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A method of localizing the spectra of sequences of orthogonal polynomials [1]

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Abstract

One of the trends in the theory of orthogonal polynomials is to get as much information on their behaviour as possible from the recurrence relation they satisfy. Our intention is to propose a method which in any particular case allows to localize the spectra of polynomial sequences orthogonal either on the real line or on the complex plane.

Keywords: Orthogonal polynomials; Recurrence relation; Support of orthogonality

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A sequence $\{p_n\}_{n=0}^{\infty} \subset \mathbb{R}[X]$ of orthogonal polynomials on the real line is uniquely determined by its recurrence relation while the orthogonality measure may not be. However, in the case of orthogonality on a bounded set it is and, consequently, so is its support.

In the real line case the support of orthogonality is usually determined or localized in terms of zeroes of polynomials and by means of chain sequences (cf. [1, 2, 4]). What we propose here is a *method* based on some results from operator theory. Though the method can be applicable in the case of polynomials in a single variable (both the real line and the complex case) as well as in several variables, we do not treat the latter case here.

The recurrence relation coefficients determine uniquely the moments of the measure of orthogonality (even more, there is a direct finite matrix algorithm leading from the coefficients to the moments) and vice versa. Thus, though we use in fact the moments to localize the support of orthogonality, one may think of this being done in terms of the coefficients as well.

The related question is to localize the support of orthogonality neglecting some of its isolated points (cf. [3, 5, 6] and references therein). We would like to contribute a bit to this as well.

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1. The real case

1.1. Our method is based on some observation from operator theory (see [9] as well as [10, 11]). Suppose A is a cyclic densely defined symmetric operator in a separable Hilbert space $\mathcal{H}$; cyclicity means that there is a vector f_0, $\|f_0\| = 1$, such that the domain $\mathcal{D}(A)$ of A is the linear span of $\{A^n f_0;\ n = 0, 1, \ldots\}$. Thus $\mathcal{D}(A)$ is invariant for A which implies that for any polynomial p and any $f \in \mathcal{D}(A)$, $p(A)f$ is well defined and is again in $\mathcal{D}(A)$.

Because, by the Schwarz inequality, we have

$$\|Af\|^2 = |\langle A^2 f, f\rangle| \leqslant \|A^2 f\|\,\|f\|, \quad f \in \mathcal{D}(A),$$

induction gives us

$$\|Af\|^{2^n} \leqslant \|A^{2^n} f\|\,\|f\|^{2^n}, \quad f \in \mathcal{D}(A), \quad n = 1, \ldots .$$

Thus, in particular,

$$\|Af_0\| \leqslant a\|f_0\| = a,$$

where

$$a = \liminf_{n\to\infty} \|A^{2^n} f_0\|^{2^{-n}} = \liminf_{n\to\infty} \langle A^{2^{n+1}} f_0, f_0\rangle^{2^{-n-1}}. \tag{1}$$

On the other hand, for any $f \in \mathcal{D}(A)$ there is a polynomial p such that $f = p(A)f_0$. Then, because A is symmetric, we have

$$\|A^{2^n} f\|^2 = \|A^{2^n} p(A) f_0\|^2 = \langle A^{2^{n+1}} f_0, p(A)^* p(A) f_0\rangle \leqslant \|A^{2^{n+1}} f_0\|\,\|p^*(A)p(A)f_0\|.$$

All this gives

$$\|Af\| \leqslant a\|f\|, \quad f \in \mathcal{D}(A).$$

If $a < +\infty$, the operator A is bounded with $\|A\| \leqslant a$. In this case, due to (1), $a \leqslant \|A\|$ and, consequently,

$$\|A\| = a.$$

Because now A is a bounded self-adjoint operator, there is a λ in the spectrum $\sigma(A)$ of A such that $|\lambda| = \|A\|$. Thus we arrived at

(A) *If A is a cyclic symmetric operator with the cyclic vector f_0 and a defined by* (1) *is finite, then the smallest interval centered at* 0 *and containing $\sigma(A)$ is* $[-a, a]$.

1.2. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials which is orthonormal on the real line. Then they satisfy the three-term recurrence relation

$$Xp_n = a_{n+1} p_{n+1} + b_n p_n + a_n p_{n-1}, \quad n = 0, 1, \ldots, \quad p_{-1} = 0,\ p_0 = 1, \tag{2}$$

with

$$b_n \in \mathbb{R}, \quad a_n > 0, \quad n \geqslant 0. \tag{3}$$

The Favard theorem (which goes back to Stieltjes, as noticed in [5]) gives the converse: if the sequence $\{p_n\}_{n=0}^{\infty}$ satisfies the recurrence relation (2) together with (3), then $\{p_n\}_{n=0}^{\infty}$ is orthonormal over the real line. Take $b \in \mathbb{R}$ and consider, instead of (2), the perturbed relation

$$Xp_n^{(b)} = a_{n+1}p_{n+1}^{(b)} + (b_n - b)p_n^{(b)} + a_n p_{n-1}^{(b)}, \quad n = 0, 1, \ldots, \quad p_{-1}^{(b)} = 0,\ p_0^{(b)} = 1. \tag{4}$$

Denoting by μ the measure of orthogonality of $\{p_n\}_{n=0}^{\infty}$ and supposing it is uniquely determined we deduce immediately that

(B) *a measure $\mu^{(b)}$ of orthogonality of $\{p_n^{(b)}\}_{n=0}^{\infty}$ is uniquely determined and $\mu^{(b)}(\beta) = \mu(\beta - b)$ for any Borel subset β of $\mathbb{R}$.*

1.3. Set

$$J(b) = \begin{pmatrix} b_0 - b & a_1 & 0 & 0 & 0 & 0 & \ldots \\ a_1 & b_1 - b & a_2 & 0 & 0 & 0 & \ldots \\ 0 & a_2 & b_2 - b & a_3 & 0 & 0 & \ldots \\ 0 & 0 & a_3 & b_3 - b & a_4 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and $J = J(0)$; $J(b)$ is nothing but the Jacobi matrix of $\{p_n^{(b)}\}_{n=0}^{\infty}$. The relationship between spectral theory of Jacobi matrices as operators in ℓ^2 and orthonormal polynomials they are attached to has been described in great detail in [8]. Taking into account what has been known since then and making use of our observations (A) and (B) the method of localization we are going to propose can be developed as follows: set

$$j_n(b) = \langle J(b)^n f_0, f_0 \rangle,$$

where $f_0 = (1, 0, 0, \ldots)$. Thus $j_n(b)$ is precisely the top-left entry of the matrix $J(b)^n$ or, in other words, the nth moment of the orthogonality measure $\mu^{(b)}$. Taking $A = J(b)$ we define $a = a(b)$ according to the formula (1), that is

$$a(b) = \liminf_{n \to \infty} (j_{2^n}(b))^{2^{-n}}.$$

Let b_* be such that

$$a(b_*) = \inf_{b \in \mathbb{R}} a(b).$$

Then, under the notation introduced so far, we come to [2]

Theorem 1. *If $a(b)$ is finite for some b, then the smallest interval containing the support of the measure of orthogonality of $\{p_n\}_{n=0}^{\infty}$ is $[b_* - a(b_*), b_* + a(b_*)]$.*

1.4. A slightly modified question to localizing the whole support (sometimes called the "true" interval of orthogonality) is that which neglects a finite number of isolated points of the spectrum.

[2] It may be tempting, instead of applying what has been worked out in Section 1, to make a shortcut here using for instance an $\mathcal{L}^{\infty}$-argument. Just a warning: in the complex case there is no other way than this coming from operator theory. Because our intention is to put emphasis on exposing the method rather than on simplifying local arguments, we have resisted this temptation.

The latter question concerns localization of the essential spectrum of the associated Jacobi matrix. Thus our operator approach can be used to contribute to what has been already done in this matter so far.

Suppose $b_n \to b$. Set $B = \mathrm{diag}(b_i)_{i=0}^{\infty}$ and $A = J - B$. Then apparently

$$A = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & \dots \\ a_1 & 0 & a_2 & 0 & 0 & 0 & \dots \\ 0 & a_2 & 0 & a_3 & 0 & 0 & \dots \\ 0 & 0 & a_3 & 0 & a_4 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5}$$

and

$$J = A + bI + B - bI.$$

Since $B - bI$ is compact, due to the Weyl theorem, we have

$$\sigma_e(J) = \sigma_e(A + bI) = \sigma_e(A) + b.$$

Since $\sigma_e(A)$ is contained in the interval $[-a, a]$ where a is given by (1) for A defined by (5), we get another localization result,

Theorem 2. *If $b_n \to b$, then the essential spectrum*[3] *of the orthogonality measure μ of $\{p_n\}_{n=0}^{\infty}$ is contained in the interval $[b - a, b + a]$.*

Invoking the definition of the essential spectrum we restate the above as

Corollary. *If $b_n \to b$, then for any $\varepsilon > 0$ the set*

$$\operatorname{supp} \mu \setminus (b - a - \varepsilon, b + a + \varepsilon)$$

is finite.

2. The complex case

2.1. Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials in $\mathbb{C}[Z]$. Supposing $\deg p_n = n$ we get

$$Zp_n = a_{n,n+1}p_{n+1} + \dots + a_{n,0}p_0, \quad n = 0, 1, \dots \tag{6}$$

with

$$a_{n,n+1} \neq 0, \quad n = 0, 1, \dots . \tag{7}$$

Thus (6) becomes a recurrence relation. In general, in contrast to the real line case, no particular form of the recurrence relation (6) can be deduced from orthogonality.

[3] That is the essential spectrum of the spectral measure of J.

Attach to this relation the Hessenberg matrix

$$H = \begin{pmatrix} a_{00} & a_{01} & a_{0,2} & a_{03} & \cdots & & \\ a_{10} & a_{11} & a_{12} & a_{13} & \cdots & & \\ 0 & a_{21} & a_{22} & a_{23} & \cdots & & \\ 0 & 0 & a_{32} & a_{33} & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then the matrix H acts as an operator (denoted by the same letter) in ℓ^2 according to the formula

$$He_n = \sum_{k=0}^{\infty} a_{nk} e_k,$$

where $e_n = (\delta_{ni})_{i=0}^{\infty}$ is the usual zero–one sequence. Define the domain $\mathcal{D}(H)$ of the operator H by

$$\mathcal{D}(H) = \operatorname{lin}\{e_n;\ n = 0, 1, \ldots\}$$

Then, due to (7), H is a cyclic operator, which means that

$$\mathcal{D}(H) = \operatorname{lin}\{H^n e_0;\ \ n = 0, 1, \ldots\}.$$

Defining the inner product in $\mathbb{C}[Z]$ according to the formula

$$\langle p_m, p_n \rangle_{\mathbb{C}[Z]} = \delta_{m,n},$$

we get that $\mathbb{C}[Z]$ is unitarily equivalent to ℓ^2 and the operator Z of multiplication by the independent variable in $\mathbb{C}[Z]$ becomes unitarily equivalent to H. More precisely, if U is the unitary operator which maps each p_n into e_n, then

$$Z = U^{-1} H U$$

and

$$\langle Z^m, Z^n \rangle_{\mathbb{C}[Z]} = \langle H^m e_0, H^n e_0 \rangle_{\ell^2}.$$

Thus putting

$$c_{i,j} = \langle H^i f_0, H^j f_0 \rangle_{\ell^2} \tag{8}$$

one may think of $\{c_{m,n}\}_{m,n=0}^{\infty}$ as of "moments" of $\{p_n\}_{n=0}^{\infty}$. If the sequence $\{p_n\}_{n=0}^{\infty}$ is orthonormal over the complex plane, that is if there is a nonnegative measure μ on (possibly a subset of) $\mathbb{C}$ such that

$$\langle p_m, p_n \rangle_{\mathbb{C}}[Z] = \int_{\mathbb{C}} p_m(z)\overline{p_n(z)}\mu(\mathrm{d}x\,\mathrm{d}y) = \delta_{m,n}, \quad m, n = 0, 1, \ldots, (z = x + \mathrm{i}y), \tag{9}$$

then $\{c_{m,n}\}_{m,n=0}^{\infty}$ is a moment (bi)sequence of the measure μ and, consequently, μ is a measure of orthogonality (or, rather, orthonormality) of $\{p_n\}_{n=0}^{\infty}$. We now want to study the converse problem.

What can be deduced from (9) is that

$$\sum_{mn,pq=0}^{N} c_{m+q,n+p}\xi_{m,n}\overline{\xi_{p,q}} \geqslant 0, \quad \{\xi_{i,j}\}_{i,j=0}^{N} \subset \mathbb{C}. \tag{10}$$

This and (7) is something which corresponds to (3) in the real case. Again in contrast to the real case, the condition (10) does not imply orthogonality of $\{p_n\}_{n=0}^{\infty}$ (cf. [13] for this and related questions).

2.2. In the bounded case, however, (7) and (10) imply orthogonality of $\{p_n\}_{n=0}^{\infty}$. Adapting the method developed in the real case to the present circumstances we can get both orthogonality and localization of the spectrum simultaneously. First some definitions come: a densely defined operator N in a Hilbert space $\mathcal{H}$ is said to be *formally normal* if $\mathcal{D}(N) \subset \mathcal{D}(N^*)$ and for any $f \in \mathcal{D}(N)$, $\|Nf\| = \|N^*f\|$. It is called *normal* if $\mathcal{D}(N) = \mathcal{D}(N^*)$. Though these two notions look much alike, they may have nothing in common; more precisely, there exist formally normal operators having no normal extension. Thus the couple "formally normal–subnormal" differs a lot from its classical, better known counterpart "symmetric–selfadjoint". Two other notions are related: a densely defined operator S in a Hilbert space $\mathcal{H}$ is said to be *formally subnormal* if there is a Hilbert superspace $\mathcal{K}$ of $\mathcal{H}$ and a formally normal operator N in it such that $\mathcal{D}(S) \subset \mathcal{D}(N)$ and for $f \in \mathcal{D}(S)$, $Sf = Nf$; S is called *subnormal* if N is normal. Not every formally subnormal operator is subnormal (the theory is far from being complete; for some recent results see [13]).

There are two notions of cyclicity: we say that a *formally subnormal* operator S is *cyclic* if there is a vector f_0 such that

$$\mathcal{D}(S) = \operatorname{lin}\{S^n f_0;\ n = 0, 1, \ldots\}$$

while a *formally normal* operator N is **-cyclic* if there a vector f_0 such that

$$\mathcal{D}(N) = \operatorname{lin}\{N^{*n} N^m f_0;\ m, n = 0, 1, \ldots\}.$$

In our situation, we have, cf. [7],

(i) $\{c_{m,n}\}_{m,n=0}^{\infty}$ satisfies (10) if and only if there is Hilbert space $\mathcal{H}$, a cyclic formally subnormal operator S in $\mathcal{H}$ with a cyclic vector f_0 such that

$$c_{m,n} = \langle S^m f_0, S^n f_0 \rangle_{\mathcal{H}}, \quad m, n = 0, 1, \ldots .$$

If this happens, there is another Hilbert space $\mathcal{K}$, containing $\mathcal{H}$, and a *-cyclic formally normal operator N in it with the *same* cyclic vector f_0 such that $\mathcal{D}(S) \subset \mathcal{D}(N)$, for $f \in \mathcal{D}(S)$, $Sf = Nf$ and, consequently,

$$c_{m,n} = \langle N^m f_0, N^n f_0 \rangle_{\mathcal{K}}, \quad m, n = 0, 1, \ldots ;$$

(ii) $\{c_{m,n}\}_{m,n=0}^{\infty}$ is a complex moment sequence if and only if S in the above is subnormal or, equivalently, N has a normal extension;

(iii) under the circumstances of (ii) the measure μ representing $\{c_{m,n}\}_{m,n=0}^{\infty}$ is precisely $\langle E(\mathrm{d}x\,\mathrm{d}y) f_0, f_0 \rangle$, where E is the spectral measure of the normal extension N of S satisfying

$$\mathcal{K} = \overline{\operatorname{lin}\{E(\sigma)f : f \in \mathcal{D}(S),\ \sigma \text{ a Borel subset of } \mathbb{C}\}}$$

(here the overline denotes the closure).

Notice that the linear space $\operatorname{lin}\{N^{*n} N^m,\ m, n = 0, 1, \ldots\}$ is invariant for such an N.

It is a matter of direct verification that if $\{c_{m,n}\}_{m,n=0}^{\infty}$ is defined by (8), then Gram–Schmidt orthonormalization applied to the sequence $\{S^n f_0\}_{n=0}^{\infty}$ of linearly independent vectors (cyclicity of

S!) leads to the sequence $\{p_n(S)f_0\}_{n=0}^{\infty}$. Thus there is a unitary isomorphism between ℓ^2 and $\mathcal{H}$ under which H is equivalent to S. This links our concrete problem concerning the operator H and the general theory of subnormal operators.

2.3. Suppose N is a formally normal operator such that $N\mathcal{D}(N) \subset \mathcal{D}(N)$. Then $A = N^*N$ is a symmetric operator with domain $\mathcal{D}(A) = \mathcal{D}(N)$. Exploiting what has been worked out in Section 1, we get

$$\|N^*Nf_0\| \leqslant a\|f_0\| = a,$$

where

$$a = \liminf_{n\to\infty} \|(N^*N)^{2^n} f_0\|^{2^{-n}} = \liminf_{n\to\infty} \langle (N^*N)^{2^{n+1}} f_0, f_0\rangle^{2^{-n-1}}$$

and $f_0 = \{1, 0, 0, \ldots\}$ as so far, and, because f_0 is a cyclic vector of N,

$$\|(N^*N)\| = a$$

provided $a < +\infty$. Consequently,

$$\|N\| = c,$$

where for c we have now

$$c = \sqrt{a} = \liminf_{n\to\infty} \langle (N^*N)^{2^n} f_0, f_0\rangle^{2^{-n-1}}$$

and, according to (i),

$$c = \liminf_{n\to\infty} (c_{2^n,2^n})^{2^{-n-1}}. \tag{11}$$

If c is finite, the formally normal operator becomes bounded and hence normal. Thus we can apply what is in (i)–(iii).

Now for any $b \in \mathbb{C}$, set

$$H(b) = H - bI,$$

denote the corresponding moment sequence defined by (8) as $\{c_{m,n}^{(b)}\}_{m,n=0}^{\infty}$ and the corresponding number defined by (11) as $c(b)$. Let b_* be such that

$$c(b_*) = \inf_{b\in\mathbb{C}} c(B).$$

Then we arrive at

Theorem 3. *Suppose* (7) *and* (8) *are satisfied. If* c *defined*[4] *by* (11) *is finite, then the smallest disc containing the support of the measure* μ *of orthogonality of* $\{p_n\}_{n=0}^{\infty}$ *is the disk with centre* b_* *and radius* $c(b_*)$.

[4] Notice that there is a finite matrix algorithm leading from the matrix H to the moments $\{c_{m,n}\}_{m,n=0}^{\infty}$: in fact the entry c_{mm} follows via simple recursion from the top-left corner of the matrix H of size $m \times m$.

2.4. Supposing the diagonal entries of H tend to zero, that is

$$a_{n,n} \to b \tag{12}$$

we get an analogue of Theorem 2 and its Corollary.

Theorem 4. *Suppose* (7) *and* (8) *are satisfied and* $c < +\infty$. *Suppose, moreover,* (12) *holds. Then for any* $\varepsilon > 0$ *the set*

$$\operatorname{supp}\mu \setminus \{z \in \mathbb{C};\ |z - b| < c + \varepsilon\}$$

is finite, where c — *defined by* (11) — *corresponds now to the matrix* $H - bI$ *instead of* H *and is finite.*

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