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Yukawan Potential Theory*

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This paper concerns the Yukawa equation $\Delta u = \mu^2 u$ where μ is a real constant. Given a solution $u(x, y)$ of this equation then there is a conjugate function $v(x, y)$ satisfying the same equation and related to $u(x, y)$ by a generalization of the Cauchy–Riemann equations. This gives rise to interesting analogies with logarithmic potential theory and with complex function theory. In particular there are generalizations of holomorphic functions, Taylor series, Cauchy's formula, and Rouche's theorem. The resulting formulae contain Bessel functions instead of the logarithmic functions which appear in the classical theory. However, as $\mu \to 0$ the formulae revert to the classical case. A convolution product for generalized holomorphic functions is shown to produce another generalized holomorphic function.

1. INTRODUCTION

In the Newtonian potential theory, the potential of a unit point charge is $1/r$ where r is the distance to the point. Then the potential u of any distribution of charges satisfies the Laplace equation $\Delta u = 0$ at points of free space.

In order to have a nuclear force potential which decays rapidly at infinity, Yukawa proposed that the potential of a point charge be $e^{-\mu r}/r$. Here μ is a positive constant and Yukawa assumed that μ^{-1} was of the order of magnitude of a nuclear radius. It results that the potential u of a charge distribution satisfies the Yukawa equation $\Delta u = \mu^2 u$ at points of free space.

Thus the Yukawa potentials, like those of Newton are governed by a second order differential equation which is invariant under the group of rotations and translations of space. This very important property would be lost if an arbitrary function were selected for the potential of a point charge.

Another important property of Yukawa potentials is that they approach those of Newton as μ approaches zero. This property will be manifest in the formulas studied in this paper.

The function $e^{-\mu r}/r$ is a member of the Bessel family of functions. Various other Bessel functions enter into the analysis of the Yukawa equation. Thus

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this theory should be appealing to Besselian scholars. For example in the Newtonian theory the potential of a line charge is $\log r^{-1}$. In the Yukawan theory the potential of a line charge is $K_0(\mu r)$ where K_0 is the modified Bessel function of the second kind. For small r the function $K_0(\mu r)$ is asymptotic to $\log r^{-1}$; However, for large r it is asymptotic to $e^{-ur}(\pi/2\mu r)^{1/2}$.

This paper is primarily concerned with the potentials of line charges. In other words the potentials u are functions of only two variables x and y. Many of the properties of these two-dimensional Yukawan potentials follow directly from corresponding properties of three-dimensional Newtonian potentials by appeal to a simple mapping. For example, an analog of Poisson's integral formula is derived in this way.

In the Yukawan theory the Cauchy–Riemann equations have the analogy $u_x - v_y = \alpha u + \beta v$, $u_y + v_x = \beta u - \alpha v$ where α and β are real constants such that $\alpha^2 + \beta^2 = \mu^2$. Then $f(x, y) = u(x, y) + iv(x, y)$ is termed a *ν-regular function* where $\nu = \alpha + i\beta$. The ν-regular functions are analogous to holomorphic functions of a complex variable. In particular the ν-regular functions can be expanded in a series analogous to the Taylor series.

It proves possible to define contour integrals of ν-regular functions. This leads to an integral which is ν-regular. Also for a closed contour there is a direct analog of the Cauchy integral formula.

The behavior of a ν-regular function at a zero point is essentially the same as that of a holomorphic function at a zero point. Pursuing this idea shows that the theorem of Rouche for holomorphic functions applies without change to ν-regular functions.

Unfortunately, a product of ν-regular functions is not a ν-regular function, so here the analogy with holomorphic functions breaks down. A similar difficulty was encountered in the theory of discrete analytic functions (a difference equation theory). There the difficulty was mitigated by the introduction of a convolution product [4, 5]. This work, together with the work of Lewy [7] suggested a resolution of the "product problem". Thus if f and g are two ν-regular functions, a convolution product $f * g$ is so devised to again be a ν-regular function.

By means of a transformation some of the results in this paper may be related to the very general theory of pseudo-analytic functions developed by Bers, Vekua and others [1]. However the equations treated here are simpler. It results that the proofs given here are much simpler and yield formulae in explicit analytic form. By another transformation some of the concepts treated here can be related to regular quaternion functions [3]. It would be of some interest to study these comparisons but this will not be attempted.

It is worth noticing that Bessel potentials have certain advantages over Newtonian potentials in functional analysis. Such questions have been treated with great generality by Aronszajn and Donoghue [2].

2. A Correspondence Principle

Let a function $u(x, y)$ satisfy the Yukawa equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \mu^2 u \tag{1}$$

where μ is a non-negative constant. Then in this paper we say that u is *panharmonic at a point* (x, y) if its second derivatives are continuous and satisfy the Yukawa equation in some neighborhood of the point. The function u is said to be *panharmonic in a closed region* if u is continuous in the region and panharmonic at interior points. A region is regarded as the closure of a domain. If $\mu = 0$ these are the standard definitions for harmonic functions [6, p. 211].

Panharmonic functions in two variables x and y are in one to one correspondence with a subclass of harmonic functions in three variables $x, y,$ and z. Thus given that $u(x, y)$ is panharmonic, this correspondence is defined by the mapping

$$U(x, y, z) = \cos \mu z \; u(x, y). \tag{2}$$

Then clearly

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \tag{3}$$

so U is a harmonic function. The virtue of this correspondence is that panharmonic functions thereby inherit many of the well-known properties of harmonic functions. The following four theorems are consequences of the correspondence principle.

THEOREM 1. *If u is panharmonic in a compact region R and if for a constant $M > 0$*

$$u(x, y) \leqslant M \qquad \textit{at boundary points of } R. \tag{4}$$

Then

$$u(x, y) < M \qquad \textit{at interior points of } R. \tag{5}$$

Proof. Consider the cylindrical region R_c of three space defined as

$$(x, y) \subset R \qquad \text{and} \qquad \frac{-\pi}{2\mu} \leqslant z \leqslant \frac{\pi}{2\mu}\,. \tag{6}$$

On the top and bottom of this cylinder the corresponding harmonic function U vanishes. On the sides of the cylinder $U \leqslant M$. Thus by the maximum principle for harmonic functions it follows that at interior points of the cylinder R_c

$$U(x, y, z) < M \tag{7}$$

unless possibly U is constant. But if U is constant, then $U \equiv 0$ because U vanishes on the ends of this cylinder. Thus (7) is always true. Then $U(x, y, 0) = u(x, y)$ so if (x, y) is an interior point of R we see that (5) follows from (7) and the proof is complete.

THEOREM 2. *If u is panharmonic at a point it can be expanded in a power series about that point.*

Proof. Let the point be (a, b) then the harmonic function $U(x, y, z)$ can be expanded in a power series in the variables $(x - a)$, $(y - b)$, and z [6, p. 220]. Putting $z = 0$ gives a power series for $u(x, y)$.

THEOREM 3. *Let R be a compact region and let u_1, u_2, u_3,... be a sequence of panharmonic functions in R. If the sequence converges uniformly on the boundary of R then it converges uniformly throughout R, and its limit u is panharmonic in R.*

Proof. For harmonic functions this is Harnack's first convergence theorem [6, p. 248]. Applying the correspondence principle to the cylindrical region R_c shows that it is true for panharmonic functions.

THEOREM 4. *Let $u_1(P)$, $u_2(P)$, $u_3(P)$,... be an infinite sequence of functions, panharmonic in a domain T, such that for every P in T, $u_n(P) \leqslant u_{n+1}(P)$, $n = 1, 2, 3,\ldots$. Then if the sequence is bounded at a single point 0 of T, it converges uniformly in any compact region R in T to a function which is panharmonic in T.*

Proof. For harmonic functions this is Harnack's second convergence theorem [6, p. 263]. Apply the correspondence principle for a cylindrical domain T_c with $-1 < \mu z < 1$ and for a cylindrical region R_c with $-\frac{1}{2} \leqslant \mu z \leqslant \frac{1}{2}$. But $\cos \mu z > 0$ so $U_n(P) \leqslant U_{n+1}(P)$ for every P in T_c. This is seen to complete the proof.

The next theorem gives an analog of Poisson's integral formula.

THEOREM 5. *Let $u(x, 0)$ be a continuous function such that for constants A and B*

$$A \leqslant u(x, 0) \leqslant B \tag{8}$$

for all real x. Then for $y > 0$ a panharmonic function $u(x, y)$ is defined by the integral formula

$$u(x, y) = \pi^{-1} \int_{-\infty}^{\infty} yp[(x - x')^2 + y^2]\, u(x', 0)\, dx' \tag{9}$$

where the kernel function p is defined as

$$p(t) = \int_0^{\infty} \frac{\cos(\mu z)\, dz}{[z^2 + t]^{3/2}}, \qquad t > 0. \tag{10}$$

Also

$$u(x, y) \to u(x, 0) \qquad as \qquad y \to 0^+ \tag{11}$$

$$p(t) \geqslant 0, \tag{12}$$

and

$$A \leqslant e^{\mu y} u(x, y) \leqslant B. \tag{13}$$

For $y > 0$ the inequalities (13) are strict unless $u(x, 0)$ is a constant.

Proof. Let $U(x, 0, z)$ be continuous and uniformly bounded for all x and z. Then for $y > 0$ a harmonic function $U(x, y, z)$ is defined by Poisson's integral formula,

$$U(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{yU(x', 0, z')\, dz'\, dx'}{[(x - x')^2 + y^2 + (z - z')^2]^{3/2}}, \tag{14}$$

$$U(x, y, z) \to U(x, 0, z) \qquad \text{as} \qquad y \to 0^+. \tag{15}$$

In particular let us substitute $U(x, 0, z) = \cos \mu z\; u(x, 0)$ in this formula to obtain

$$U(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{y \cos \mu z' u(x', 0)\, dz'\, dx'}{[(x - x')^2 + y^2 + (z - z')^2]^{3/2}}.$$

Make the change of variable $z' = z'' + z$ so

$$\cos \mu z' = \cos \mu z \cos \mu z'' - \sin \mu z \sin \mu z''.$$

Then integrating with respect to z'' it is only necessary to retain the part of the integrand which is even with respect to z''. Thus

$$U(x, y, z) = \frac{\cos \mu z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{y \cos \mu z'' u(x', 0)\, dz''\, dx'}{[(x - x')^2 + y^2 + (z'')^2]^{3/2}}. \tag{16}$$

The integrand is absolutely convergent so we may evaluate (16) as an iterated integral. Thus by the definition of $p(t)$ and $u(x, y)$ we see that (16) simply states that

$$U(x, y, z) = \cos \mu z u(x, y). \tag{17}$$

Since $U(x, y, z)$ is harmonic it follows from (17) that $u(x, y)$ is panharmonic. This proves (9), the correspondent of Poisson's formula. By taking $z = 0$ we see that (15) and (17) prove (11).

To prove (12) suppose, on the contrary, that $p(y_0^2) < 0$ for some value $y_0 > 0$. Then it follows from (9) that there is a function $u_0(x, 0) \geqslant 0$ and of compact support such that for some $M > 0$

$$-u_0(0, y_0) \geqslant 2M. \tag{18}$$

It follows from the definition (10) that

$$|p(x^2 + y^2)| \leqslant \int_0^\infty \frac{dz}{[z^2 + x^2 + y^2]^{3/2}} = \frac{c}{x^2 + y^2}\,. \tag{19}$$

This inequality together with the fact that $u_0(x, 0)$ is of compact support shows that

$$u_0(x, y) \to 0 \qquad \text{as} \qquad x^2 + y^2 \to \infty.$$

Thus the panharmonic function $-u_0(x, y)$ satisfies the condition $-u_0(x, y) \leqslant M$ on the boundary of a large semicircle with center at the origin and containing the point $(0, y_0)$. Thus by Theorem 1 $-u_0(0, y_0) \leqslant M$. This contradicts (18) and thereby proves (12).

Making use of the hypothesis (8) and the fact just proved that $p \geqslant 0$ we find

$$u(x, y) \leqslant \pi^{-1} \int_{-\infty}^{\infty} y p[(x - x')^2 + y^2]\, B\, dx'. \tag{20}$$

The function on the right is of the form $v(y)$ and is panharmonic, thus

$$\frac{d^2v}{dy^2} = \mu^2 v. \tag{21}$$

Hence

$$v = c_1 e^{-\mu y} + c_2 e^{\mu y}. \tag{22}$$

But (19) and (20) show that v is bounded as $y \to +\infty$, hence $c_2 = 0$. It now follows from (20) that $c_1 = B$, so $v = Be^{-\mu y}$. This proves the right side of inequality (13). Moreover if $u(x, 0)$ is not constant, then (20) is a strict inequality, so (13) is a strict inequality.

The left side of inequality (13) follows by an analogous argument, and this completes the proof of Theorem 5. There are several other theorems which follow from the correspondence principle but these questions will not be pursued.

3. Modified Bessel Functions

The following result is the analog of the Gauss mean value theorem.

THEOREM 6. *Let $u(x, y)$ be panharmonic in the circular disk*

$$(x - x_0)^2 + (y - y_0)^2 \leqslant a^2.$$

Then

$$u(x_0, y_0) = \frac{1}{2\pi I_0(\mu a)} \int_0^{2\pi} u(x_0 + a \cos \theta, y_0 + a \sin \theta)\, d\theta \tag{1}$$

where $I_0(r)$ is the modified Bessel function.

Proof. The panharmonic equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \mu^2 u. \tag{2}$$

Integrating this equation with respect to θ gives

$$\frac{d^2 \bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} = \mu^2 \bar{u} \tag{3}$$

where

$$\bar{u}(r) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos \theta, y_0 + r \sin \theta)\, d\theta. \tag{4}$$

The general solution of equation (3) is

$$\bar{u} = a I_0(\mu r) + b K_0(\mu r). \tag{5}$$

Here I_0 is the modified Bessel function of the first kind. Of course

$$I_0(x) = J_0(ix)$$

and

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \frac{x^6}{2^2 4^2 6^2} + \cdots. \tag{6}$$

It is easy to check that letting

$$K_0(x) = I_0(x) \int_x^\infty \frac{dx}{xI^2(x)} \tag{7}$$

gives an independent solution of Eq. (3). The function $K_0(x)$ is the modified Bessel function of the second kind. It follows from (6) and (7) that

$$K_0(x) \cong \log x^{-1} \qquad \text{as} \qquad x \to 0^+. \tag{8}$$

But $\bar{u}(r)$ is continuous at the origin so $b = 0$ in (5). Then since

$$\bar{u}(0) = u_0(x_0, y_0)$$

it results that

$$\bar{u}(r) = u(x_0, y_0) I_0(\mu r).$$

Putting $r = a$ gives (1) and the proof of Theorem 6 is complete.

THEOREM 7. *The kernel function p of the Poisson formula of Theorem 4 is given by*

$$p(r^2) = -\frac{\mu}{r} K_0'(\mu r) \tag{9}$$

where K_0 is the modified Bessel function of the second kind. Moreover K_0 is a positive decreasing function for $r > 0$.

Proof. The following formula is a well known integral expression for the modified Bessel function of the second kind.

$$K_0(\mu r) = \int_0^\infty \frac{\cos \mu z}{(r^2 + z^2)^{1/2}} dz. \tag{10}$$

It is easy to prove this by substitution in the Bessel equation (3). Differentiating (10) with respect to r gives

$$-\mu K_0'(\mu r) = r \int_0^\infty \frac{\cos \mu z}{(r^2 + z^2)^{3/2}} dz.$$

Comparing the right side with the kernel function p of Poisson's formula proves (9).

According to Theorem 4, $p \geqslant 0$. This together with (9) proves the last statement of Theorem 7.

The next result is a sharpened form of the maximum principle given in Theorem 1.

THEOREM 8. *Let $u(x, y)$ be panharmonic in a compact region R. On the boundary of R suppose $u \leqslant M$ for some constant $M > 0$. If (x_0, y_0) is an interior point of R it is also interior to a circle C contained in R. Then*

$$u(x_0, y_0) \leqslant M \frac{I_0(\mu r)}{I_0(\mu a)} \tag{11a}$$

where a is the radius of C and r is the distance from (x_0, y_0) to the center of the circle. It is a corollary of (11a) *that*

$$u(x_0, y_0) \leqslant M \frac{1}{I_0(\mu d)} \tag{11b}$$

where d is the distance from (x_0, y_0) to the boundary of R.

Proof. By virtue of Theorem 1 we have $u \leqslant M$ on C. Let $w = MI_0(\mu r)/I_0(\mu a)$, so w is a panharmonic function of r. Then the function $U = u - w$ is panharmonic and $U \leqslant 0$ on C. By Theorem 1 this implies that $U(x_0, y_0) \leqslant 0$. This proves (11a). Taking (x_0, y_0) to be the center of a circle of radius d proves (11b) because $a = d$, $r = 0$, and $I_0(0) = 1$.

It is worth noticing that Theorems 5, 6 and 8 bring to light a characteristic property of panharmonic functions. The inference is that these functions decay exponentially as a point moves away from the boundary into the interior of a region because

$$I_0(x) \sim \frac{e^x}{(2\pi x)^{1/2}}.$$

The next theorem gives an analog of Harnack's inequality [6, p. 262].

THEOREM 9. *Let u be panharmonic and non-negative in the disk*

$$(x - x_0)^2 + (y - y_0)^2 < a^2$$

where $a > 0$. Then

$$\left| \frac{\partial u}{\partial x} \right|_{(x_0, y_0)} \leqslant u(x_0, y_0) \frac{a I_0(\mu a)}{\int_0^a r I_0(\mu r)\, dr}. \tag{12}$$

Proof. By integrating the mean value relation (1) we obtain

$$u(x_0, y_0) \int_0^b r I_0(\mu r)\, dr = \frac{1}{2\pi} \int_0^b \int_0^{2\pi} u r\, dr\, d\theta. \tag{13}$$

Here we take $0 < b < a$. But

$$\left(\frac{\partial u}{\partial x}\right)_0 = \lim_{h\to 0} \frac{u(x_0 + h, y) - u(x_0, y_0)}{h}.$$

This together with (13) gives

$$\left(\frac{\partial u}{\partial x}\right)_0 \int_0^b rI_0(\mu r)\, dr = \frac{1}{2\pi} \int_0^{2\pi} ub \cos\theta \, d\theta.$$

Thus

$$\left|\frac{\partial u}{\partial x}\right|_0 \int_0^b rI_0(\mu r)\, dr \leqslant \frac{b}{2\pi} \int_0^{2\pi} u \, d\theta = u(x_0, y_0)\, bI_0(\mu b).$$

Allowing b to approach a in this relation proves the Harnack type inequality (12).

In the limit as $\mu \to 0$ we see that relation (11) becomes

$$\left|\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} \leqslant u(x_0, y_0) \frac{2}{a}. \tag{14}$$

This is a Harnack type inequality for harmonic functions.

The modified Bessel function of order n is defined by the series

$$I_n(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n \left[1 + \frac{(x/2)^2}{1\cdot(n+1)} + \frac{(x/2)^4}{1\cdot 2\cdot(n+1)(n+2)} + \cdots\right]. \tag{15}$$

It satisfies the Bessel equation

$$y'' + \frac{y'}{x} = \left(1 + \frac{n^2}{x^2}\right) y. \tag{16}$$

The corresponding modified Bessel function of the second kind is defined as

$$K_n(x) = I_n(x) \int_x^\infty \frac{dx}{xI_n^2(x)}. \tag{17}$$

It may be checked that K_n also satisfies equation (16). Clearly K_n has a singularity at the origin.

THEOREM 10. *If $u(r, \theta)$ is panharmonic in the circle $x^2 + y^2 \leqslant a^2$ then for $0 \leqslant r < a$*

$$u(r, \theta) = \sum_{-\infty}^{\infty} c_n I_{|n|}(\mu r)\, e^{in\theta} \tag{18}$$

where

$$c_n = \frac{1}{2\pi I_{|n|}(\mu a)} \int_0^{2\pi} u(a, \theta)\, e^{-in\theta}\, d\theta. \tag{19}$$

Proof. By virtue of the smoothness guaranteed by Theorem 2 it is possible to obtain the convergent Fourier series

$$u(r, \theta) = \sum_{-\infty}^{\infty} f_n(r)\, e^{in\theta}, \tag{20}$$

$$f_n(r) = (2\pi)^{-1} \int_0^{2\pi} u(r, \theta)\, e^{-in\theta}\, d\theta. \tag{21}$$

Letting f_n' denote differentiation with respect to r we obtain

$$\begin{aligned} f_n'' &+ r^{-1} f_n' - \mu^2 f_n - n^2 r^{-2} f_n \\ &= (2\pi)^{-1} \int_0^{2\pi} [u'' + r^{-1}u' - \mu^2 u - n^2 r^{-2} u]\, e^{-in\theta}\, d\theta. \end{aligned}$$

But u satisfies equation (2) so the right side becomes

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\frac{\partial^2 u}{\partial \theta^2} + n^2 u\right] e^{-in\theta}\, d\theta.$$

An integration by parts shows that this vanishes. Hence f_n satisfies the Bessel Eq. (16). But since $f_n(r)$ is finite as $r \to 0$ we must have $f_n(r) = c_n I_{|n|}(\mu r)$. This completes the proof.

4. Analogs of the Cauchy–Riemann Equations

The replacement of a second order differential equation by a system of first order equations has been of great importance in mathematics and physics. For example the Cauchy–Riemann equations replace the Laplace equation, Maxwell's equations replace the wave equation, and Dirac's equations replace the Klein–Gordon equation. Yukawa's theory of nuclear force gave strong impetus to further studies of first order systems like those of Dirac. Among other things these studies have led to interesting algebraic structures [8].

Here we wish to replace the two-dimensional Yukawa equation by two first order equations. More precisely the real Yukawa equation is to be replaced by a first order complex equation. To this end the following operator notation is convenient:

$$L = \frac{\partial}{\partial x} + \frac{i\partial}{\partial y} = \frac{\partial}{\partial z^*},$$

$$L^* = \frac{\partial}{\partial x} - \frac{i\partial}{\partial y} = \frac{\partial}{\partial z},$$

$$LL^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta.$$

Let ν be a complex number termed the *polarization vector* and let $|\nu| = \mu$. Then we say that a function $f(x, y)$ is *ν-regular* at a point if it has continuous second derivatives and satisfies the equation

$$Lf = \nu f^* \tag{1}$$

in the neighborhood of the point. The function f is said to be ν-regular in a closed region if f is continuous in the region and is ν-regular at all interior points.

THEOREM 11. *If f is ν-regular then f is panharmonic.*

Proof. $\Delta f = L^*Lf$ and

$$L^*Lf = \nu L^*f^* = \nu\nu^*f = \mu^2 f.$$

Q.E.D.

THEOREM 12. *If u is panharmonic then*

$$f = \nu u + (Lu)^* \tag{2}$$

is ν-regular.

Proof.

$$Lf = \nu Lu + LL^*u^* = \nu Lu + \mu^2 u^*$$
$$= \nu(Lu + \nu^* u^*) = \nu f^*.$$

Q.E.D.

If f is ν-regular let

$$f = u + iv, \qquad \nu = \alpha + i\beta \tag{3}$$

where u and v are real functions and α and β are real constants. Then separating the regularity Eq. (1) into real and imaginary parts gives:

$$\begin{aligned}\frac{\partial u}{\partial x} - \alpha u &= \frac{\partial v}{\partial y} + \beta v,\\ \frac{\partial u}{\partial y} - \beta u &= -\frac{\partial v}{\partial x} - \alpha v.\end{aligned} \tag{4}$$

Clearly these are the analogs of the Cauchy–Riemann equations. Then we may term (u, v) a pair of *conjugate panharmonic functions with polarization* (α, β).

The regularity equations may be written in a third form which is also significant. Let a complex number j, termed the *current vector*, be defined as $j = j_x + ij_y$ where

$$j_x = \alpha u - \frac{\partial u}{\partial x}, \qquad j_y = \beta u - \frac{\partial u}{\partial y}. \tag{5}$$

Let a complex number k, termed the *conjugate current vector*, be defined as $k = k_x + ik_y$ where

$$k_x = -\alpha v - \frac{\partial v}{\partial x}, \qquad k_y = -\beta v - \frac{\partial v}{\partial y}. \tag{6}$$

THEOREM 13. *The function f is ν-regular if and only if the current j and the conjugate current k satisfy*

$$k = ij. \tag{7}$$

Geometrically this signifies that the current streamlines and the conjugate current streamlines form orthogonal vector fields.

Proof. The Cauchy–Riemann equations (4) state that $j_x = k_y$ and $j_y = -k_x$. Q.E.D.

The transformation between different directions of polarization is brought out in the following two theorems.

THEOREM 14. *If f is ν-regular and c is a constant then $g = cf$ is ν'-regular where $\nu' = \nu(c/c^*)$.*

Proof. $Lg = cLf = \nu cf^* = (\nu c/c^*)\, g^*$. Thus $Lg = \nu' g^*$ and the proof is complete.

THEOREM 15. *If $f(z)$ is ν-regular and c is a constant then $h = f(cz)$ is ν'' regular where $\nu'' = \nu c^*$.*

Proof. The notation $f(z)$ is a short notation for $f(x, y)$. It is not meant to imply that f is a holomorphic function of $z = x + iy$. Thus if $c = a + ib$

$$\begin{aligned} h &= f(X, Y) = f(ax - by, bx + ay) \\ Lh &= (af_X + bf_Y) + i(-bf_X + af_Y) \\ Lh &= (a - ib)(f_X + if_Y) = c^* \nu h^*. \end{aligned}$$

This completes the proof.

Because of the last two theorems it suffices to work with one direction of polarization. Thus the principal direction of polarization is taken to be $\nu = \mu$, a positive number. In this case we shall say that f is *right regular*. If $\nu = -\mu$ we say that f is *left regular*. If $f(z)$ is right regular it follows from Theorem 14 that $if(z)$ is left regular and it follows from Theorem 15 that $f(-z)$ is left regular.

The right regularity condition is

$$Lf = \mu f^*. \tag{8}$$

The corresponding Cauchy–Riemann equations are

$$\begin{aligned} \frac{\partial u}{\partial x} - \mu u &= \frac{\partial v}{\partial y}, \\ \frac{\partial v}{\partial x} + \mu v &= -\frac{\partial u}{\partial y}. \end{aligned} \tag{9}$$

5. Contour Integration and Cauchy's Formula

Of concern now are contour integrals of continuous functions in the complex plane. The definition of contours and contour integration given in elementary complex analysis is sufficient for the purpose at hand.

Lemma 1. *At points inside and on a simple closed contour Γ suppose that a function $h(z)$ has continuous first derivatives and that*

$$\operatorname{Re} Lh(z) = 0. \tag{1}$$

Then

$$\operatorname{Im} \int_\Gamma h(z)\, dz = 0. \tag{2}$$

Proof. $Lh = h_x + ih_y$ and $(Lh)^* = h_x^* - ih_y^*$. Thus if $h = p + iq$ where p and q are real,

$$2\operatorname{Re}(Lh) = Lh + (Lh)^* = 2p_x - 2q_y.$$

Hence $p_x = q_y$ and by Green's theorem

$$0 = \int_\Gamma q\,dx + p\,dy = \operatorname{Im} \int_\Gamma (p + iq)\,(dx + i\,dy).$$

This is the same as (2) so the lemma is proved.

THEOREM 16. *At points inside and on a simple closed contour Γ suppose that:*

$$\textit{the function } f(z) \textit{ is } \nu\textit{-regular,} \tag{3}$$

$$\textit{the function } g(z) \textit{ is } -\nu^*\textit{-regular.} \tag{4}$$

Then

$$\operatorname{Im} \int_\Gamma f(z)\,g(z)\,dz = 0. \tag{5}$$

Proof. Let us define $h = fg$ then

$$Lh = gLf + fLg = \nu g f^* - \nu^* f g^*.$$

Hence $\operatorname{Re} Lh = 0$ so Lemma 1 applies and the proof is complete. Theorem 16 is an analog of the Cauchy integral theorem.

THEOREM 17. *At points inside and on a simple closed contour Γ suppose that:*

$$\textit{the function } f(z) \textit{ is right regular,} \tag{6}$$

$$\textit{the function } g_+(z) = p(z) + q(z) \textit{ is right regular,} \tag{7}$$

$$\textit{the function } g_-(z) = p(z) - q(z) \textit{ is left regular.} \tag{8}$$

Then

$$\int_\Gamma f(z)\,p(z)\,dz + \left(\int_\Gamma f(z)\,q(z)\,dz\right)^* = 0. \tag{9}$$

Proof. According to Theorem 14 the function ig_+ is left regular. Write

$$\int_\Gamma fg_-\,dz = \int_\Gamma fp\,dz - \int_\Gamma fq\,dz = P - Q \tag{10}$$

$$\int_\Gamma fig_+\,dz = i\int_\Gamma fp\,dz + i\int_\Gamma fq\,dz = iP + iQ. \tag{11}$$

According to the previous theorem, relation (10) defines a real number so $(P - Q) - (P - Q^* = 0$. Likewise (11) defines a real number so $(P + Q) + (P + Q)^* = 0$. Adding gives $P + Q^* = 0$. Q.E.D.

Corollary 1. *If $f(z)$ is right regular and $u(z)$ is panharmonic inside and on Γ*

$$\int_\Gamma f(z)\,[u_x(z) - iu_y(z)]\,dz + \left[\int_\Gamma f(z)\,\mu u^*(z)\,dz\right]^* = 0. \tag{12}$$

Proof. By Theorem 12 the functions

$$g_- = L^*u - \mu u^*, \qquad g_+ = L^*u + \mu u^* \tag{13}$$

are left and right regular so Theorem 17 applies.

The next result is an analog of Cauchy's integral formula.

Theorem 18. *Let $f(z)$ be right regular inside and on a simple closed contour Γ. If $r = | z - a |$ then*

$$f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{a - z}\,\mu r K_0'(\mu r)\,dz + \left[\frac{1}{2\pi i}\int_\Gamma f(z)\,\mu K_0(\mu r)\,dz\right]^* \tag{14}$$

provided a is inside Γ. If a is outside Γ the right side vanishes.

Proof. First suppose a is outside Γ then Corollary 1 applies with $u = K_0(\mu r)$. Thus if $a = x_0 + iy_0$ then

$$L^*K_0(\mu r) = \mu K_0'(\mu r)\left(\frac{x - x_0}{r} - i\frac{y - y_0}{r}\right) = \frac{\mu r K_0'(\mu r)}{z - a}.$$

Substituting in relation (12) gives

$$0 = \int_\Gamma \frac{f(z)}{z - a}\,\mu r K_0'(\mu r)\,dz + \left[\int_\Gamma f(z)\,\mu K_0(\mu r)\,dz\right]^*.$$

This proves the last statement of the theorem.

If a is inside Γ let it be the center of a circular contour γ of radius ϵ. Then a is outside of the contour $\Gamma_\epsilon = \Gamma - \gamma$. It is then sufficient to show that (14) holds for γ. It is seen from formula (3–7) that

$$K_0(x) \sim \log\frac{1}{x} \qquad \text{as} \qquad x \to 0^+, \tag{15}$$

$$K_0'(x) \sim -\frac{1}{x} \qquad \text{as} \qquad x \to 0^+. \tag{16}$$

From (15) it follows that

$$\int_\gamma f(z)\,\mu K_0(\mu r)\,dz \to 0 \qquad \text{as} \qquad \epsilon \to 0.$$

From (16) it follows that as $\epsilon \to 0$

$$\int_{\gamma} \frac{f(z)}{a-z}\, \mu r K_0'(\mu r)\, dz \to \int \frac{f(z)}{z-a}\, dz \to 2\pi i f(a). \qquad \text{Q.E.D.}$$

6. Convolution Integrals

Of concern now are functions defined by contour integrals on open contours.

THEOREM 19. *Let $f(z)$ be right regular in a simply connected domain D. Let an integration contour connect two points z and a in D. Then a right regular function $F(z)$ is defined by*

$$F(z) = \int_a^z \cosh \mu(y-y') f(z')\, dz' + \left[i \int_a^z \sinh \mu(y-y') f(z')\, dz'\right]^*. \tag{1}$$

Proof. Let a function $g_+(z')$ be defined as

$$g_+(z') = \cosh \mu(y-y') + i \sinh \mu(y-y'). \tag{2}$$

Then we see that

$$\frac{\partial}{\partial y'} g_+(z') = -\mu \sinh \mu(y-y') - i\mu \cosh \mu(y-y'),$$
$$Lg_+(z') = -\mu i \sinh \mu(y-y') + \mu \cosh \mu(y-y').$$

Thus $g_+(z')$ is right regular. Likewise

$$g_-(z') = \cosh \mu(y-y') - i \sinh \mu(y-y') \tag{3}$$

is left regular. Then by Theorem 17

$$\int_\Gamma \cosh \mu(y-y') f(z')\, dz' + \left[\int_\Gamma i \sinh \mu(y-y') f(z')\, dz'\right]^* = 0 \tag{4}$$

for any closed contour Γ in D because D is simply connected. Relation (4) validates a standard argument about line integrals. The conclusion of this argument is that $F(z)$ is a single valued function, independent of the choice of the contour connecting a fixed point a and a variable point z.

First continue the contour beyond the point z as a straight line parallel to the x axis. Then we see from (1) that

$$\frac{\partial F}{\partial x} = f(z). \tag{5}$$

Next continue the contour beyond the point z as a straight line parallel to the y-axis. Then

$$\begin{aligned}\frac{\partial F}{\partial y} &= if(z) + \mu \int_0^z \sinh \mu(y - y') f(z')\, dz' \\ &\quad + \left[i\mu \int_0^z \cosh \mu(y - y') f(z')\, dz' \right]^*.\end{aligned} \tag{6}$$

It then follows from (5) and (6) that $LF = \mu F^*$. Thus F is right regular and the proof is complete.

It is worth noting that if a and z are both real then (1) becomes

$$F(x) = \int_a^x f(x')\, dx'. \tag{7}$$

Thus we can term F an integral of f.

THEOREM 20. *Let $f(z)$ and $g(z)$ be right regular in a simply connected domain D. Let z and 0 be two points of D and let a convolution functional φ_1 be defined as*

$$\varphi_1 = \int_0^z f(z')\, g(z - z')\, dz' \tag{8}$$

where the integral is evaluated along a contour Γ_1 such that if z' is on Γ_1 then z' and $z - z'$ are both in D. Then

$$w = \operatorname{Im} \varphi_1 \tag{9}$$

is independent of the contour joining 0 and z and is a panharmonic function of z.

Proof. As a function of z' it is seen that $f(z')$ is right regular and $g(z - z')$ is left regular. Thus Theorem 16 is applicable and for a closed contour Γ

$$\operatorname{Im} \int_\Gamma f(z')\, g(z - z')\, dz' = 0. \tag{10}$$

Suppose w had a different value for two contours connecting 0 and z, say Γ_1 and Γ_2. Then employing a standard argument, relation (10) would be contradicted. This proves the first part of the theorem.

The contour Γ_1 is arbitrary, so choose it such that contiguous to z it becomes a line segment parallel to the x axis. Then at z

$$\begin{aligned}\frac{\partial \varphi_1}{\partial x} &= f(z)\, g(0) + \int_0^z f(z')\, g_x(z - z')\, dz', \\ \frac{\partial^2 \varphi_1}{\partial x^2} &= f_x(z)\, g(0) + f(z)\, g_x(0) + \int_0^z f(z')\, g_{xx}(z - z')\, dz'.\end{aligned} \tag{11}$$

Likewise choose a contour Γ_2 which contiguous to the point z is a line segment parallel to the y-axis. This gives the functional φ_2 and we see that

$$\frac{\partial \varphi_2}{\partial y} = if(z)\, g(0) + \int_0^z f(z')\, g_y(z - z')\, dz',$$
$$\frac{\partial^2 \varphi_2}{\partial y^2} = if_y(z)\, g(0) + if(z)\, g_x(0) + \int_0^z f(z')\, g_{yy}(z - z')\, dz'. \tag{12}$$

Since $f_x + if_y = \mu f^*$ and $g_x + ig_y = \mu g^*$ it follows from (11) and (12) that

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial y^2} = \mu f^*(z)\, g(0) + \mu f(z)\, g^*(0)$$
$$+ \int_0^z f(z')\, g_{xx}(z - z')\, dz' + \int_0^z f(z')\, g_{yy}(z - z')\, dz'. \tag{13}$$

Here the first integral is over Γ_1 and the second integral is over Γ_2. But g_{yy} is also right regular so the imaginary part of these integrals can be evaluated by an arbitrary contour. Taking the imaginary part of (13) gives

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \operatorname{Im} \int_0^z f(z')\, \mu^2 g(z - z')\, dz' = \mu^2 w. \tag{14}$$

The continuity of the second derivatives can be shown easily, so the proof of Theorem 2 is complete.

COROLLARY 2. *Under the hypotheses of Theorem 20 a convolution product of $f(z)$ and $g(z)$ is denoted by*

$$h(z) = f(z) * g(z) = U + iV \tag{15}$$

and is defined by

$$U = \operatorname{Im} \left[\mu \int_0^z f(z')\, g(z - z')\, dz' + f(z)\, g(0) + \int_0^z f(z')\, g_x(z - z')\, dz'\right] \tag{16}$$

$$V = \operatorname{Im} \left[- if(z)\, g(0) - \int_0^z f(z')\, g_y(z - z')\, dz'\right]. \tag{17}$$

Then the product $h(z)$ is commutative and is a right regular function of z.

Proof. If w is a real panharmonic function let

$$U = \mu w + w_x, \qquad V = - w_y. \tag{18}$$

Then it is seen to be a consequence of Theorem 12 that $h = U + iV$ is a right regular function. If w is defined as in Theorem 20, then (18) proves relations (16) and (17).

The proof of commutativity follows from

$$\varphi_1 = \int_0^z f(z')\, g(z - z')\, dz' = \int_0^z f(z - z'')\, g(z'')\, dz''$$

when the change of variable $z'' = z - z'$ is made in the first integral along the contour Γ_1 giving the second integral along the contour Γ_2. Then Γ_2 is the reflection of Γ_1 in the point $z/2$.

7. Expansion in Pseudo-Powers

We have seen above that the function $u = ce^{in\theta}I_n(\mu r)$ is panharmonic for any constant c. Thus according to Theorem 12 the function $z^{(n)} = \mu u + (Lu)^*$ is right regular. Take $c = n!\, 2^n \mu^{-n-1}$ and

$$\begin{aligned} I_n(x) &= \frac{1}{n!}\left(\frac{x}{2}\right)^n \left[1 + \frac{(x/2)^2}{1(n+1)} + \frac{(x/2)^4}{1 \cdot 2(n+1)(n+2)} + \cdots\right] \\ &= \frac{1}{n!}\left(\frac{x}{2}\right)^n \Psi_n(x). \end{aligned} \tag{1}$$

Then

$$u = n!\, 2^n \mu^{-n-1} e^{in\theta} I_n(\mu r) = z^n \frac{\Psi_n(\mu r)}{\mu}.$$

Note that $Lz^n = 0$ and that

$$\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = e^{i\theta}\left(\frac{\partial}{\partial r} + i \frac{\partial}{\partial \theta}\right)$$

so $Lz^n\Psi_n = z^n L\Psi_n$ and

$$z^{(n)} = z^n \Psi_n(\mu r) + (z^*)^{n+1} \frac{\Psi_n'(\mu r)}{r}.$$

The following is a familiar Bessel identity

$$\Psi_n'(x) = \frac{x}{2(n+1)} \Psi_{n+1}(x). \tag{2}$$

Substituting in the previous formula gives

$$z^{(n)} = z^n\Psi_n(\mu r) + \mu(z^*)^{n+1}\frac{\Psi_{n+1}(\mu r)}{2(n+1)}. \tag{3}$$

The right regular function $z^{(n)}$ may be termed a *pseudopower*. Likewise a left regular function is given by

$$z_-^{(n)} = z^n\Psi_n(\mu r) - \mu(z^*)^{n+1}\frac{\Psi_{n+1}(\mu r)}{2(n+1)}. \tag{4}$$

THEOREM 21. *Let $f(z)$ be right regular in the circular region $|z| \leqslant b$. Then for z in this circle the function $f(z)$ can be expanded in the pseudo-power series.*

$$f(z) = \sum_0^\infty a_n z^n\Psi_n(\mu r) + \left[\sum_0^\infty a_n \frac{\mu}{2(n+1)} z^{n+1}\Psi_{n+1}(\mu r)\right]^* \tag{5}$$

where the coefficients are defined as

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}f(b,\theta)\,d\theta/b^n\Psi_n(\mu b). \tag{6}$$

Proof. Of course the function $f(z)$ is panharmonic in the circle. Thus according to Theorem 10 there is an expansion of the form

$$f = \sum_0^\infty a_n z^n\Psi_n(\mu r) + \sum_1^\infty a_{-n}(z^*)^n\,\Psi_n(\mu r), \tag{7}$$

$$a_n = \frac{1}{2\pi}\int_0^\infty e^{-in\theta}f(b,\theta)\,d\theta/b^{|n|}\Psi_{|n|}(ur). \tag{8}$$

Let

$$2p = z^{(n)} + z_-^{(n)} \qquad \text{and} \qquad 2q = z^{(n)} - z_-^{(n)}$$

then $z^{(n)} = p + q$ and satisfies the condition of Theorem 17. Thus

$$\int_{\Gamma} z^n\Psi_n(\mu r)f(z)\,dz + \frac{\mu}{2(n+1)}\left[\int_\Gamma (z^*)^{n+1}\,\Psi_{n+1}(\mu r)f(z)\,dz\right]^* = 0.$$

Taking the contour Γ to be the circle $|z| = b$ gives

$$\frac{\int_0^{2\pi} e^{i(n+1)\theta}f(b,\theta)\,d\theta}{b^{n+1}\Psi_{n+1}(\mu b)} = \frac{\mu\left[\int_0^{2\pi} e^{-in\theta}f(b,\theta)\,d\theta\right]^*}{2(n+1)\,b^n\Psi_n(\mu r)}.$$

Referring to (8) we see that this means

$$a_{-n-1} = \frac{\mu}{2(n+1)} a_n{}^*, \qquad n = 0, 1, 2, \ldots . \tag{9}$$

But

$$\sum_1^\infty a_{-n}(z^*)^n \Psi_n(\mu r) = \sum_0^\infty a_{-n-1}(z^*)^{n+1} \Psi_{n+1}(\mu r). \tag{10}$$

Substituting (9) in (10) proves (5).

THEOREM 22. *Suppose that $f(z)$ is right regular and $f(\alpha) = 0$ but that $f(z)$ does not vanish identically in the neighborhood of the point α. Then for some positive integer m and some constant $a_m \neq 0$:*

$$f(z) = a_m(z - \alpha)^m + 0 \mid z - \alpha \mid^{m+1} \tag{11}$$

and

$$\frac{\partial f(z)}{\partial x} = m a_m(z - \alpha)^{m-1} + 0 \mid z - \alpha \mid^m . \tag{12}$$

Proof. Clearly it is sufficient to prove (11) and (12) in the case $\alpha = 0$. Then the pseudo-power series (5) of Theorem 21 is valid for a sufficiently small radius b. Thus $a_0 = 0$ but not all the coefficients a_n vanish because $f(z)$ does not vanish identically. Let a_m be the first coefficient which does not vanish. Then (11) is seen to be a direct consequence of (5). Moreover (5) shows that the two terms on the right side of (11) are power series in $X = x - \operatorname{Re} \alpha$ and $Y = y - \operatorname{Im} \alpha$. Differentiating (11) with respect to x proves (12).

The conditions of Theorem 22 are said to define a *zero of multiplicity m.*

THEOREM 23. *Let $f(z)$ be right regular in a domain D and suppose that there is an infinite sequence of distinct zero points $\{\alpha_n\}$ of $f(z)$ which have a limit point α in D. Then actually $f(z)$ vanishes at all points of D.*

Proof. By continuity it follows that $f(\alpha) = 0$. Then it follows directly from Theorem 22 that $f(z)$ vanishes identically in a neighborhood of α. Let β be another point of D and let P be a polygonal path connecting α to β. Moving along P from α let γ be the first point with the property that $f(z)$ does not vanish identically in the neighborhood of the point. But $f(z) = 0$ at all the points of P from α to γ. Thus by what has just been proved it follows that f vanishes in a neighborhood of γ. This contradiction shows that $f(\beta) = 0$.

Q.E.D.

8. The Principle of the Argument

For holomorphic functions the principle of the argument is based on contour integrals of the form $\int (f_x/f)\, dz$. The next theorem is aimed at developing an analogy of such integrals for ν-regular functions.

Theorem 24. *Let $f(z)$ be a right regular function at every point of a region whose boundary is a finite set of simple closed contours Γ^*. Let $\phi(w)$ be a holomorphic function at corresponding points $w = f(z)$. Then*

$$\operatorname{Im} \int_\Gamma \{\phi(f) f_x - \mu \operatorname{Re}[\phi(f) f^*]\}\, dz = 0. \tag{1}$$

Proof. In order to employ Lemma 1 of Section 5 let

$$h(z) = 2\phi f_x - \mu\phi f^* - \mu\phi^* f. \tag{2}$$

But

$$L^*f = L^*f + Lf - \mu f^* = 2f_x - \mu f^*$$

so:

$$\begin{aligned} h &= \phi L^*f - (\phi Lf)^*, \\ Lh &= L\phi L^*f + \phi LL^*f - (L^*\phi Lf + \phi L^*Lf)^*, \\ Lh &= \phi' LfL^*f + \phi LL^*f - (\phi' L^*fLf + \phi L^*Lf)^*. \end{aligned}$$

It is clear from this last relation that $\operatorname{Re} Lh = 0$. Then the proof of Theorem 24 follows from the extension of Lemma 1 to multiple contours.

Theorem 25. *Let $f(z)$ be right regular inside and on a simple closed contour Γ and suppose $f(z) \neq 0$ for points on Γ. Then the total number of zeros of $f(z)$ inside Γ is*

$$N = \operatorname{Re}\left[\frac{1}{2\pi i} \int_\Gamma \left(\frac{f_x}{f} - \frac{\mu f^*}{2f} - \frac{\mu f}{2f^*}\right) dz\right]. \tag{3}$$

Proof. First suppose that f has no zeros inside Γ. Then employ Theorem 24 with $\phi(w) = 1/w$. It follows that the expression N vanishes and this proves Theorem 2 in this case.

Next suppose $f(z)$ has zeros. Then by virtue of Theorem 23 there are a finite number of points $\alpha_1, \alpha_2, \ldots, \alpha_k$ where f vanishes. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be circular contours of radius $\epsilon > 0$ centered at the zero points. Let

$$\Gamma^* = \Gamma - \Gamma_1 - \Gamma_2 - \cdots - \Gamma_k. \tag{4}$$

Then $f(z)$ does not vanish inside the contour Γ^* so

$$\operatorname{Re} \frac{1}{2\pi i} \int_{\Gamma^*} = \operatorname{Re} \frac{1}{2\pi i} \int_{\Gamma_1} + \cdots + \operatorname{Re} \frac{1}{2\pi i} \int_{\Gamma_k} . \tag{5}$$

By virtue of Theorem 22 it is seen that

$$\frac{f_x}{f} - \frac{\mu}{2} \frac{f^*}{f} - \frac{\mu f}{2f^*} = \frac{m_1}{(z - \alpha_1)} + 0(1) \qquad \text{as} \qquad z \to \alpha_1$$

where m_1 is the multiplicity of the zero at α_1. Thus

$$\frac{1}{2\pi i} \int_{\Gamma_1} \left(\frac{f_x}{f} - \frac{\mu f^*}{2f} - \frac{\mu f}{2f^*} \right) dz = m_1 + 0(\epsilon). \tag{6}$$

Substituting (6) in (5) and letting $\epsilon \to 0$ gives

$$\operatorname{Re} \left(\frac{1}{2\pi i} \int_{\Gamma^*} \right) = m_1 + m_2 + \cdots + m_k .$$

Here the right side is the total number of zeros. Q.E.D.

THEOREM 26. *The number N of Theorem 25 is equal to the winding number of f relative to the contour Γ.*

Proof. Write

$$f = u + iv = \rho e^{i\varphi} \tag{7}$$

where u, v, ρ, and φ are real. Then

$$d\varphi = \frac{u\, dv - v\, du}{u^2 + v^2} . \tag{8}$$

The *winding number* N' is the net number of revolutions of the vector f as z traverses Γ once in the positive direction so

$$2\pi N' = \int_\Gamma \left[\frac{uv_x - vu_x}{u^2 + v^2} dx + \frac{uv_y - vu_y}{u^2 + v^2} dy \right] \tag{9}$$

The Cauchy–Riemann equations for a right regular function give the relation

$$uv_y - vu_y = uu_x + vv_x - \mu(u^2 - v^2). \tag{10}$$

Substituting this relation in (9) we get $2\pi N' = A - \mu B$ where

$$A = \int_\Gamma \frac{uv_x - vu_x}{u^2 + v^2}\, dx + \frac{uu_x + vv_x}{u^2 + v^2}\, dy, \tag{11}$$

$$B = \int_\Gamma \frac{u^2 - v^2}{u^2 + v^2}\, dy. \tag{12}$$

Note that

$$(u - iv)(u_x + iv_x) = uu_x + vv_x + i(uv_x - vu_x)$$

so

$$A = \operatorname{Im} \int_\Gamma \frac{(u - iv)(u_x + iv_x)}{u^2 + v^2}\, dz = \operatorname{Im} \int_\Gamma \frac{f_x}{f}\, dz. \tag{13}$$

Also note that

$$2(u^2 - v^2) = (u - iv)^2 + (u + iv)^2$$

so

$$B = \operatorname{Im} \int_\Gamma \frac{(u - iv)^2 + (u + iv)^2}{2u^2 + 2v^2}\, dz = \operatorname{Im} \int_\Gamma \left(\frac{f^*}{f} + \frac{f}{f^*}\right) \frac{dz}{2}. \tag{14}$$

Then (13) and (14) prove $N' = N$. Q.E.D.

The next result shows that the statement of Rouche's theorem for holomorphic functions holds for ν-regular functions without change.

THEOREM 4. *Let $f(z)$ and $g(z)$ be ν-regular inside and on a simple closed contour Γ. Suppose*

$$|f(z)| > |g(z)| \qquad \text{on } \Gamma. \tag{15}$$

Then $f(z)$ and $f(z) - g(z)$ have the same number of zeros inside Γ.

Proof. Let N_λ be the number of zeros of

$$f_\lambda(z) = f(z) - \lambda g(z), \qquad 0 \leqslant \lambda \leqslant 1 \tag{16}$$

inside Γ. Then the integrand of the expression (3) for N_λ is a continuous function of λ. Hence N_λ is a continuous function of λ. But a continuous function which is an integer must be a constant so $N_0 = N_1$. Q.E.D.

Papers [9] and [10] in the following list of references have been added only to show that the electrical or thermal flow in a plate with leakage is governed by the Yukawa equation.

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