The extension of the Krein–Šmulian theorem for Orlicz sequence spaces and convex sets

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Abstract

If $X$ is a Banach space and $C \subset X^{**}$ a convex subset, for $x^{**} \in X^{**}$ and $A \subset X^{**}$ let $d(x^{**}, C) = \inf\{\|x^{**} - x\|: x \in C\}$ be the distance from $x^{**}$ to $C$ and $d(A, C) = \sup\{d(a, C): a \in A\}$. In this paper we prove that if $\varphi$ is an Orlicz function, $I$ an infinite set and $X = \ell^\varphi(I)$ the corresponding Orlicz space, equipped with either the Luxemburg or the Orlicz norm, then for every $w^*$-compact subset $K \subset X^{**}$ we have $d(\overline{co} w^*(K), X) = d(K, X)$ if and only if $\varphi$ satisfies the $\Delta_2$-condition at 0. We also prove that for every Banach space $X$, every nonempty convex subset $C \subset X$ and every $w^*$-compact subset $K \subset X^{**}$ then $d(\overline{co} w^*(K), C) \leq 9d(K, C)$ and, if $K \cap C$ is $w^*$-dense in $K$, then $d(\overline{co} w^*(K), C) \leq 4d(K, C)$.


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1. Introduction

If $X$ is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of $X$, respectively, and $X^*$ its topological dual. If $C \subset X^{**}$ is a convex subset, for $x^{**} \in X^{**}$ and $A \subset X^{**}$ let $d(x^{**}, C) = \inf\{\|x^{**} - x\|: x \in C\}$ be the distance from $x^{**}$ to $C$ and $d(A, C) = \sup\{d(a, C): a \in A\}$. Observe that:

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(i) \(d(\text{co}(A), C) = d(A, C)\) where \(\text{co}(A)\) is the convex hull of \(A\);
(ii) if \(X^\perp = \{z \in X^{**}: z(x) = 0, \forall x \in X\}\) and \(Q : X^{**} \to X^{**}/X\) is the canonical quotient mapping, then:
\[
d(x^{**}, X) = \sup\{z(x^{**}): z \in S(X^\perp)\} = \|Qx^{**}\|.
\]

With this terminology, the Krein–Šmulian theorem (see [2, p. 51]) states the following: if \(X\) is a Banach space and \(K \subseteq X^{**}\) a \(w^*-\)compact subset such that \(d(K, X) = 0\) (thus, \(K\) is a weakly compact subset of \(X\)), then \(d(\overline{\text{co}}w^*(K), X) = 0\), that is, \(\overline{\text{co}}w^*(K) \subseteq X\) and \(\overline{\text{co}}w^*(K) = \overline{\text{co}}(K)\) is also a weakly compact subset of \(X\) (\(\overline{\text{co}}(K) = \|\cdot\|\)-closure of \(\text{co}(K)\) and \(\overline{\text{co}}w^*(K) = w^*-\)closure of \(\text{co}(K)\)). So, in view of this situation, we can pose two natural questions:

(A) If \(K \subseteq X^{**}\) is a \(w^*-\)compact subset, does the equality \(d(\overline{\text{co}}w^*(K), X) = d(K, X)\) always hold?

The answer to this question is negative. In fact, we constructed (under the continuum hypothesis in [4] and without axiomatic assumptions in [6]):

(i) a \(w^*-\)compact subset \(K \subseteq B(X^{**})\) such that \(K \cap X\) is \(w^*\)-dense in \(K\), \(d(K, X) = \frac{1}{2}\) and \(d(\overline{\text{co}}w^*(K), X) = 1\);
(ii) a \(w^*-\)compact subset \(K \subseteq B(X^{**})\) such that \(d(K, X) = \frac{1}{2}\) and \(d(\overline{\text{co}}w^*(K), X) = 1\).

(B) Does there exist a universal constant \(1 \leq M < \infty\) such that always \(d(\overline{\text{co}}w^*(K), X) \leq Md(K, X)\)?

The answer to this question is affirmative. In [4] we proved the following result, which extends the Krein–Šmulian theorem: if \(K \subseteq X^{**}\) is a \(w^*-\)compact subset and \(Z \subseteq X\) a subspace of \(X\) then \(d(\overline{\text{co}}w^*(K), Z) \leq 5d(K, Z)\) and, if \(Z \cap K\) is \(w^*-\)dense in \(K\), then \(d(\overline{\text{co}}w^*(K), Z) \leq 2d(K, Z)\). So, in view of these results we have:

(i) the universal constant \(M\) of our extension of the Krein–Šmulian theorem satisfies \(3 \leq M \leq 5\);
(ii) for the category of \(w^*-\)compact subsets \(K \subseteq X^{**}\) such that \(X \cap K\) is \(w^*-\)dense in \(K\), the constant \(M\) is exactly \(M = 2\).

Although the answer to question (A) is, in general, negative there are many Banach spaces \(X\) for which \(d(\overline{\text{co}}w^*(K), X) = d(K, X)\). This is the case (see [4]), for instance, if \(\ell_1 \not\subseteq X^*\), if the unit ball \(B(X^*)\) of the dual \(X^*\) is \(w^*-\)angelic (for example, if \(X\) is weakly compactly generated (WCG) or weakly Lindelöf determined (WLD)), if \(X = \ell_1(I)\), if \(K\) is fragmented by the norm of \(X^{**}\), etc. In this paper we enlarge this class of Banach spaces (for which \(d(\overline{\text{co}}w^*(K), X) = d(K, X)\) for every \(w^*-\)compact subset \(K \subseteq X^{**}\) with the Orlicz sequence spaces \(\ell_\varphi(I)\) when \(\varphi\) satisfies the \(\Delta_2\) condition at 0. In fact, we prove that if \(\varphi\) is an Orlicz function, \(I\) an infinite set and \(X = \ell_\varphi(I)\) the corresponding Orlicz space, equipped with either the Luxemburg or the Orlicz norm, then for every \(w^*-\)compact subset \(K \subseteq X^{**}\) we have \(d(\overline{\text{co}}w^*(K), X) = d(K, X)\) if and only if \(\varphi\) satisfies the \(\Delta_2\) condition at 0.

We also prove that if \(X\) is a Banach space, \(C \subseteq X\) a nonempty convex subset and \(K \subseteq X^{**}\) a \(w^*-\)compact subset, then \(d(\overline{\text{co}}w^*(K), C) \leq 9d(K, C)\) and, if \(K \cap C\) is \(w^*-\)dense in \(K\), then
2. Orlicz sequence spaces

Let \( \varphi : \mathbb{R} \rightarrow [0, +\infty] \) denote an Orlicz function, i.e., a convex function which is even, non-decreasing and left continuous for \( x \geq 0, \varphi(0) = 0 \) and \( \varphi(x) \rightarrow \infty \) as \( x \rightarrow \infty \) (see [1,7]). Define \( a(\varphi) = \sup\{t \geq 0 : \varphi(t) = 0\} \). The complementary function of \( \varphi \) is a new Orlicz function \( \psi \) defined for \( u \geq 0 \) as \( \psi(u) = \sup\{tu - \varphi(t) : 0 \leq t < \infty\} \). The Orlicz function \( \varphi \) satisfies the \( \Delta_2 \) condition at 0 (for short, \( \varphi \in \Delta_2^0 \)) if \( \limsup_{t \to 0} \frac{\varphi(2t)}{\varphi(t)} < \infty \).

If \( I \) is an infinite set, let \( \beta I \) be the Stone–Čech compactification of \( I \) and \( I^* = \beta I \setminus I \). If \( J \subset I \) we denote \( J^* = \overline{J} \setminus J \subset I^* \). \( J^* \) is homeomorphic to \( \beta J \setminus J \). For \( x \in \mathbb{R}^I \), define \( I_\varphi(x) = \sum_{i \in I} \varphi(x_i) \). Let \( \ell_\varphi(I) \) be the corresponding Orlicz space, i.e., \( \ell_\varphi(I) = \{x \in \mathbb{R}^I : \exists \lambda > 0 \text{ such that } I_\varphi(x/\lambda) < \infty\} \). In \( \ell_\varphi(I) \) we consider the Luxemburg norm \( \| \cdot \|_O \):

\[
\|x\|_O = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq 1\},
\]

as well as the Orlicz norm \( \| \cdot \|_L \):

\[
\|x\|_L = \sup \left\{ \sum_{i \in I} x_i y_i : y \in \mathbb{R}^I, I_\varphi(y) \leq 1 \right\}.
\]

It is known that, \( \forall x \in \ell_\varphi(I) \), \( \|x\|_L \leq \|x\|_O \leq 2\|x\|_L \) and that with both norms \( \ell_\varphi(I) \) is a Banach space. We denote \( (\ell_\varphi(I), \| \cdot \|_L) =: \ell_\varphi^L(I) \) and \( (\ell_\varphi(I), \| \cdot \|_O) =: \ell_\varphi^O(I) \). Define \( h_\varphi(I) \) by

\[
h_\varphi(I) = \{x \in \ell_\varphi(I) : \forall \lambda > 0, I_\varphi(x/\lambda) < \infty\},
\]

and recall that \( h_\varphi(I) \) is the closed subspace (and the closed ideal) generated by the vectors \( e_j, j \in I, \) where \( e_j(i) = 1, \) if \( j = i, \) and \( e_j(i) = 0, \) otherwise. In fact, the family \( \{e_j\}_{j \in I} \) is a symmetric basis of \( h_\varphi(I) \). Denote \( h_\varphi^L(I), \| \cdot \|_L := h_\varphi^L(I) \) and \( h_\varphi^O(I), \| \cdot \|_O := h_\varphi^O(I) \). Recall that \( h_\varphi(I) = \ell_\varphi(I) \) iff \( \varphi \in \Delta_2^0 \).

Let see the dual of an Orlicz space. It is well known that \( (\ell_\varphi^O(I))^* = \ell_\varphi^O(I) \) and \( (h_\varphi^O(I))^* = \ell_\varphi^L(I) \). As \( \ell_\varphi(I) \) is a Köthe function space, the dual of \( \ell_\varphi(I) \) is the monotone direct sum of the space of integral functionals \( \ell_\varphi(I) \) and the space of singular functionals \( (h_\varphi(I))^\perp \). In more detail, we have

\[
(\ell_\varphi^L(I))^* = \ell_\varphi^O(I) \oplus F(I) \quad \text{and} \quad (\ell_\varphi^O(I))^* = \ell_\varphi^L(I) \oplus F(I),
\]

where \( F(I) = (h_\varphi(I))^\perp \). The norm of \( F(I) \) (see [5]) is the same, considered whether a subspace of \( (\ell_\varphi^O(I))^* \) or whether a subspace of \( (\ell_\varphi^L(I))^* \), and with this norm

\[
F(I) \cong \left( \frac{\ell_\varphi^L(I)}{h_\varphi^L(I)} \right)^* \cong \left( \frac{\ell_\varphi^O(I)}{h_\varphi^O(I)} \right)^*.
\]

(\( \cong \) means isometric isomorphism). The structure of \( F(I) \) depends on the number \( a(\varphi) \) (see [5]):

(A) If \( a(\varphi) > 0 \), then \( F(I) \) is isometrically isomorphic to the space \( M_R(I^*) \) of Radon measures on \( I^* \), i.e., \( F(I) = M_R(I^*) = (C(I^*))^* \).

(B) If \( a(\varphi) = 0 \), then \( F(I) \) is isometrically isomorphic to the subspace of Radon measures \( \nu \) on \( \beta I \) such that \( \nu(i) = 0, \forall i \in I \) (i.e., \( \nu \in M_R(I^*) \)), and there exists a sequence \( \{G_k\}_{k \geq 1} \) of pairwise disjoint finite subsets of \( I \) satisfying:

Then there exist $z \in \mathbb{R}^d$ such that $|G_k| = \text{card}(G_k)$.

(ii) $\sum_{k \geq 1} \varphi(\frac{1}{k}) \cdot |G_k| < \infty$, where $|G_k| = \text{card}(G_k)$.

(iii) $\sum_{k \geq 1} \varphi(\frac{1}{k}) \cdot |G_k \cap E| = \infty$, for every $n \geq 1$ and every $E \subseteq I$ such that $|v|(|I| \setminus I) > 0$.

If $I \subset I$ is an infinite subset, we consider the space $Y_I := h^1_{\varphi}(I)$ as a complemented subspace of $h^1_{\varphi}(I)$, with projection $P_I : h^1_{\varphi}(I) \rightarrow h^1_{\varphi}(I)$ such that, $\forall f \in h^1_{\varphi}(I)$, $P_I(f) = f|_I = \text{restriction of } f \text{ to } I$. Notice that $\|P_I\| = 1$. Clearly, $(h^1_{\varphi}(I))^{**} = \ell^1_{\varphi}(I) \oplus F(I)$, $Y_I^{**} = \ell^1_{\varphi}(I) \oplus F(I)$ and the space $Y_I^{**}$ can be considered as a complemented subspace of $(h^1_{\varphi}(I))^{**}$ by means of the projection $P_I^{**} : (h^1_{\varphi}(I))^{**} \rightarrow (h^1_{\varphi}(I))^{**}$ such that, for every $f = v + w$ with $v \in \ell^1_{\varphi}(I)$ and $w \in F(I)$, then $P_I^{**}(f) = v|_I + w|_I$. So, $\|P_I^{**}\| = 1$ and $P_I^{**}$ is $w^*-w^*$-continuous.

If instead of the Luxemburg norm $\|\cdot\|$, we consider the Orlicz norm $\|\cdot\|_o$ on the spaces $h^0_{\varphi}(J)$ and $h^1_{\varphi}(I)$, the behaviour of $P_J$ with this norm is similar. In particular, $\|P_J\| = \|P_J^{**}\| = 1$ with the Orlicz norm.

3. The extension of the Krein–Šmulian theorem for the spaces $h_{\varphi}(I)$

The following lemma is a reduced version of [4, Lemma 13]. We include it for the sake of completeness.

**Lemma 1.** Let $X$ be a Banach space such that there exist $a, b > 0$ and a $w^*$-compact subset $K \subset B(X^{**})$ with $d(K, X) < a < b < d(\overline{co}^{w^*}(K), X)$. Then there exist $z_0 \in S(X^\perp)$ and a $w^*$-compact subset $\emptyset \neq H \subset K$ such that for every $w^*$-open subset $V$ with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{co}^{w^*}(V \cap H)$ with $z_0(\xi) > b$.

**Proof.** Since $b < d(\overline{co}^{w^*}(K), X)$, there exists $u_0 \in \overline{co}^{w^*}(K)$ such that $d(u_0, X) > b$. Thus, we can find $z_0 \in S(X^\perp)$ so that $z_0(u_0) > b + \epsilon$ for some $\epsilon > 0$. By the Bishop–Phelps theorem, there exists $z_1 \in S(X^{***})$ such that $\|z_0 - z_1\| \leq \epsilon/4$ and $z_1$ attains its maximum on $\overline{co}^{w^*}(K)$ in some $u_1 \in \overline{co}^{w^*}(K)$. So,

\[ z_1(u_1) \geq z_1(u_0) = z_0(u_0) + (z_1 - z_0)(u_0) > b + \epsilon - \frac{1}{4} \epsilon = b + \frac{3}{4} \epsilon, \]

\[ z_0(u_1) = z_1(u_1) + (z_0 - z_1)(u_1) > b + \frac{3}{4} \epsilon - \frac{1}{4} \epsilon = b + \frac{1}{2} \epsilon, \]

and for every $k \in K$,

\[ z_1(k) = z_0(k) + (z_1 - z_0)(k) \leq d(k, X) + \frac{1}{4} \epsilon < a + \frac{1}{4} \epsilon < b + \frac{3}{4} \epsilon < z_1(u_1). \]

Since $z_0(u_1) > b + \frac{1}{2} \epsilon$ and $\sup \{|z_0(k)| : k \in K\} \leq d(K, X) < a < b$, we get that $u_1 \notin K$. If $v$ is a Radon probability on $K$, denote by $r(v)$ the barycenter of $v$ and recall that: (i) $r(v) \in \overline{co}^{w^*}(K)$; (ii) for every $u \in \overline{co}^{w^*}(K)$ there exists a Radon probability $v$ on $K$ with $r(v) = u$. So, as $u_1 \in \overline{co}^{w^*}(K)$, we can find a Radon probability $\mu$ on $K$ such that $r(\mu) = u_1$. 

\[ z_0(u_1) > b + \frac{1}{2} \epsilon \text{ and } \sup \{|z_0(k)| : k \in K\} \leq d(K, X) < a < b, \text{ we get that } u_1 \notin K. \text{ If } v \text{ is a } \text{Radon probability on } K, \text{ denote by } r(v) \text{ the barycenter of } v \text{ and recall that: (i) } r(v) \in \overline{co}^{w^*}(K); \text{ (ii) for every } u \in \overline{co}^{w^*}(K) \text{ there exists a Radon probability } v \text{ on } K \text{ with } r(v) = u. \text{ So, as } u_1 \in \overline{co}^{w^*}(K), \text{ we can find a Radon probability } \mu \text{ on } K \text{ such that } r(\mu) = u_1. \]
Claim. \( \mu \) is atomless.

Indeed, suppose that \( \mu \) has mass \( 0 < \lambda \leq 1 \) on some \( k_0 \in K \), i.e., \( \mu = \lambda \cdot \delta_{k_0} + \mu_1, \mu_1 \geq 0 \). If \( \lambda = 1 \) then \( \mu = \delta_{k_0} \), whence \( r(\mu) = k_0 \in K \), which is impossible because \( r(\mu) = u_1 \notin K \). So, \( 0 < \lambda < 1 \), i.e., \( \mu_1 \neq 0 \) and \( \|\mu_1\| = 1 - \lambda > 0 \). Then \( \mu = \lambda \cdot \delta_{k_0} + (1 - \lambda) \frac{\mu_1}{\|\mu_1\|} \) and

\[
u_1 = r(\mu) = \lambda k_0 + (1 - \lambda)r \left( \frac{\mu_1}{\|\mu_1\|} \right).
\]

So, as \( z_1(k_0) < z_1(u_1) \) and \( z_1(r(\frac{\mu_1}{\|\mu_1\|})) \leq z_1(u_1) \), we get

\[
z_1(u_1) = \lambda z_1(k_0) + (1 - \lambda)z_1 \left( r \left( \frac{\mu_1}{\|\mu_1\|} \right) \right) < \lambda z_1(u_1) + (1 - \lambda)z_1(u_1) = z_1(u_1),
\]
a contradiction.

Let \( H = \text{supp}(\mu) \) be the support of \( \mu \) and suppose that there exists a \( w^* \)-open subset \( V \subset X^{**} \) with \( V \cap H \neq \emptyset \) such that \( z_0(\xi) \leq b \), for every \( \xi \in \overline{co}w^*(V \cap H) \). Denote \( \mu_1 = \mu |_{V \cap H} \) and \( \mu_2 := \mu - \mu_1 \). Observe that \( \mu_1 \neq 0 \) (because \( V \) is \( w^* \)-open and \( \emptyset \neq V \cap H = V \cap \text{supp}(\mu) \)) and \( \mu_2 \neq 0 \) (if \( \mu_2 = 0 \), then \( \mu = \mu_1 \) and \( u_1 = r(\mu) \in \overline{co}w^*(V \cap H) \), which is not true because \( z_0(u_1) > b + \frac{1}{4} \varepsilon \)). Then we have the decomposition \( \mu = \mu_1 + \mu_2 \) with \( \|\mu_1\| > 0 \), \( \|\mu_2\| > 0 \) and \( 1 = \|\mu\| = \|\mu_1\| + \|\mu_2\| \). So,

\[
u_1 = r(\mu) = \|\mu_1\| \cdot r \left( \frac{\mu_1}{\|\mu_1\|} \right) + \|\mu_2\| \cdot r \left( \frac{\mu_2}{\|\mu_2\|} \right).
\]

Since \( r(\frac{\mu_1}{\|\mu_1\|}) \in \overline{co}w^*(V \cap H) \), then \( z_0 \left( r(\frac{\mu_1}{\|\mu_1\|}) \right) \leq b \), whence \( z_1 \left( r(\frac{\mu_1}{\|\mu_1\|}) \right) \leq b + \frac{1}{4} \varepsilon \) because \( \|z_0 - z_1\| \leq \frac{1}{4} \varepsilon \). Therefore,

\[
z_1(u_1) = \|\mu_1\| z_1 \left( r \left( \frac{\mu_1}{\|\mu_1\|} \right) \right) + \|\mu_2\| z_1 \left( r \left( \frac{\mu_2}{\|\mu_2\|} \right) \right) \\
\leq \|\mu_1\| \left( b + \frac{1}{4} \varepsilon \right) + \|\mu_2\| z_1(u_1) < \|\mu_1\| z_1(u_1) + \|\mu_2\| z_1(u_1) = z_1(u_1),
\]
a contradiction. Thus, for every \( w^* \)-open subset \( V \) with \( V \cap H \neq \emptyset \) there exists \( \xi \in \overline{co}w^*(V \cap H) \) with \( z_0(\xi) > b \). \( \Box \)

Proposition 2. Let \( I \) be an infinite set, \( \varphi \) an Orlicz function and either \( X = h^L_\varphi(I) \) or \( X = h^0_\varphi(I) \). Then for every \( w^* \)-compact subset \( K \subset X^{**} \) we have \( d(K, X) = d(\overline{co}w^*(K), X) \).

Proof. Denote by \( \psi \) the complementary Orlicz function of \( \varphi \). We consider different cases.

Case 1. \( a(\varphi) > 0 \). In this case \( X \) is a WCG (= weakly compactly generated) space because it is isomorphic to \( c_0(I) \). So, the result holds because it is true for the class of WCG Banach spaces (see [4]).

Case 2. \( a(\varphi) = 0 = a(\psi) \).

(A) We first consider the case \( X = h^L_\varphi(I) \). Now \( X^* = \ell^0_\varphi(I) \) and \( X^{**} = \ell^L_\varphi(I) \oplus F(I) \), where \( F(I) \) is a subspace of \( M_R(I^*) \) as described before.

Suppose that there exist a \( w^* \)-compact subset \( K \subset X^{**} \) and a vector \( z_0 \in \overline{co}w^*(K) \) such that \( d(z_0, X) > b > a > d(K, X) \). Let \( z_0 = v_0 + w_0 \) with \( v_0 \in \ell^L_\varphi(I) \) and \( w_0 \in F(I) \). Since \( a(\varphi) = 0 \),
the support \(\text{supp}(v_0) = \{i \in I: v_0(i) \neq 0\}\) is countable. Also, as \(a(\psi) = 0\), there exists a countable subset \(I_0 \subset I\) such that, in fact, \(w_0 \in M_R(I_0^*)\). Let \(J = \text{supp}(v_0) \cup I_0\), a countable subset of \(I\). If \(Y_J = h^{I}_{\psi}(J)\), consider \(Y_J\) as a complemented subspace of \(X\) and \(Y_J^{**}\) as a complemented subspace of \(X^{**}\) by means of the projection \(P_{J}^{**}: X^{**} \to X^{**}\) described in Section 2. Notice that \(P_{J}^{**}(z_0) = z_0\). Clearly, \(L := P_{J}^{**}(K)\) is a \(w^*\)-compact subset of \(X^{**}\) (in fact, of \(Y_J^{**} := (h^{I}_{\psi}(J))^{**}\) such that \(z_0 \in \overline{co}w^*(L)\) and \(d(z_0, Y_J) \geq d(z_0, X) > b\). On the other hand, for every \(k \in K\) there exists \(x \in X\) with \(\|k - x\| < a\), whence \[
\|P_{J}^{**}(k) - P_{J}^{**}(x)\| \leq \|k - x\| < a.
\]
So, \(d(L, Y_J) \leq d(K, X) < a\). Since \(Y_J = h^{I}_{\psi}(J)\) is separable, we obtain a contradiction (because for every separable Banach space \(Z\) and every \(w^*\)-compact subset \(K \subset Z^{**}\) we have \(d(K, Z) = d(\overline{co}w^*(K), Z)\) (see [4])).

(B) The case \(X = h^{I}_{\psi}(I)\) follows using analogous arguments.

**Case 3.** \(a(\varphi) = 0, a(\psi) > 0\). Let \(X = h^{I}_{\psi}(I)\) and \(\|\cdot\|\) be either the Luxemburg norm or the Orlicz norm. We have:

(a) Since \(a(\psi) > 0\), then \(\varphi \in \Delta^0\), \(a(\varphi) = 0\) and the right derivative \(\varphi'_d\) of \(\varphi\) at 0 satisfies \(\varphi'_d(0) = a(\psi) > 0\), whence \(X \simeq \ell_1(I)\) (isomorphism). In fact, the canonical inclusion \(i : h^{I}_{\psi}(I) \to \ell_1(I)\) is an isomorphism. Hence the adjoint operator \(i^* : \ell_\infty(I) \to X^* = \ell_{\varphi}(I)\) is also an isomorphism.

(b) We know that \(X^{**} = X \oplus F(I)\) with \(F(I) = M_R(I^*)\). So, if \(f = v + w \in X^{**}\) with \(v \in X\) and \(w \in M_R(I^*)\), then \(\|f\| \geq \sup\{\|v\|, \|w\|\}\), because the direct sum \(X \oplus F(I)\) is monotone. Thus, \(d(f, X) = \|w\|\).

Suppose that there exists a \(w^*\)-compact subset \(K \subset B(X^{**})\) such that \(d(\overline{co}w^*(K), X) > b > a > d(K, X)\). By Lemma 1 we have:

**Fact.** There exist \(z \in S(X^{\perp})\) and a \(w^*\)-compact subset \(\emptyset \neq H \subset K\) such that for every \(w^*\)-open subset \(V\) with \(V \cap H \neq \emptyset\) there exists \(\xi \in \overline{co}w^*(V \cap H)\) with \(z(\xi) > b\).

**Step 1.** By the Fact there exists \(\xi_1 \in \overline{co}w^*(H)\) with \(z(\xi_1) > b\). As \(B(X^{**})\) is \(w^*\)-dense in \(B(X^{***})\), we can find a vector \(x_1 = x_1^* (\xi_1) > b\). Since \(\xi_1 \in \overline{co}w^*(H)\) and \(x_1 = x_1^*(\xi_1) > b\), we can choose \(\eta_1 \in H\) such that \(x_1^* (\eta_1) > b\). If \(\eta_1 = v_1 + w_1\) with \(v_1 \in X\) and \(w_1 \in M_R(I^*)\), then \(a > d(\eta_1, X) = \|w_1\|\) and \(\|v_1\| > b - a\), because \(\|\eta_1\| \geq x_1^*(\eta_1) > b\). So, as \(v_1 \in X\) is isomorphic to \(\ell_1(I)\) and \(\|v_1\| > b - a\), we can find \(y_1 \in B(X^{**})\) (\(X^{**}\) is isomorphic to \(\ell_\infty(I)\) with finite support \(\text{supp}(y_1)\) (we say \(\text{supp}(y_1) = \{y_1(1), \ldots, y_1(p_1) \subset I\}\) such that \(y_1(v_1) > b - a\). Since \(y_1 \in h_{\psi}(I)\) and \(w_1 \in F(I) = (h_{\psi}(I))^\perp\), we have \(y_1(\eta_1) = y_1(v_1) > b - a\).

**Step 2.** Let \(V_1 = \{u \in X^{**}: y_1(u) > b - a\}\), which is a \(w^*\)-open subset of \(X^{**}\) with \(V_1 \cap H \neq \emptyset\), because \(\eta_1 \in V_1 \cap H\). By the Fact there exists \(\xi_2 \in \overline{co}w^*(V_1 \cap H)\) with \(z(\xi_2) > b\). Since \(z(\xi_2) > b\) and \(z(e_{\gamma_1}) = 0, 1 \leq i \leq p_1\) (where \(e_i \in X = h_{\psi}(I)\) is the unit vector such that \(e_i(i) = 1, i = j, p_1\)) otherwise, we can find \(x_2 = x_2^*(\xi_2) > b\) and \(x_2(e_{\gamma_1}) = 0, 1 \leq i \leq p_1\). Clearly, we can choose \(\eta_2 \in V_1 \cap H\) such that \(x_2^*(\eta_2) > b\) and, also, \(y_1(\eta_2) > b - a\) because \(\eta_2 \in V_1\). Let \(\eta_2 = v_2 + w_2\), with \(v_2 \in X\), \(w_2 \in M_R(I^*)\) and \(\|w_2\| = d(\eta_2, X) < a\). Since \[
|\gamma_2(v_2)| = |\gamma_2(\eta_2) - x_2^*(w_2)| \geq |\gamma_2(\eta_2)| - |x_2^*(w_2)| > b - a
\]
and \( x_2^* = 0 \) on \( \text{supp}(y_1) \), we can find \( y_2 \in B(X^*) \) with finite support disjoint from \( \text{supp}(y_1) \) (we say \( \text{supp}(y_2) = \{y_{21}, \ldots, y_{2p_2}\} \subseteq I \setminus \text{supp}(y_1) \)) such that \( y_2(\eta_2) = y_2(\nu_2) > b - a \).

By reiteration, we obtain a sequence \( \{y_i\}_{i \geq 1} \subseteq B(X^*) \) with pairwise disjoint supports, a sequence of \( w^* \)-open subsets \( \{V_k\}_{k \geq 1} \) with \( V_k = \{u \in X^*: y_i(u) > b - a, \; i = 1, 2, \ldots, k\} \) and \( V_k \cap H \neq \emptyset \), and a sequence \( \{\eta_k\}_{k \geq 1} \) with \( \eta_{k+1} \in V_k \cap H \subseteq B(X^*) \) such that \( y_n(\eta_k) > b - a \) for \( k \geq n \).

Since \( X^* = \ell_\psi(I) \) is canonically isomorphic to \( \ell_\infty(I) \) and the elements of the sequence \( \{y_k\}_{k \geq 1} \subseteq B(X^*) \) have pairwise disjoint supports, there exists a real number \( 0 < M < \infty \) such that \( \|\sum_{i=1}^n y_i\| \leq M, \forall n \geq 1 \). On the other hand, \( (\sum_{i=1}^n y_i)(\eta_n) > n(b - a), \forall n \geq 1 \), which implies \( \|\eta_n\| > \frac{n}{M}(b - a), \forall n \geq 1 \), a contradiction because \( \|\eta_n\| \leq 1, \forall n \geq 1 \). \( \square \)

4. The extension of the Krein–Šmulian theorem for the spaces \( \ell_\varphi(I) \)

First we consider the special Orlicz sequence space \( X = \ell_\infty(I) \). In [6] it is asked whether every \( w^* \)-compact subset \( K \subseteq (\ell_\infty(I))^{**} \) satisfies \( d(K, \ell_\infty(I)) = d(\overline{\sigma w^*}(K), \ell_\infty(I)) \). In the following proposition we show that the answer is negative.

**Proposition 3.** Let \( \Gamma \) be an infinite set and \( X = \ell_\infty(\Gamma) \) with the supremum norm \( \|f\| = \sup\{|f(i)|: \; i \in \Gamma, \; \forall f \in \ell_\infty(\Gamma)\} \). Then there exists a \( w^* \)-compact subset \( K \subseteq B(X^{**}) \) such that \( d(K, X) \leq \frac{1}{2} \) but \( d(\overline{\sigma w^*}(K), X) \geq \frac{2}{3} \).

**Proof.** Without loss of generality we suppose that \( \Gamma = \mathbb{N} \) and denote \( \ell_\infty(\mathbb{N}) =: \ell_\infty \). Set \( I = [0, 1] \) and let \( \lambda \) denote the Lebesgue measure on \( I \). We use the family of continuous functions \( g_{[i_1, \ldots, i_n]}: I \rightarrow I \) introduced in [6], namely, for every \( n \geq 1 \) consider the family of \( n^{2n} \) continuous functions \( g_{[i_1, \ldots, i_n]}: I \rightarrow I, \; i_j \in \{0, \ldots, n^2 - 1\}, \) such that:

\[
\begin{align*}
(a) & \quad g_{[i_1, \ldots, i_n]} \left( \frac{ij}{n^2}, \frac{i_j + 1}{n^2} \right) = [0] \quad \text{and} \\
(b) & \quad g_{[i_1, \ldots, i_n]}(t) = 1 \quad \text{whenever} \quad \min_{1 \leq j \leq n} \left| t - \frac{i_j}{n^2} \right| \geq \frac{2}{n^2}.
\end{align*}
\]

It is easy to verify that:

(A) \( \lambda(g_{[i_1, \ldots, i_n]}^{-1}(1)) \geq 1 - \frac{4}{n} \).
(B) For every \( t_1, \ldots, t_k \in I \) pairwise distinct, every subset \( A \subseteq \{t_1, \ldots, t_k\} \) and every \( m \geq 1 \) there exist \( n \geq m \) and \( g_{[i_1, \ldots, i_n]} \) such that:

\[
g_{[i_1, \ldots, i_n]}(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \in \{t_1, \ldots, t_k\} \setminus A. \end{cases}
\]

To see (B), it suffices to take \( n \) such that \( \frac{4n^2}{n^2} < \min_{i \neq j} |t_i - t_j| \) and apply conditions (a) and (b).

Reindex the collection \( \{g_{[i_1, \ldots, i_n]}: n \in \mathbb{N}, \; i_j \in \{0, \ldots, n^2 - 1\}\} \) as \( \{h_n\}_{n \in \mathbb{N}} \). Consider \( B(\ell_\infty) \) equipped with the topology \( \tau_p \) of the pointwise convergence. Notice that \( \tau_p \) coincides on \( B(\ell_\infty) \) with the \( w^* \)-topology \( \sigma(\ell_\infty, \ell_1) \). So, \( (B(\ell_\infty), \tau_p) \) is a metric compact space. Let \( H: [0, 1] = I \rightarrow (B(\ell_\infty), \tau_p) \) be such that \( H(t) = (h_n(t))_{n \geq 1}, \forall t \in I \). Since \( H \) is injective and continuous, \( L := H(I) \) is a compact subset homeomorphic to \( I \). Denote by \( \mu \) the Radon probability on \( L \), image by \( H \) of the Lebesgue probability \( \lambda \) on \( I \), and let \( z_0 \in \overline{\sigma w^*}(L) \) be the barycenter of \( \mu \).
If \( f \in \ell_\infty \), denote by \( \hat{f} \in C(\beta^N) \) the Stone–Čech continuous extension of \( f \) to \( \beta^N \). By (B), for every finite subset \( F \subset L \) we have \((\cap_{f \in F} \hat{f}^{-1}(0)) \cap N^* \neq \emptyset \) (\( N^* = \beta^N \setminus N \)). So, by compactness we get \( O := (\cap_{f \in L} \hat{f}^{-1}(0)) \cap N^* \neq \emptyset \).

**Claim 1.** \( \forall \delta > 0, \exists n_0 \in \mathbb{N} \) such that, \( \forall n \geq n_0, z_0(n) \geq 1 - \delta \). So, \( \hat{z}_0 = 1 \) on \( N^* \).

Indeed, by (A) there exists \( n_0 = n_0(\delta) \) such that, \( \forall n \geq n_0, \lambda(h_{n^{-1}}) \geq 1 - \delta \). So, for \( n \geq n_0 \) we have

\[
\begin{align*}
z_0(n) &= \int \limits_L x(n) \cdot d\mu = \int \limits_f H(t)(n) \cdot d\lambda = \int \limits_f h_n(t) \cdot d\lambda \geq \lambda(h_{n^{-1}}(1)) \geq 1 - \delta.
\end{align*}
\]

If \( X = \ell_\infty \), then \( X^* = \ell_1 \oplus M_R(N^*) \) and \( X^{**} = \ell_\infty \oplus M_R(N^*)^* \). Denote by \( \pi_1 : X^{**} \to \ell_\infty \), \( \pi_2 : X^* \to M_R(N^*)^* \) the canonical projections. So, if \( u \in X^{**} \), then \( u = (u_1, u_2) \) with \( u_1 = \pi_1(u) \) and \( u_2 = \pi_2(u) \). Observe that, if \( j : X \to X^{**} \) is the canonical embedding of \( X \) in its bidual and \( f \in X \), then \( j(f) = (f_1, f_2) \) with \( f_1 = \pi_1 \circ j(f) = f \) and \( f_2 = \pi_2 \circ j(f) = \hat{f}|\mathbb{N}^* \), where \( \hat{f}|\mathbb{N}^* \) is considered as an element of \( M_R(N^*)^* \).

Let \( \phi : \ell_\infty \to X^{**} \) be such that, \( \forall f \in \ell_\infty, \phi(f) = (f, 0) \), which is a linear \( w^*-w^* \)-continuous and \( \| \cdot \| \)-continuous mapping. Denote \( L_0 := \phi(L) = \{(f, 0) : f \in L \} \subset B(X^{**}) \). Looking at \( \frac{1}{2}1_{N^* \setminus O} \) and \( \frac{1}{2}1_{O} \) (recall that \( O := (\cap_{f \in L} \hat{f}^{-1}(0)) \cap N^* \)) as elements of \( M_R(N^*)^* \), consider the subset \( K = L_0 + (0, \frac{1}{2}1_{N^* \setminus O} - \frac{1}{2}1_{O}) \subset B(X^{**}) \), which is a \( w^*- \)compact subset of \( B(X^{**}) \) homeomorphic to \( L_0 \). Notice that \( (z_0, \frac{1}{2}1_{N^* \setminus O} - \frac{1}{2}1_{O}) \in \partial w^*(K) \).

**Claim 2.** \( d(K, X) \leq \frac{1}{3} \).

Indeed, pick \((f, \frac{1}{2}1_{N^* \setminus O} - \frac{1}{2}1_{O}) \in K \) with \( f \in L \). Since \( \hat{f}|O = 0 \) we have

\[
\left\| \left( f, \frac{1}{3}1_{N^* \setminus O} - \frac{1}{3}1_{O} \right) - \frac{2}{3}j(f) \right\| = \sup \left\{ \left\| \frac{1}{2}f \right\|, \left\| \frac{1}{3}1_{N^* \setminus O} \right\| \right\} \frac{1}{3} = \frac{1}{3}.
\]

**Claim 3.** \( d((z_0, \frac{1}{2}1_{N^* \setminus O} - \frac{1}{2}1_{O}), X) = \frac{2}{3} \).

Indeed, by Claim 1 we have \( \hat{z}_0 = 1 \) on \( N^* \), whence

\[
\left\| \left( z_0, \frac{1}{3}1_{N^* \setminus O} - \frac{1}{3}1_{O} \right) - \frac{2}{3}j(z_0) \right\| = \left\| \left( \frac{2}{3}z_0 - \frac{2}{3}1_{O} \right) \right\| = \frac{2}{3}.
\]

On the other hand, if \( c \in O \) and \( f \in \ell_\infty \), then

\[
\left\| \left( z_0, \frac{1}{3}1_{N^* \setminus O} - \frac{1}{3}1_{O} \right) - j(f) \right\| \geq \sup \left\{ |1 - \hat{f}(c)|, \left| \frac{1}{3} + \hat{f}(c) \right| \right\} \geq \frac{2}{3}.
\]

**Proposition 4.** Let \( I \) be an infinite set and \( \varphi \) an Orlicz function.

1. If \( \varphi \in \Delta_2^0 \) and either \( X = \ell_{\varphi}^p(I) \) or \( X = \ell_{\varphi}^w(I) \), then for every \( w^* \)-compact subset \( K \subset X^{**} \) we have \( d(\partial w^*(K), X) = d(K, X) \).
2. If \( \varphi \notin \Delta_2^0 \) and \( X = \ell_{\varphi}^p(I) \), then there exists a \( w^* \)-compact subset \( K \subset B(X^{**}) \) such that \( d(\partial w^*(K), X) \geq 2d(K, X) > 0 \).
(3) If \( \varphi \notin \Delta^o_2 \) and \( X = \ell^o_\varphi(I) \), then for every \( \epsilon > 0 \) there exists a \( w^* \)-compact subset \( K_\epsilon \subset B(X^{**}) \) such that \( d(\overline{\text{co}}w^*(K_\epsilon), X) \geq (2 - \epsilon)d(K_\epsilon, X) > 0 \).

**Proof.** (1) As \( \varphi \in \Delta^o_2 \), then \( \ell^o_\varphi(I) = h_\varphi(I) \). So, this part follows from Proposition 2.

(2) In this case it is well known that there exists in \( X \) a complemented isometric copy \( Y \) of \( \ell_\infty \) with projection \( P : X \to Y \) such that \( \|P\| = 1 \). So, if \( Y^{**} \) is considered as a subspace of \( X^{**} \) (in fact, \( Y^{**} = Y^{w^*} \) inside \( X^{**} \)) and \( K \subset Y^{**} \) is the \( w^* \)-compact set constructed in Proposition 3, then \( d(\overline{\text{co}}w^*(K), X) \geq 2d(K, X) > 0 \), because for every \( u \in Y^{**} \) we have \( d(u, Y) = d(u, X) \).

(3) Since \( \varphi \notin \Delta^o_2 \), it is well known that there exists in \( \ell^o_\varphi(I) \) an isomorphic copy of \( \ell_\infty \). In [8] it has been proved that any Banach space isomorphic to \( \ell_\infty \) contains subspaces arbitrarily nearly isometric to \( \ell_\infty \). So, in this case for every \( \delta > 0 \) there exists in \( X \) a complemented subspace \( Y_\delta \) which is \( (1 + \delta) \)-isometric to \( \ell_\infty \), with projection \( P_\delta \) such that \( \|P_\delta\| \leq 1 + \delta \). Thus, using the same argument as in (2), for every \( \epsilon > 0 \) we can construct a \( w^* \)-compact subset \( K_\epsilon \subset X^{**} \) such that \( d(\overline{\text{co}}w^*(K_\epsilon), X) \geq (2 - \epsilon)d(K_\epsilon, X) > 0 \). Notice that if the Orlicz function \( \varphi \) is strictly convex and \( (\varphi(t)/t) \to 0 \) as \( t \to 0 \), then the Orlicz norm of \( \ell^o_\varphi(I) \) is strictly convex (see [1, p. 55]) and \( \ell^o_\varphi(I) \) cannot contain an isometric copy of \( \ell_\infty \), although it contains an isomorphic copy if \( \varphi \notin \Delta^o_2 \). \( \square \)

5. The extension of the Krein–Šmulian theorem for convex subsets

If \( X \) is a Banach space, \( K \subset X^{**} \) a \( w^* \)-compact subset and \( C \subset X^{**} \) a convex subset, we consider in this section the distances \( d(K, C) \) and \( d(\overline{\text{co}}w^*(K), C) \). In view of our extension of the Krein–Šmulian theorem, which deals with this type of distances when \( C \) is a subspace of \( X \), it is natural to ask whether there exists some constant \( 1 \leq M < \infty \) such that always \( d(\overline{\text{co}}w^*(K), C) \leq Md(K, C) \), when \( C \) is a convex subset. The answer to this question is the following:

(a) For the category of convex subsets \( C \subset X^{**} \) there is not such a constant. In fact, if \( L \subset B(\ell_\infty) \) is the \( w^* \)-compact constructed in Proposition 3, \( \mathcal{O} := (\bigcap_{f \in L} \tilde{f}^{-1}(0)) \cap \mathbb{N}^* \) and \( Y_\mathcal{O} \) is the closed subspace of \( \ell_\infty \) such that \( Y_\mathcal{O} = \{ f \in \ell_\infty : \tilde{f}|\mathcal{O} = 0 \} \), then, clearly, \( L \subset Y_\mathcal{O} \) and \( d(L, Y_\mathcal{O}) = 0 \). On the other hand, since \( \tilde{z}_0 = 1 \) on \( \mathbb{N}^* \) (see Proposition 3), then \( d(z_0, Y_\mathcal{O}) = 1 \) and, so, \( d(\overline{\text{co}}w^*(L), Y_\mathcal{O}) \geq 1 \). In [4, Corollary 12] we proved that if \( I \) is an infinite set, \( H \subset I^* \) a regular compact subset (\( H \) is regular in \( I^* \) if and only if \( \text{int}(H) \) is dense in \( H \)) and \( Y_H = \{ f \in \ell_\infty : \tilde{f}|H = 0 \} \), then for every \( w^* \)-compact subset \( K \subset \ell_\infty \) we have \( d(K, Y_H) = d(\overline{\text{co}}w^*(K), Y_H) \). So, the subset \( \mathcal{O} = (\bigcap_{f \in L} \tilde{f}^{-1}(0)) \cap \mathbb{N}^* \) is not regular in \( \mathbb{N}^* \).

(b) For the category of convex subsets \( C \subset X \), we prove in the following that for every \( w^* \)-compact subset \( K \subset X^{**} \) we have \( d(\overline{\text{co}}w^*(K), C) \leq 9d(K, C) \) and, if \( C \cap K \) is \( w^* \)-dense in \( K \), then \( d(\overline{\text{co}}w^*(K), C) \leq 4d(K, C) \).

**Lemma 5.** Let \( X \) be a Banach space and \( D \subset C \subset X \) two convex subsets of \( X \). Then for every \( z \in \overline{D} \subset X^{**} \) we have

\[
d(z, C) \leq d(z, D) \leq 2d(z, C).
\]

**Proof.** Fix some \( z \in \overline{D} \). Clearly, \( d(z, C) \leq d(z, D) \). So, prove that \( d(z, D) \leq 2d(z, X) \). Assume that \( d(z, D) > 2d(z, X) \). Then:

(i) for some \( a > 0 \) we have \( d(z, D) > 2a > 2d(z, X) \) and

(ii) there exists \( w \in X \setminus \overline{D} \) such that \( d(w, D) > a \) and \( \|w - z\| < a \).
Since \( d(w, D) > a \), there exists \( x^* \in S(X^*) \) such that \( \inf \{ x^*(w - d): d \in D \} > a \). Choose a net \( \{d_i\}_{i \in I} \subset D \) such that \( d_i \xrightarrow{w^*} z \). Then \( w - d_i \xrightarrow{w^*} w - z \) and, so, \( x^*(w - d_i) \to x^*(w - z) \). Since \( x^*(w - z) < a \) (because \( \|w - z\| < a \)), there exists \( i_0 \in I \) such that \( \forall i \geq i_0 \) we have \( x^*(w - d_i) < a \). But by construction \( a < x^*(w - d_i), \forall i \in I \). So, we get a contradiction which proves that \( d(z, D) \leq 2d(z, X) \).

Finally, observe that \( d(z, X) \leq d(z, D) \). \( \square \)

**Proposition 6.** Let \( X \) be a Banach space, \( C \subset X \) a convex subset of \( X \) and \( K \subset X^{**} \) a \( w^* \)-compact subset. Then \( d(\overline{co}w^*(K), C) \leq 9d(K, C) \).

**Proof.** Without restriction, we suppose that \( 0 \in C \). Assume that

\[
d(\overline{co}w^*(K), C) > b > 9a > 9d(K, C)
\]

and choose \( z_0 \in \overline{co}w^*(K) \) such that \( d(z_0, C) > b \). So, there exists \( \psi \in S(X^{**}) \) such that \( \inf \{\psi(z_0 - c): c \in C\} > b \).

**Step 1.** Let \( D_0 = \{0\} \). Since \( \psi(z_0) > b \) and \( B(X^*) \) is \( w^* \)-dense in \( B(X^{**}) \), there exists \( x_i^* \in S(X^*) \) such that \( x_i^*(z_0) > b \). So, as \( z_0 \in \overline{co}w^*(K) \) we can find \( \eta_i = \sum_{i=1}^{n_1} \lambda_{i1} \eta_{i1} \in co(K), \eta_{i1} \in K, \lambda_{i1} \geq 0, \sum_{i=1}^{n_1} \lambda_{i1} = 1 \), such that \( x_i^*(\eta_{i1}) > b \). Since \( d(\eta_{i1}, C) < a \) we have the decomposition \( \eta_{i1} = \eta_{i1}^1 + \eta_{i1}^2 \) with \( \eta_{i1}^1 \in C \) and \( \eta_{i1}^2 \in aB(X^*) \).

**Step 2.** Let \( D_1 = \{\eta_{i1}^1: 1 \leq i \leq n_1\} \cup D_0 \subset C \). Since \( D_1 \) is finite and \( \min \{\psi(z_0 - y): y \in D_1\} > b \), there exists \( x_2^* \in S(X^*) \) such that \( \min \{x_2^*(z_0 - y): y \in D_1\} > b \). So, as \( x_1^*(z_0) > b \), \( \min \{x_1^*(z_0 - y): y \in D_1\} > b \), \( D_1 \) is finite and \( z_0 \in \overline{co}w^*(K) \), we can find \( \eta_2 = \sum_{i=1}^{n_2} \lambda_{2i} \eta_{2i} \in co(K), \eta_{2i} \in K, \lambda_{2i} \geq 0, \sum_{i=1}^{n_2} \lambda_{2i} = 1 \), such that \( x_1^*(\eta_{2i}) > b \) and \( \min \{x_2^*(\eta_{2i} - y): y \in D_1\} > b \). Since \( d(\eta_{2i}, C) < a \) we have the decomposition \( \eta_{2i} = \eta_{2i}^1 + \eta_{2i}^2 \) such that \( \eta_{2i}^1 \in C \) and \( \eta_{2i}^2 \in aB(X^*) \).

By reiteration, we obtain the sequences \( \{x_i^*\}_{i \geq 1} \subset S(X^*), \eta_k = \sum_{i=1}^{n_k} \lambda_{ki} \eta_{ki} \in co(K), \eta_{ki} \in K, \lambda_{ki} \geq 0, \sum_{i=1}^{n_k} \lambda_{ki} = 1 \), \( D_k = \{\eta_{ki}: 1 \leq i \leq n_k\} \cup D_{k-1} \), \( \eta_{ki} = \eta_{ki}^1 + \eta_{ki}^2 \) with \( \eta_{ki}^1 \in C \) and \( \eta_{ki}^2 \in aB(X^*) \), \( k \geq 1 \), such that \( \min \{x_i^*(\eta_k - y): y \in D_{k-1}\} > b \) for every \( k \geq i \).

Let \( D = \overline{co}(\bigcup_{k \geq 1} D_k) \subset C \) and

\[
K_1 = \left\{ \eta_{ji}^1: i \geq 1, 1 \leq j_i \leq n_i \right\} \subset (K + aB(X^*)) \cap \overline{D}w^*.
\]

Let \( \eta_0 \) be a \( w^* \)-limit point of \( \{\eta_k\}_{k \geq 1} \).

**Claim 1.** \( d(\eta_0, D) < 9a \).

Indeed, clearly \( \eta_0 \in \overline{co}w^*(K_1) + aB(X^*) \). Now observe that:

(i) Since \( K_1 \subset K + aB(X^*) \), we get \( d(K_1, C) \leq d(K, C) + a < 2a \).

(ii) Since \( K_1 \cap X \) is \( w^* \)-dense in \( K_1 \), by [4, Theorem 6], [3, Theorem 2] we obtain

\[
d(\overline{co}w^*(K_1), X) \leq 2d(K_1, X) \leq 2d(K_1, C) < 4a.
\]
(iii) Since $\overline{co}^{w^*}(K_1) \subset \overline{D}^{w^*}$, by Lemma 5 we get
\[ d(\overline{co}^{w^*}(K_1), D) \leq 2d(\overline{co}^{w^*}(K_1), X) < 8a. \]

So, as $\eta_0 \in \overline{co}^{w^*}(K_1) + aB(X^{**})$, we finally get $d(\eta_0, D) < 9a$.

**Claim 2.** $d(\eta_0, D) \geq b$.  

Indeed, let $\phi \in B(X^{***})$ be a $w^*$-limit point of $\{x^*_n\}_{n \geq 1}$. Since $x^*_n(\eta_k - y) > b$ if $k \geq n$ and $y \in D_{n-1}$, then $x^*_n(\eta_0 - y) \geq b$, $\forall n \geq 1$, $\forall y \in D_{n-1}$. Hence $\phi(\eta_0 - y) \geq b$, $\forall y \in D$, and, so, $d(\eta_0, D) \geq b$.

Since $b > 9a$ we get a contradiction, which completes the proof.  

**Proposition 7.** Let $X$ be a Banach space, $C \subset X$ a convex subset of $X$ and $K \subset X^{**}$ a $w^*$-compact subset such that $K \cap C$ is $w^*$-dense in $K$. Then $d(\overline{co}^{w^*}(K), C) \leq 4d(K, C)$.

**Proof.** Suppose that there exists a $w^*$-compact subset $K \subset B(X^{**})$ with $\bigcap \{x^*_n\} \cap C \cap K$ $w^*$-dense in $K$ such that $d(\overline{co}^{w^*}(K), C) > 4d(K, C)$, i.e., there exist $z_0 \in \overline{co}^{w^*}(K)$ and $a, b > 0$ such that $d(z_0, C) > b > 4a > 4d(K, C)$. Pick $\psi \in S(X^{***})$ such that $\inf\{\psi(z_0 - c) : c \in C\} > b$. We follow the argumentation of Proposition 6 with the following changes:

(i) as $C \cap K$ is $w^*$-dense in $K$ we choose $\eta_k \in co(C \cap K)$, i.e., $\eta_k = \sum_{i=1}^{n_k} \lambda_{ki} \eta_{ki}$ with $\eta_{ki} \in C \cap K$ and $\lambda_{ki} \geq 0$, $\sum_{i=1}^{n_k} \lambda_{ki} = 1$;

(ii) we define:
\[ D_k = \{\eta_{kj} : 1 \leq j \leq n_k\} \cup D_{k-1}, \quad D = \overline{co}\left(\bigcup_{k \geq 1} D_k\right) \quad \text{and} \quad K_1 = \{\eta_{ij} : i \geq 1, 1 \leq j \leq n_i\}^{w^*} \subset \overline{D}^{w^*} \cap K. \]

Clearly, $d(K_1, X) \leq d(K_1, C) \leq d(K, C) < a$, whence $d(\overline{co}^{w^*}(K_1), X) \leq 2d(K_1, X) < 2a$ and $d(\overline{co}^{w^*}(K_1), D) < 4a$. Finally, every $w^*$-limit point $\eta_0$ of $\{\eta_k\}_{k \geq 1}$ satisfies $\eta_0 \in \overline{co}^{w^*}(K_1)$, $d(\eta_0, D) < 4a$ and $d(\eta_0, D) \geq b$, a contradiction.  

**References**