebra over a field K and M be a finitely generated R-module. Let m be a maximal ideal of R. Suppose $\operatorname{Supp} M = \{m\}$ and $K \simeq R/m$ via the natural map. Then $\ell_R(M) = \dim_K M$. Here $\dim_K M$ denote the dimension of M as a K-vector space.*

Fact 2.2. *Let R be a commutative ring and M be a finitely generated R-module. Let $\widehat{R_m}$ be the m-adic completion of R_m where m is a maximal ideal of R. If $\operatorname{Supp}_R M = \{m\}$, then*

$$\ell_R(M) = \ell_{R_m}(M_m) = \ell_{\widehat{R_m}}(\widehat{M_m}).$$

Lemma 2.3. *Let $R = K[X, Y, U, V]/(XY - UV)$ where K is a field of characteristic $p > 0$ and X, Y, U, V are indeterminates. Consider K as a module over R in the obvious way. Then $\operatorname{Hom}_R(K, F_R^n(K))$ is a K-vector space and*

$$\dim_K \operatorname{Hom}_R\big(K, F_R^n(K)\big) \geqslant p^n.$$

Proof. To simplify our notations, we use x, y, u, v to denote the images of X, Y, U, V respectively in any quotient ring of $K[X, Y, U, V]$ if there is no confusion about that ambient quotient ring. $\operatorname{Hom}_R(K, F_R^n(K))$ is a K-vector space consisting of all the elements of $F_R^n(K)$ which are killed by the maximal ideal (x, y, u, v). Let $\mathcal{A} = \{x^{p^n-1} y^i u^{p^n-1-i} \mid 0 \leqslant i \leqslant p^n - 1\}$, which is a subset of

$$F_R^n(K) = \frac{K[X, Y, U, V]}{(X^{p^n}, Y^{p^n}, U^{p^n}, V^{p^n}, XY - UV)}.$$

It is easy to verify that $\mathcal{A} \subset \operatorname{Hom}_R(K, F_R^n(K))$. We will show that elements in $\mathcal{A}$ are linearly independent over K which gives us the desired inequality.

Let $\{\lambda_i\}_{0 \leqslant i \leqslant p^n-1}$ be elements in K such that

$$\sum_{i=0}^{p^n-1} \lambda_i x^{p^n-1} y^i u^{p^n-1-i} = 0 \in F_R^n(K). \tag{5}$$

Let

$$S = \frac{K[X, Y, U, V]}{(X^{p^n}, Y^{p^n}, U^{p^n}, V^{p^n})}.$$

Then $R = S/(xy - uv)$. Lift the relation (5) to a relation in S. Since S is a K-vector space with basis $\{x^i y^j u^k v^l \mid 0 \leqslant i, j, k, l \leqslant p^n - 1\}$, we obtain

$$\sum_{i=0}^{p^n-1} \lambda_i x^{p^n-1} y^i u^{p^n-1-i} = \left(\sum_{0 \leqslant i,j,k,l \leqslant p^n-1} \mu_{i,j,k,l} x^i y^j u^k v^l \right) (xy - uv) \in S, \tag{6}$$

where the $\mu_{i,j,k,l}$ are elements of K. Define

$$\lambda_{i,j,k,l} = \begin{cases} \lambda_j, & \text{if } i = p^n - 1,\ j + k = p^n - 1 \text{ and } l = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We also define $\mu_{i,j,k,l} = 0$ if one of i, j, k, l is negative.

By comparing the coefficients on both sides of (6), we obtain that

$$\lambda_{i,j,k,l} = \mu_{i-1,j-1,k,l} - \mu_{i,j,k-1,l-1}, \quad \forall i, j, k, l \leqslant p^n - 1.$$

Using the above formula repeatedly, noticing that $\lambda_{i,j,k,l} = 0$ if $i < p^n - 1$, we get

$$\begin{aligned} \lambda_i &= \lambda_{p^n-1,i,p^n-1-i,0} \\ &= \mu_{p^n-2,i-1,p^n-1-i,0} \ + \ 0 \\ &= \mu_{p^n-3,i-2,p^n-i,1} \\ &= \mu_{p^n-4,i-3,p^n-i+1,2} \\ &\ \ \vdots \\ &= \mu_{p^n-i-1,0,p^n-2,i-1} \\ &= 0 \end{aligned}$$

for all $i = 0, 1, \ldots, p^n - 1$. □

The following is an example where the inequality (2) in Proposition 1.2 fails when codim $N = 3$. The complex $F_\bullet$ is taken to be a free resolution of K and $i = 0$.

Proposition 2.4. *Let $R = K[[X, Y, U, V]]/(XY - UV)$ where K is a field of characteristic $p > 0$ and X, Y, U, V are indeterminates. Then*

$$\ell\big(\operatorname{Tor}_3^R\big(F^n(K), R/(x, y, u + v)\big)\big) \geqslant p^n.$$

Proof. Since $\{x, y, u + v\}$ forms an R-sequence, it follows that

$$\operatorname{Tor}_3^R\big(F^n(K), R/(x, y, u + v)\big) \simeq \operatorname{Hom}_R\big(R/(x, y, u + v), F^n(K)\big).$$

Since there is a surjection $R/(x, y, u+v) \twoheadrightarrow K$, by applying $\mathrm{Hom}_R(-, F^n(K))$, we obtain an injection

$$\mathrm{Hom}_R\big(K, F^n(K)\big) \hookrightarrow \mathrm{Hom}_R\big(R/(x, y, u+v), F^n(K)\big).$$

From Facts 2.1, 2.2 and Lemma 2.3, we have

$$\ell\big(\mathrm{Hom}_R\big(K, F^n(K)\big)\big) \geqslant p^n.$$

Therefore,

$$\ell\big(\mathrm{Tor}_3^R\big(F^n(K), R/(x, y, u+v)\big)\big) \geqslant p^n. \qquad \square$$

Remark 2.5. Using the same method, one can show that over the hypersurface ring $R = K[[X_1, \ldots, X_t, Y_1, \ldots, Y_t]]/(\sum_{i=1}^t X_i Y_i)$, $\ell(\mathrm{Hom}_R(K, F^n(K)))$ is unbounded.

3.

In [D4], Dutta gave an asymptotic length condition over Gorenstein local rings of positive characteristic for the nonnegativity of $\chi_\infty(M, N)$ when $\dim M = 2$. In this section, we will construct examples to show that over the local hypersurface R discussed in Corollary 2.4, this length condition fails to hold.

Let R be a local ring in characteristic $p > 0$. Let M and N be two finitely generated modules such that $\ell(M \otimes_R N) < \infty$, $\dim M + \dim N \leqslant \dim R$ and $\operatorname{proj\,dim} M < \infty$. In [D1], Dutta defined

$$\chi_\infty(M, N) = \lim_{n\to\infty} \chi\big(F^n(M), N\big)/p^{n \operatorname{codim} M}.$$

For properties of χ_∞, see [D1,D2,R,Se]. Dutta [D4] established the following criterion for nonnegativity of χ_∞ over a local Gorenstein rings of positive characteristic.

Theorem 3.1 (Dutta). *Let R be a local Gorenstein ring in characteristic $p > 0$. Let M and N be finitely generated modules of finite projective dimension such that $\ell(M \otimes N) < \infty$. Suppose $\dim M + \dim N = \dim R$, $\dim N = \operatorname{depth} N + 1 = s$ and $\dim M = \operatorname{depth} M + 1 = 2$. Then $\chi_\infty(M, N) \geqslant 0$, if*

$$\lim_{n\to\infty} \ell\big(\mathrm{Ext}^3(N, R) \otimes H_m^0\big(F^n\big(\mathrm{Ext}^{s+1}(M, R)\big)\big)^\vee\big)/p^{ns} = 0.$$

Here $(-)^\vee$ denotes the Matlis duality $\mathrm{Hom}_R(-, E)$ where E is the injective hull of the residue field of R.

The following is an example where the length criterion in Theorem 3.1 fails.

Theorem 3.2. *Let $R = K[[X, Y, U, V]]/(XY - UV)$ where K is a field of characteristic $p > 0$ and X, Y, U, V are indeterminates. There exist finitely generated modules M, N over R satisfying the conditions in Theorem* 3.1 *such that*

$$\lim_{n\to\infty} \ell\big(\mathrm{Ext}^3(N, R) \otimes H^0_m\big(F^n\big(\mathrm{Ext}^2(M, R)\big)\big)^\vee\big)/p^n > 0.$$

Proof. We are going to construct modules M and N satisfying the conditions in Theorem 3.1 with $s = 1$, such that $\mathrm{Ext}^2_R(M, R) \simeq K$ and $\mathrm{Ext}^3_R(N, R) \simeq K$.

Let x, y, u, v denote the images of X, Y, U, V in R. Take a minimal free resolution of K over R

$$\cdots \to R^t \xrightarrow{\psi} R^4 \xrightarrow{\phi} R \to K \to 0$$

where ϕ can be written as a matrix $[x, y, u, v]$ with respect to the standard bases for R^4 and R. Let $(-)^*$ denote $\mathrm{Hom}_R(-, R)$. Apply $(-)^*$ to the above exact sequence. Since depth $R = 3$, $K^* = 0$ and we obtain the following exact sequence

$$0 \to R \xrightarrow{\phi^*} R^4 \to M' \to 0$$

where $M' = \mathrm{coker}\,\phi^*$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be a standard basis for R^4, it follows that $M' = R^4/R(x\mathbf{e}_1 + y\mathbf{e}_2 + u\mathbf{e}_3 + v\mathbf{e}_4)$. Note that if $r \in \mathrm{Ann}_R M'$, then there exists an $a \in R$ such that

$$r(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) = a(x\mathbf{e}_1 + y\mathbf{e}_2 + u\mathbf{e}_3 + v\mathbf{e}_4).$$

It follows that $ax = ay = r$. But R is a domain and $x \neq y$ in R, thus $r = 0$. Therefore $\mathrm{Ann}_R M' = (0)$ whence $\dim M' = \dim R = 3$. Moreover, since $\mathrm{Ext}^1_R(K, R) = 0$, $M' = \mathrm{Im}\,\psi^*$, which is a submodule of R^t and therefore torsion-free. Hence, $x \in R$ is a nonzero divisor on M'.

Let $M = M'/xM' = R^4/(xR^4 + R(x\mathbf{e}_1 + y\mathbf{e}_2 + u\mathbf{e}_3 + v\mathbf{e}_4))$. It follows that $\dim M = 2$. One can also prove that $\mathrm{proj\,dim}\, M = 2$ since $\mathrm{proj\,dim}\, M' = 1$ and x is both M'-regular and R-regular. Therefore by Auslander–Buchsbaum formula, depth $M = 1$. Moreover,

$$\mathrm{Ext}^2_R(M, R) \simeq \mathrm{Ext}^1_R(M', R) \simeq K.$$

In order to construct N, let $\bar{R} = R/(y, u + v)$. Then $\dim \bar{R} = 1$, depth $\bar{R} = 1$. Take a minimal resolution of K over $\bar{R}$

$$\bar{R}^2 \xrightarrow{\zeta} \bar{R} \to K \to 0.$$

Apply $\mathrm{Hom}_{\bar{R}}(-, \bar{R})$. Let $N = \mathrm{coker}\,\zeta^*$ and we obtain a free resolution of N over $\bar{R}$

$$0 \to \bar{R} \xrightarrow{\zeta^*} \bar{R}^2 \to N \to 0.$$

Use a similar argument as before, $\mathrm{Ann}_{\bar{R}} N = (0)$. Hence $\dim_{\bar{R}} N = 1$, $\mathrm{proj\,dim}_{\bar{R}} N = 1$ and $\mathrm{depth}_{\bar{R}} N = 0$. Therefore $\dim_R N = 1$, $\mathrm{depth}_R N = 0$ and $\mathrm{proj\,dim}_R N = 3$. Note that

$\ell(M \otimes_R N) < \infty$ since the annihilator of $M \otimes_R N$ contains $(x, y, u+v)$ which is primary to the maximal ideal (x, y, u, v). Moreover, $\mathrm{Ext}^3_R(N, R) \simeq \mathrm{Ext}^1_{\bar{R}}(N, \bar{R}) \simeq K$.

Finally, to check

$$\lim_{n\to\infty} \ell\big(\mathrm{Ext}^3(N, R) \otimes H^0_m\big(F^n\big(\mathrm{Ext}^2(M, R)\big)\big)^\vee\big)/p^n > 0,$$

it is enough to notice that

$$\begin{aligned}
&\ell\big(\mathrm{Ext}^3(N, R) \otimes H^0_m\big(F^n\big(\mathrm{Ext}^2(M, R)\big)\big)^\vee\big)\\
&\quad = \ell\big(\big(\mathrm{Ext}^3(N, R) \otimes H^0_m\big(F^n\big(\mathrm{Ext}^2(M, R)\big)\big)^\vee\big)^\vee\big)\\
&\quad = \ell\big(\mathrm{Hom}\big(\mathrm{Ext}^3(N, R), H^0_m\big(F^n\big(\mathrm{Ext}^2(M, R)\big)\big)\big)\big)\\
&\quad = \ell\big(\mathrm{Hom}\big(\mathrm{Ext}^3(N, R), H^0_m\big(F^n(K)\big)\big)\big)\\
&\quad = \ell\big(\mathrm{Hom}\big(K, F^n(K)\big)\big)\\
&\quad \geqslant p^n. \qquad \square
\end{aligned}$$

Remark 3.3. Although the length criterion does not hold in general, there do exist local Gorenstein rings such that the length criterion holds for all M and N. It would be nice to have a general method to identify such rings.

Acknowledgments

I am indebted to my thesis advisor Sankar Dutta for his direction and many inspiring discussions on the subject of this paper. I also thank the referee for the valuable suggestions and comments.

References

[D1] S.P. Dutta, Frobenius and multiplicities, J. Algebra 85 (2) (1983) 424–448.
[D2] S.P. Dutta, A special case of positivity, Proc. Amer. Math. Soc. 103 (2) (1988) 344–346.
[D3] S.P. Dutta, Ext and Frobenius, II, J. Algebra 186 (1995) 724–735.
[D4] S.P. Dutta, Intersection multiplicity of modules in the positive characteristics, J. Algebra 280 (2004) 394–411.
[K] E. Kunz, Characterization of regular local rings for characteristic p, Amer. J. Math. 91 (1969) 772–784.
[R] P. Roberts, Multiplicities and Chern Classes in Local Algebra, Cambridge Univ. Press, Cambridge, UK, 1998.
[Se] G. Seibert, Complexes with Homology of Finite Length and Frobenius Functors, J. Algebra 125 (1989) 278–287.
[S] J.-P. Serre, Algébre locale, multiplicités, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin, 1975.

Further reading

[H] J. Herzog, Ringe de Charakteristik p und Frobenius-Funktoren, Math. Z. 140 (1974) 67–68.
[M] P. Monsky, The Hilbert–Kunz function, Math. Ann. 263 (1983) 43–49.