Steklov–Neumann eigenproblems and nonlinear elliptic equations with nonlinear boundary conditions

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\textbf{Article info}

\textbf{Abstract}

We study the solvability of nonlinear second order elliptic partial differential equations with nonlinear boundary conditions. We introduce the notion of “eigenvalue-lines” in the plane; these eigenvalue-lines join each Steklov eigenvalue to the first eigenvalue of the Neumann problem with homogeneous boundary condition. We prove existence results when the nonlinearities involved asymptotically stay, in some sense, below the first eigenvalue-lines or in a quadrilateral region (depicted in Fig. 1) enclosed by two consecutive eigenvalue-lines. As a special case we derive the so-called nonresonance results below the first Steklov eigenvalue as well as between two consecutive Steklov eigenvalues. The case in which the eigenvalue-lines join each Neumann eigenvalue to the first Steklov eigenvalue is also considered. Our method of proof is variational and relies mainly on minimax methods in critical point theory.

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\textbf{1. Introduction}

We are concerned with existence results for nonlinear second order elliptic partial differential equation with (possibly) nonlinear boundary conditions

\begin{equation}
\begin{cases}
-\Delta u + c(x)u = f(x, u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = g(x, u) \quad \text{on } \partial \Omega,
\end{cases}
\end{equation}

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where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with boundary \( \partial \Omega \) of class \( C^{0,1} \), and \( \partial / \partial \nu := v \cdot \nabla \) is the outward (unit) normal derivative on \( \partial \Omega \).

Throughout this paper we shall assume that \( n \geq 2 \) and that the function \( c : \Omega \to \mathbb{R} \), and the nonlinearities \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) and \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) satisfy the following conditions.

(C1) \( c \in L^p(\Omega) \) with \( p \geq n/2 \) when \( n \geq 3 \) (\( p \geq 1 \) when \( n = 2 \)), and \( c \geq 0 \) a.e. on \( \Omega \) with strict inequality on a set of positive measure; that is, \( \int_{\Omega} c(x) \, dx > 0 \).

(C2) \( g \in C(\Omega \times \mathbb{R}) \) and \( f \in C(\Omega \times \mathbb{R}) \).

(C3) There exist constants \( a_1, a_2 > 0 \) such that

\[ |g(x, u)| \leq a_1 + a_2|u|^s \quad \text{with} \quad 0 \leq s < \frac{n}{n-2}. \]

(C3’) There exist constants \( b_1, b_2 > 0 \) such that

\[ |f(x, u)| \leq b_1 + b_2|u|^s \quad \text{with} \quad 0 \leq s < \frac{n+2}{n-2}. \]

The purpose of this paper is to study the existence of weak solutions of problem (1) in which the nonlinearities interact, in some sense, with the Steklov and Neumann spectra. We relate the boundary nonlinearity \( g(x, u) \) to the Steklov spectrum, while the reaction nonlinearity \( f(x, u) \) is related to the Neumann spectrum (see below). By a weak solution of Eq. (1) we mean a function \( u \in H^1(\Omega) \) such that

\[
\int \nabla u \nabla v + \int c(x) u v = \int f(x, u) v + \oint g(x, u) v \quad \text{for any} \quad v \in H^1(\Omega),
\]

where \( \int \) denotes the (volume) integral on \( \Omega \), \( \oint \) denotes the (surface) integral on \( \partial \Omega \), and throughout this paper, \( H^1(\Omega) \) denotes the usual real Sobolev space of functions on \( \Omega \).

The nonlinear problem, Eq. (1), has been considerably studied by many authors in the framework of sub and super-solutions method. We refer e.g. to Amann [2], Mawhin and Schmitt [14], and references therein. Restricting the domain of the nonlinearities (through a slightly modified problem) to the sub and super solutions interval, the methods used in that framework reduce the problem essentially considering bounded nonlinearities and then using a priori estimates and fixed points or topological degree arguments. Since it is based on (the so-called) comparison techniques, the (ordered) sub–super solutions method does not apply when the nonlinearities are compared with higher eigenvalues.

In recent years much work has been devoted to the study of the solvability of elliptic boundary value problems (with linear homogeneous boundary condition) where the reaction nonlinearity in the differential equation interacts with the eigenvalues of the corresponding linear differential equation with linear homogeneous boundary condition (resonance and nonresonance problems). For some recent results in this direction we refer e.g. to the papers by Castro [6], de Figueiredo and Gossez [8], Nkashama and Robinson [17], Rabinowitz [18], and the bibliography therein.

Concerning problem (1) with boundary eigenparameters, there are some (scattered) results in the literature by several authors. For the linear case, we mention the work by Steklov [19] who initiated the problem on a disk in 1902, Amann [2], Bandle [5], and more recently Auchmuty [4]. To the best of our knowledge, not much has been done for the nonlinear problem (1) in the framework of the Steklov spectrum. A few results on a disk (\( n = 2 \)) were obtained by Klingelhöfer [10] and Cushing [7]. (The results in [10] were significantly generalized to higher dimensions in [2] in the framework of sub and super-solutions method as aforementioned.) We also refer to Klingelhöfer [11] where monotonicity methods were used for nonlinearities near the first eigenvalue.

In this paper, we introduce the notion of “eigenvalue-lines” in the plane. These eigenvalue-lines, herein referred to as Steklov-to-Neumann eigenvalue lines, join each Steklov eigenvalue to the
first eigenvalue of the Neumann problem with homogeneous boundary condition. We prove existence results when the nonlinearities involved asymptotically stay, in some sense, below the first eigenvalue-line(s) or in a quadrilateral region (depicted in Fig. 1) enclosed by two consecutive Steklov-to-Neumann eigenvalue-lines. As a special case we derive the so-called nonresonance results below the first Steklov eigenvalue and as well as between two consecutive Steklov eigenvalues. The case where the eigenvalue-lines join each Neumann eigenvalue to the first Steklov eigenvalue (herein referred to as Neumann-to-Steklov eigenvalue-lines) is also considered. Our method of proof is variational and relies mainly on a priori estimates and minimax methods in critical point theory.

This paper is organized as follows. To put our results into context, we have collected in Section 2 some relevant preliminary results on linear Steklov and Neumann eigenproblems which are needed for our purposes. (The proofs of these auxiliary results for the Steklov case may be found in a recent paper of Auchmuty [4].) Section 3 is devoted to the statements of our main results which consist of relating the asymptotic behavior of the nonlinearities involved with the Steklov-to-Neumann or Neumann-to-Steklov eigenvalue-line segments. (The regions involved are depicted in Fig. 1.) In Section 4 we provide some auxiliary results on Critical Point Theory that are needed for the proofs of our main results. Section 5 is devoted to the proofs of our main results, and a few remarks to relate our results to the previous ones in the literature. The case \( c \equiv 0 \) is briefly discussed in Remark 2 (see Section 5). Unlike some previous approaches to problems with nonlinear boundary conditions, all of our results are based upon minimax methods in Critical Point Theory (see e.g. Rabinowitz [18] and references therein).

2. Some preliminaries on Steklov and Neumann problems

To put our results into context, we have collected in this short section some relevant results on linear Steklov and Neumann eigenproblems which are needed for our purposes. We refer to a very recent and interesting paper of Auchmuty [4] for the proofs of the results regarding Steklov eigenproblems.

Consider the linear problem

\[
\begin{cases}
-\Delta u + c(x)u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \mu u & \text{on } \partial \Omega.
\end{cases}
\]

(3)

The Steklov eigenproblem is to find a pair \((\mu, \varphi) \in \mathbb{R} \times H^1(\Omega), \varphi \neq 0 \), such that

\[
\int \nabla \varphi \nabla v + \int c(x) \varphi v = \mu \int \varphi v \quad \text{for any } v \in H^1(\Omega).
\]

Picking \( v = \varphi \), and subsequently \( v \in H^1_0(\Omega) \) one immediately sees that if there is such an eigenpair, then \( \mu > 0 \) and \( \varphi \perp H^1_0(\Omega) \) in the \( H^1-c \)-inner product defined by

\[
(u, v)_c = \int \nabla u \nabla v + \int c(x) uv,
\]

(4)

with the associated norm denoted by \( \|u\|_c \); which is equivalent to the standard norm on \( H^1(\Omega) \). This implies that one can split

\[
H^1(\Omega) = H^1_0(\Omega) \oplus_c \left[ H^1_0(\Omega) \right]^\perp
\]

(5)

as a direct orthogonal sum (in the sense of \( H^1-c \)-inner product).

Besides the Sobolev spaces, we shall make use, in what follows, of the real Lebesgue spaces \( L^q(\partial \Omega), 1 \leq q \leq \infty \), and the compactness of the trace operator \( \Gamma : H^1(\Omega) \to L^q(\partial \Omega) \) for \( 1 \leq q < \frac{2n-1}{n-2} \) (see e.g. Kufner, John and Fučík [12, Chapter 6], Adams and Fournier [1] and references therein).
Sometimes we will just use \( u \) in place of \( \Gamma u \) when considering the trace of a function on \( \partial \Omega \).

Throughout this paper we denote the \( L^2(\partial \Omega) \)-inner product by

\[
(u, v)_{\partial} = \oint uv
\]

and the associated norm by \( \|u\|_{\partial} \).

Assuming that the above assumptions are satisfied, Auchmuty [4] recently proved that, for \( n \geq 2 \), the Steklov eigenproblem (3) has a sequence of real eigenvalues

\[
0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_j \leq \cdots \to \infty, \quad \text{as } j \to \infty,
\]

each eigenvalue has a finite-dimensional eigenspace. The eigenfunctions \( \varphi_j \) corresponding to these eigenvalues form a complete orthonormal family in \([H^1_0(\Omega)]^\perp\), which is also complete and orthogonal in \( L^2(\partial \Omega) \). Moreover, the trace inequality

\[
\mu_1 \oint (\Gamma u)^2 \leq \int |\nabla u|^2 + \int c(x)u^2
\]

holds for all \( u \in H^1(\Omega) \), where \( \mu_1 > 0 \) is the least Steklov eigenvalue for Eq. (3). If equality holds in (7), then \( u \) is a multiple of an eigenfunction of Eq. (3) corresponding to \( \mu_1 \).

Of course for the linear elliptic problem with homogeneous Neumann boundary condition

\[
\begin{aligned}
-\Delta u + c(x)u &= \lambda u \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

it is very well known (see e.g. [9,15]) that Eq. (8) has a sequence of eigenvalues

\[
0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to \infty, \quad \text{as } j \to \infty,
\]

with finite-dimensional eigenspaces such that

\[
\lambda_1 \int u^2 \leq \int |\nabla u|^2 + \int c(x)u^2
\]

for all \( u \in H^1(\Omega) \), where \( \lambda_1 > 0 \) is the first eigenvalue associated with Eq. (8). If equality holds, then \( u \) is a multiple of an eigenfunction of Eq. (8) corresponding to \( \lambda_1 \).

Let us finally mention that no direct comparison between the Steklov and Neumann eigenvalues is available in general.

3. Main results

In this section, we state the main results which consist of relating, in some sense, the asymptotic behavior of the nonlinearities involved with the first Steklov–Neumann eigenvalue-lines, then subsequently with two consecutive higher Steklov-to-Neumann eigenvalue-lines, and finally with two consecutive higher Neumann-to-Steklov eigenvalue-lines (see Fig. 1).

**Theorem 3.1** (Below the first Steklov–Neumann eigenvalue-line(s)). Suppose that the assumptions (C1)–(C3’)

are met. Let the potentials \( F(x, u) = \int_0^u f(x, s) \, ds \) and \( G(x, u) = \int_0^u g(x, s) \, ds \) be such that the following conditions hold.
(C4) There exist $\lambda, \mu \in \mathbb{R}$ such that
\[
\limsup_{|u| \to \infty} \frac{2G(x, u)}{u^2} \leq \mu < \mu_1 \quad \text{and} \quad \limsup_{|u| \to \infty} \frac{2F(x, u)}{u^2} \leq \lambda < \lambda_1
\]
uniformly for $x \in \overline{\Omega}$ with
\[
\lambda_1 \mu + \mu_1 \lambda < \mu_1 \lambda_1. \tag{10}
\]

Then the nonlinear equation, Eq. (1), has at least one solution $u \in H^1(\Omega)$.

Notice that the inequality (10) implies that, in some sense, some ratios of the nonlinearities involved stay asymptotically in the plane-domain $\mathbb{R}^2 := \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda < \lambda_1, \mu < \mu_1, \lambda_1 \mu + \mu_1 \lambda < \mu_1 \lambda_1 \}$

which is depicted in Fig. 1. Therefore, in some sense, these ratios asymptotically stay below the first Steklov–Neumann “eigenvalue line-segments” $\mu = -\frac{\mu_1}{\lambda_1} \lambda + \mu_1$ joining the Neumann eigenvalue-point $(\lambda_1, 0)$ to the Steklov eigenvalue-point $(0, \mu_1)$ in the first quadrant of the $(\lambda, \mu)$-plane. Consequently, we derive the so-called nonresonance below the first eigenvalues associated with two different sets of linear problems; namely the Steklov and Neumann problems. This is a new result even in this case, since $\mu_1$ need not be equal to $\lambda_1$. (Also see the remarks at the end of this paper and [2].)

In the next result, we are concerned with the case where the asymptotic behavior of the nonlinearities is related to two consecutive Steklov-to-Neumann eigenvalue-lines. We impose conditions on the asymptotic behavior of the nonlinearities $g(x, u)$ and $f(x, u)$ directly. These conditions imply similar ones on the asymptotic behavior of the potentials $G(x, u)$ and $F(x, u)$.

**Theorem 3.2 (Between consecutive Steklov-to-Neumann eigenvalue-lines).** Suppose that the assumptions (C1)–(C3) are met, and that the following conditions hold.

(C5) There exist constants $a, b, \alpha, \beta \in \mathbb{R}$ such that
\[
\mu_j < a \leq \liminf_{|u| \to \infty} \frac{g(x, u)}{u} \leq \limsup_{|u| \to \infty} \frac{g(x, u)}{u} \leq b < \mu_{j+1},
\]

and
\[
\alpha \leq \liminf_{|u| \to \infty} \frac{f(x, u)}{u} \leq \limsup_{|u| \to \infty} \frac{f(x, u)}{u} \leq \beta
\]
uniformly for $x \in \overline{\Omega}$ with
\[
\mu_j \lambda_1 < \lambda_1 a + \mu_j \alpha \quad \text{and} \quad \lambda_1 b + \mu_{j+1} \beta < \mu_{j+1} \lambda_1. \tag{11}
\]

Then the nonlinear equation, Eq. (1), has at least one solution $u \in H^1(\Omega)$.

Notice that inequalities (11) imply that, in some sense, some ratios of the nonlinearities involved stay asymptotically in the quadrilateral region in the $(\lambda, \mu)$-plane enclosed by the horizontal line segments $\mu = \mu_j$ and $\mu = \mu_{j+1}$ and two consecutive eigenvalue-lines joining the first Neumann eigenvalue-point $(\lambda_1, 0)$ to the higher Steklov eigenvalue-points $(0, \mu_j)$ and $(0, \mu_{j+1})$, respectively. These quadrilateral regions are depicted as $S_1, S_2, \ldots$ in Fig. 1.
We now take up the case when the asymptotic behavior of the nonlinearities is related, in some sense, to two consecutive Neumann-to-Steklov eigenvalue-lines. (In this case the eigenvalue lines join each Neumann eigenvalue to the first Steklov eigenvalue.)

**Theorem 3.3** (Between consecutive Neumann-to-Steklov eigenvalue-lines). Suppose that the assumptions (C1)–(C3′) are met, and that the following conditions hold.

(C6) There exist constants \(a, b, \alpha, \beta \in \mathbb{R}\) such that

\[
a \leq \liminf_{|u| \to \infty} \frac{g(x, u)}{u} \leq \limsup_{|u| \to \infty} \frac{g(x, u)}{u} \leq b,
\]

and

\[
\lambda_j < \alpha \leq \liminf_{|u| \to \infty} \frac{f(x, u)}{u} \leq \limsup_{|u| \to \infty} \frac{f(x, u)}{u} \leq \beta < \lambda_{j+1}
\]

uniformly for \(x \in \overline{\Omega}\) with

\[
\lambda_j \mu_1 < \lambda_j a + \mu_1 \alpha \quad \text{and} \quad \lambda_{j+1} b + \mu_1 \beta < \lambda_{j+1} \mu_1.
\]  \hspace{1cm} (12)

Then the nonlinear equation, Eq. (1), has at least one solution \(u \in H^1(\Omega)\).

As above, notice that inequalities (12) imply that, in some sense, some ratios of the nonlinearities involved stay asymptotically in the quadrilateral region in the \((\lambda, \mu)\)-plane enclosed by the vertical line segments \(\lambda = \lambda_j\) and \(\lambda = \lambda_{j+1}\) and the two consecutive eigenvalue-lines joining the first Steklov eigenvalue-point \((0, \mu_1)\) to the higher Neumann eigenvalue-points \((\lambda_j, 0)\) and \((\lambda_{j+1}, 0)\), respectively. These quadrilateral regions are depicted as \(N_1, N_2, \ldots\) in Fig. 1.

We will use a variational approach to prove Theorems 3.1–3.3. We therefore need some preliminary results which are given in the next section.
4. Some auxiliary results on critical point theory

In this section we present some auxiliary results which will be needed in the sequel. We only prove those for which, to our knowledge, there is no readily available proof in the literature. The following result on the continuity of the Nemytskii operator on the boundary readily follows from the arguments similar to those used in the proof of Proposition B.1 in [18] or Theorem 2.2 in [3, Chapter I].

Lemma 4.1. Suppose that $g$ satisfies (C2) and there are constants $p, q \geq 1$ and $a_1, a_2$ such that for all $x \in \Omega, \xi \in \mathbb{R}$,

$$|g(x, \xi)| \leq a_1 + a_2 |\xi|^{p/q}.$$ 

Then the Nemytskii operator $\varphi(x) \mapsto g(x, \varphi(x))$ is continuous from $L^p(\partial \Omega)$ to $L^q(\partial \Omega)$.

Now, we consider the energy functional $I : H^1(\Omega) \to \mathbb{R}$ associated with Eq. (1) which is defined by

$$I(u) := \frac{1}{2} \int (|\nabla u|^2 + c(x)u^2) - \int F(x, u) - \int G(x, u), \quad (13)$$

where $G(x, u) = \int_0^u g(x, \xi) \, d\xi$ and $F(x, u) = \int_0^u f(x, \xi) \, d\xi$ are the potentials of $g$ and $f$, respectively.

Taking into account condition (C3) and (C3') with $0 \leq s < \frac{n}{n-2}$ and $0 \leq s < \frac{n+2}{n-2}$, respectively, the compactness of the trace operator from $H^1(\Omega)$ into $L^{s+1}(\partial \Omega)$ (see e.g. Kufner, John and Fučík [12, Chapter 6]), and Lemma 4.1 above, one can use arguments similar to those in the proof of Proposition B.10 in [18] to obtain the following result concerning the properties of the energy functional $I$. However, since the proof of the properties of the boundary-trace part of the nonlinear functional $I$ is not readily available in the literature, we give it below for the reader’s convenience.

Lemma 4.2. Assume that (C1)–(C3') hold. Then $I \in C^1(H^1(\Omega), \mathbb{R})$ and

$$I'(u)v = \int \nabla u \nabla v + \int c(x)uv - \int f(x, u)v - \int g(x, u)v \quad \text{for every } v \in H^1(\Omega), \quad (14)$$

where $I'(u)$ denotes the Fréchet derivative of $I$ at $u$. Moreover,

$$J(u) = \int F(x, u) + \int G(x, u)$$

is weakly continuous, and $J'$ is compact.

Proof. Set

$$J_1(u) := \frac{1}{2} \int (|\nabla u|^2 + c(x)u^2), \quad J_2(u) := \int F(x, u), \quad \text{and} \quad J_3(u) := \int G(x, u).$$

Then, $I(u) = J_1(u) + J_2(u) + J_3(u)$. It follows from the assumptions (C3)–(C3'), the Sobolev embedding of $H^1(\Omega)$ into $L^{2n/(n-2)}(\Omega)$, the continuity of the trace operator from $H^1(\Omega)$ into $L^{2(n-1)/(n-2)}(\partial \Omega)$ and the Hölder inequality that $I$ and $I'(u)$ are well defined. Using arguments similar to those in the proof of Proposition B.10 in [18] one sees that $J_i$ ($i = 1, 2$) belong to $C^1(H^1(\Omega), \mathbb{R})$ with Fréchet derivative given by the first three terms of $I'(u)$. Moreover, $J_2$ is weakly continuous and $J_2'$ is compact. We shall now prove that $J_3$ also belongs to $C^1(H^1(\Omega), \mathbb{R})$, that it is weakly continuous and that $J_3'(u)$ is compact.
We first prove that \( J_3 \) is Fréchet differentiable on \( H^1(\Omega) \), and that \( J'_3(u) \) is continuous. For this purpose, let \( u \in H^1(\Omega) \), we claim that given \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon, u) \) such that

\[
\left| J_3(u + v) - J_3(u) - J'_3(u)v \right| \leq \epsilon \|v\|_c
\]

for all \( v \in H^1(\Omega) \) with \( \|v\|_c < \delta \). Set

\[
\Psi \equiv \left| G(x, u + v) - G(x, u) - g(x, u)v \right|.
\]

It therefore follows that

\[
\left| J_3(u + v) - J_3(u) - J'_3(u)v \right| \leq \int \Psi.
\]

Define

\[
S_1 := \{ x \in \partial \Omega \mid |u(x)| \geq \vartheta \},
\]

\[
S_2 := \{ x \in \partial \Omega \mid |v(x)| \geq \kappa \},
\]

\[
S_3 := \{ x \in \partial \Omega \mid |u(x)| \leq \vartheta \text{ and } |v(x)| \leq \kappa \},
\]

where \( \vartheta \) and \( \kappa \) will be defined later. It then follows that

\[
\int \Psi \leq \sum_{i=1}^3 \int_{S_i} \Psi.
\]

By using the Mean Value Theorem we get that

\[
G(x, \xi + \eta) - G(x, \xi) = g(x, \xi + \theta \eta)\eta,
\]

where \( \theta \in (0, 1) \). It follows from (15) and (C3) that

\[
\int_{S_1} \left| G(x, u + v) - G(x, u) \right| \leq \int \left| g(x, u + \theta v) \right| |v| \leq \int \left[ a_1 + a_2 |u + \theta v|^s \right] |v| 
\]

\[
\leq \int \left[ a_1 + a_2 \left( |u| + |v|^s \right) \right] |v|.
\]

Using Hölder inequality we obtain

\[
\int_{S_1} \left| G(x, u + v) - G(x, u) \right| \leq a_1 |S_1|^{n/(2n-2)} \|v\|_{L^{2n-2/(n-2)}(\partial \Omega)} + a_2 \left( \int_{S_1} |u|^s |v| + \int_{S_3} |v|^s |v| \right)
\]

\[
\leq \left[ a_1 |S_1|^{n/(2n-2)} + a_2 |S_1|^{1/\sigma} \left( \|u\|_{L^{s+1}}^s + \|v\|_{L^{s+1}}^s \right) \right] \|v\|_{L^{2n-2/(n-2)}},
\]

where

\[
\frac{1}{\sigma} + \frac{s}{s+1} + \frac{n-2}{2n-2} = 1.
\]

Notice that \( \frac{s}{s+1} + \frac{n-2}{2n-2} < 1 \), so there exists a \( \sigma > 1 \) such that (16) is satisfied. Using the continuity of the trace operator from \( H^1(\Omega) \) into \( L^t(\partial \Omega) \) with \( t \leq \frac{2(n-1)}{n-2} \) we obtain
\begin{equation}
\int_{S_1} \left| G(x, u + v) - G(x, u) \right| \leq \left[ a_1 |S_1|^{n/(2n-2)} + a_3 |S_1|^{1/\sigma} \left( \|u\|_c^5 + \|v\|_c^5 \right) \right] \|v\|_c \tag{17}
\end{equation}

\begin{equation}
\leq a_4 \|v\|_c \left[ |S_1|^{n/(2n-2)} + |S_1|^{1/\sigma} \left( \|u\|_c^5 + \|v\|_c^5 \right) \right]. \tag{18}
\end{equation}

Similarly,

\begin{equation}
\int_{S_1} \left| g(x, u) v \right| \leq a_5 \|v\|_c \left[ |S_1|^{n/(2n-2)} + |S_1|^{1/\sigma} \left( \|u\|_c^5 + \|v\|_c^5 \right) \right]. \tag{19}
\end{equation}

By the continuity of the trace operator from $H^1(\Omega)$ into $L^t(\partial\Omega)$ with $t \leq \frac{2(n-1)}{n-2}$ and Hölder inequality,

\[ \|u\|_c \geq a_6 \|u\|_{L^2(S_1)} \geq a_6 \delta |S_1|^{1/2}. \]

Hence,

\[ |S_1|^{1/\sigma} \leq \left( \frac{\|u\|_c}{a_6 \delta} \right)^{2/\sigma} = M_1 \quad \text{and} \quad |S_1|^{n/(2n-2)} \leq \left( \frac{\|u\|_c}{a_6 \delta} \right)^{n/n-1} = M_2, \]

$M_1, M_2 \to 0$ as $\delta \to \infty$. Therefore, \( \int_{S_1} \Psi \leq a_7 [M_2 + M_1 (\|u\|_c^5 + \|v\|_c^5)] \|u\|_c. \)

We can assume $\delta \leq 1$ and choose $\delta$ large such that

\[ a_7 [M_2 + M_1 (\|u\|_c^5 + \|v\|_c^5)] \leq \frac{\epsilon}{3}. \]

Hence, \( \int_{S_1} \Psi \leq \frac{\epsilon}{3} \|v\|_c. \)

Similarly,

\begin{equation}
\int_{S_2} \Psi \leq a_3 \int_{S_2} \left[ 1 + \left( \|u\| + \|v\| \right)^5 \right] \|v\| \leq a_4 \left( 1 + \|u\|_c^5 + \|v\|_c^5 \right) \left( \int_{S_2} |v|^{s+1} \left( \frac{|v|}{\kappa} \right)^{m-(s+1)} \right) \frac{s}{s+1} \quad \text{with} \quad m = \frac{2(n-1)}{n-2},
\end{equation}

\begin{align*}
&\leq a_6 \kappa^{(s+1-m)/s+1} \left( 1 + \|u\|_c^5 + \|v\|_c^5 \right) \|v\|_{L^m}^{m/s+1} \\
&\leq a_6 \kappa^{(s+1-m)/s+1} \left( 1 + \|u\|_c^5 + \|v\|_c^5 \right) \|v\|_c^{m/s+1}.
\end{align*}

Since $G \in C^1(\overline{\Omega} \times \mathbb{R})$, given any $\hat{\epsilon}, \hat{\delta} > 0$, there exists a $\hat{\kappa} = \hat{\kappa}(\hat{\epsilon}, \hat{\delta})$ such that

\[ |G(x, \xi + h) - G(x, \xi) - g(x, \xi)h| \leq \hat{\epsilon} |h| \]

for all $x \in \partial\Omega$, $|\xi| \leq \hat{\delta}$, and $|h| \leq \hat{\kappa}$. In particular if $\hat{\delta} = \delta$ and $\hat{\kappa} = \kappa$, this implies

\[ \int_{S_3} \Psi \leq \hat{\epsilon} \int_{S_3} |v| \leq a_7 \hat{\epsilon} \|v\|_{L^1} \leq a_7 \hat{\epsilon} \|v\|_c. \]
Choose \( \hat{\epsilon} \) such that \( a_7 \hat{\epsilon} \leq \epsilon / 3 \). This determines \( \hat{k} = \kappa \). It follows

\[
\int_{\partial \Omega} \Psi \leq \frac{2e}{3} \| v \|_c + a_6 \kappa^{(s+1-m)/s+1}(1 + \| u \|_c^2 + \| v \|_c^2) \| v \|_c^{m/s+1}.
\]

Choose \( \delta \) small so that \( a_6 \kappa^{(s+1-m)/s+1}(1 + \| u \|_c^2 + \| v \|_c^2)\delta^{m/s+1} \leq \frac{\epsilon}{3} \).

Now, we shall prove that \( J'_3(u) \) is continuous, let \( u_m \to u \) in \( H^1(\Omega) \) then by using Hölder inequality and the continuity of the trace operator from \( H^1(\Omega) \) into \( L^s(\partial \Omega) \) with \( t \leq \frac{2(n-1)}{n-2} \), we get

\[
\| J'_3(u_m) - J'_3(u) \| = \sup_{\| v \|_c \leq 1} \left| \int g(x, u_m)v - g(x, u)v \right|
\leq \sup_{\| v \|_c \leq 1} \left| \int g(x, u_m) - g(x, u) \right| \| v \|
\leq \| g(\cdot, u_m) - g(\cdot, u) \|_{L^{s+1/3}} \| v \|_{L^{s+1}}
\leq C \| g(\cdot, u_m) - g(\cdot, u) \|_{L^{s+1/3}}.
\]

By taking into account condition (C3) and Lemma 4.1, we see that the right-hand of the above inequality tends to zero as \( m \to \infty \). Hence, \( J'_3 \) is continuous. Let \( u_n \to u \) in \( H^1(\Omega) \), it follows that \( \| u_n \|_c < C \). By the compactness of the trace operator, there exists a subsequence \( u_{n_k} \to u \) in \( L^{s+1}(\partial \Omega) \).

\[
\| J_3(u_{n_k}) - J_3(u) \| \leq \int |g(x, \xi_{n_k})| |u_{n_k} - u| \quad \text{by the Mean Value Theorem}
\leq \| g(\cdot, \xi_{n_k}) \|_{L^{s+1/3}} \| u_{n_k} - u \|_{L^{s+1}} \quad \text{by Hölder inequality.}
\]

Therefore, by Lemma 4.1 we get that \( J_3(u_{n_k}) \to J_3(u) \). We claim that \( J_3(u_n) \to J_3(u) \), hence \( J_3(u_n) \to J_3(u) \). Suppose by contradiction that \( J_3(u_n) \to J_3(u) \), then there exists a subsequence \( \{ u_{n_k} \} \) such that \( |J_3(u_{n_k}) - J_3(u)| \geq \epsilon \). But the sequence \( \{ u_{n_k} \} \) has a subsequence (we call again \( \{ u_{n_k} \} \)) which converges to \( u \) in \( L^{s+1}(\partial \Omega) \) and \( J_3(u_{n_k}) \to J_3(u) \). This leads to a contradiction. Thus, \( J_3(u_n) \to J_3(u) \).

Finally, let us prove that \( J' \) is compact. Let \( u_n \) be a bounded sequence in \( H^1(\Omega) \), then there exists a subsequence \( u_{n_k} \to u \) in \( H^1(\Omega) \). Therefore, \( u_{n_k} \to u \) in \( L^{s+1}(\partial \Omega) \). Then,

\[
\| J'_3(u_{n_k}) - J'_3(u) \| \leq C \| g(\cdot, u_{n_k}) - g(\cdot, u) \|_{L^{s+1/3}}.
\]

By Lemma 4.1 we get that \( J'_3(u_{n_k}) \to J'_3(u) \). Thus, \( J' \) is compact.

The next result concerns the Palais–Smale condition (PS) which builds some “compactness” into the functional \( I \). It requires that any sequence \( \{ u_m \} \) in \( H^1(\Omega) \) such that

(i) \( \{ I(u_m) \} \) is bounded,
(ii) \( \lim_{m \to \infty} I(u_m) = 0 \),

be precompact.
Owing to the next proposition, to get \((\text{PS})\) it suffices to show that (i)–(ii) imply that \(\{u_m\}\) is a bounded sequence.

**Proposition 4.3.** Assume that \((\text{C1})–(\text{C3}')\) hold. If \(\{u_m\}\) is a bounded sequence in \(H^1(\Omega)\) such that
\[
\lim_{m \to \infty} I'(u_m) = 0
\]
then \(\{u_m\}\) has a convergent subsequence.

**Proof.** Let \(T : H^1(\Omega) \to (H^1(\Omega))^*\) be the duality mapping defined by
\[
T(u)v = \int \nabla u \nabla v + \int c(x)uv,
\]
for all \(v \in H^1(\Omega)\). It follows from the Riesz–Fréchet Representation Theorem and the Open Mapping Theorem that
\[
T^{-1}I'(u) = u - T^{-1}J'_2(u) - T^{-1}J'_3(u).
\]

To show that the conclusion of Proposition 4.3 holds, it suffices to show that \(J'_i(u_m) (i = 2, 3)\) have a (common) convergent subsequence. Indeed, by the Open Mapping Theorem, the assumption that \(\lim_{m \to \infty} I'(u_m) = 0\) and Eq. (20) we have that
\[
u_m = T^{-1}I'(u_m) + T^{-1}J'_2(u_m) + T^{-1}J'_3(u_m) \to \lim_{m \to \infty} \left[ T^{-1}J'_2(u_m) + T^{-1}J'_3(u_m) \right].
\]

But since \(\{u_m\}\) is bounded in \(H^1(\Omega)\), and \(J'_2\) and \(J'_3\) are compact by Lemma 4.2, it follows that \(J'_i(u_m) (i = 2, 3)\) have a (common) convergent subsequence obtained by using the compactness of \(J'_2\) first subsequently followed by that of \(J'_3\). Thus \(\{u_m\}\) has a convergent subsequence and the proof is complete. \(\Box\)

### 5. Proofs of the main results

To prove Theorems 3.1–3.3, we will make use of the Saddle Point Theorem and its variant proved in [18].

**Proof of Theorem 3.1.** Observe that condition \((\text{C4})\) implies that for all \(\epsilon > 0\) there is \(r = r(\epsilon) > 0\) such that
\[
\frac{2G(x, u)}{u^2} \leq \mu + \epsilon \quad \text{and} \quad \frac{2F(x, u)}{u^2} \leq \lambda + \epsilon
\]
for all \(x \in \Omega\) and all \(u \in \mathbb{R}\) with \(|u| > r\). Combining (21) and \((\text{C3})–(\text{C3}')\) there exists a constant \(M_\epsilon > 0\) such that
\[
\forall x \in \Omega, \forall u \in \mathbb{R}, \quad G(x, u) \leq \frac{1}{2}(\mu + \epsilon)u^2 + M_\epsilon \quad \text{and} \quad F(x, u) \leq \frac{1}{2}(\lambda + \epsilon)u^2 + M_\epsilon.
\]

To prove that Eq. (1) has at least one solution, it suffices, according to Theorem 2.7 in [18, p. 8], to show that the functional \(I\) is bounded below and that it satisfies the \((\text{PS})\) condition. Under the assumptions of Theorem 3.1, we shall show that the functional \(I\) is coercive on \(H^1(\Omega);\) that is,
\[
I(u) \to \infty \quad \text{as} \quad \|u\|_c \to \infty,
\]
which would imply that \(I\) is bounded below and that the Palais–Smale is satisfied.
Now let us prove that $I$ is coercive on $H^1(\Omega)$. Assume $\|u\|_c \to \infty$, then by using the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\partial \Omega)$, we get that either $\|u\|_\partial \to \infty$ or $\|u\|_\partial < K$, where $K$ is a positive constant. We claim that in either case $I(u) \to \infty$.

First, suppose that $\|u\|_\partial < K$, since

$$I(u) \geq \frac{1}{2} \left[ 1 - \frac{\lambda}{\lambda_1} - \frac{\mu}{\lambda_1} \right] \|u\|_c^2 - \frac{1}{2} (\mu + \epsilon) \|u\|_\partial^2 - C,$$

where $C$ is a positive constant. Hence $I(u) \to \infty$ as $\|u\|_c \to \infty$, provided $\epsilon > 0$ is sufficiently small, since $\lambda < \lambda_1$.

Now, suppose that $\|u\|_\partial \to \infty$. Using inequalities (7) and (9), one has that

$$I(u) \geq \frac{1}{2} \left[ 1 - \frac{\lambda}{\lambda_1} - \frac{\mu}{\lambda_1} \right] \mu_1 \|u\|_\partial^2 - \frac{1}{2} (\mu + \epsilon) \|u\|_\partial^2 - C$$

Since $\mu_1 \lambda_1 - \mu \lambda_1 - \lambda_1 \mu > 0$ one gets that $I(u) \to \infty$ as $\|u\|_c \to \infty$, provided $\epsilon > 0$ is sufficiently small. Thus $I$ is coercive.

By combining condition (C3) and the coercivity of $I$, one deduces that $I$ is bounded from below; that is,

$$\exists K \in \mathbb{R} \text{ such that } I(u) \geq K, \quad \forall u \in H^1(\Omega).$$

To show that $I$ satisfies (PS), it suffices, according to Proposition 4.3 herein, to show that for any sequence $\{u_m\}$ in $H^1(\Omega)$ such that $\|I(u_m)\|$ is bounded and $\lim_{m \to \infty} I'(u_m) = 0$, it follows that $\{u_m\}$ is bounded. But this follows immediately from the coercivity of $I$ in (23). Hence, by Proposition 4.3, $I$ satisfies (PS). By Theorem 2.7, Chapter 2 in [18] it follows that $I$ has a critical point $u \in H^1(\Omega)$, that is, $I'(u) = 0$. Hence, $u$ satisfies Eq. (2), and thus Eq. (1) has at least one solution. The proof is complete. □

**Proof of Theorem 3.2.** Under the assumptions of Theorem 3.2 we need to show that the conditions of the Saddle Point Theorem are fulfilled. Let

$$V = \text{span}\{\varphi_k \mid k \leq j\}, \quad X = Y \oplus_c H^1_0(\Omega), \quad \text{where } Y = \overline{\text{span}}\{\varphi_k \mid k \geq j + 1\}.$$  \hfill (26)

It follows from (5) and (26) that

$$H^1(\Omega) = V \oplus_c X.$$  \hfill (27)
We need to prove that there exists a constant $r > 0$ such that

$$\sup_{\partial D} I < \inf_X I,$$  \hspace{1cm} (28)

where $D = \{ v \in V : \|u\|_c \leq r \}$. Assuming that this is the case, and the Palais–Smale condition is satisfied, we deduce by the Saddle Point Theorem [18] that $I$ has a critical point. Therefore, Eq. (1) has at least one solution.

We shall show that the functional $I$ is coercive on $X$ and $-I$ is coercive on $V$ which would imply that (28) is satisfied by choosing $r > 0$ sufficiently large.

Notice that condition (C5) implies a similar condition on the potential $G$; that is, there exist constants called again $a, b, \alpha, \beta \in \mathbb{R}$ such that

$$a \leq \liminf_{|u| \to \infty} \frac{2G(x, u)}{u^2} \leq \limsup_{|u| \to \infty} \frac{2G(x, u)}{u^2} \leq b$$  \hspace{1cm} (29)

and

$$\alpha \leq \liminf_{|u| \to \infty} \frac{2F(x, u)}{u^2} \leq \limsup_{|u| \to \infty} \frac{2F(x, u)}{u^2} \leq \beta.$$  \hspace{1cm} (30)

Combining (C3) and (29)–(30), one gets that

$$(a - \epsilon) \frac{u^2}{2} - C \leq G(x, u) \leq (b + \epsilon) \frac{u^2}{2} + C \quad \text{and} \quad (\alpha - \epsilon) \frac{u^2}{2} - C \leq F(x, u) \leq (\beta + \epsilon) \frac{u^2}{2} + C,$$  \hspace{1cm} (31)

where $C$ is a positive constant.

On the one hand, assuming without loss of generality from now on that $\alpha \leq 0$, it follows that for every $u \in V$ one has that

$$I(u) = \frac{1}{2} \|u\|_c^2 - \int F(x, u) - \oint G(x, u)$$

$$\leq \frac{1}{2} \|u\|_c^2 - \frac{1}{2}(\alpha - \epsilon) \int u^2 - \frac{1}{2}(a - \epsilon) \oint u^2 + \tilde{C}$$

$$\leq \frac{1}{2} \left[ 1 - \frac{\alpha}{\lambda_1} + \frac{\epsilon}{\lambda_1} \right] \|u\|_c^2 - \frac{1}{2}(a - \epsilon) \oint u^2 + \tilde{C}$$

$$= \frac{1}{2} \left[ 1 - \frac{\alpha}{\lambda_1} + \frac{\epsilon}{\lambda_1} \right] \|u\|_c^2 - \frac{1}{2}(a - \epsilon) \|u\|_b^2 + \tilde{C}.$$

Using the Parseval identities obtained in [4, p. 331] it follows that

$$I(u) \leq \frac{1}{2} \left[ 1 - \frac{\alpha}{\lambda_1} - \frac{a}{\mu_j} + \frac{\epsilon}{\lambda_1} + \frac{\epsilon}{\mu_j} \right] \|u\|_c^2 + \tilde{C}.$$

By the first inequality in (11) it follows that $1 - \frac{a}{\lambda_1} - \frac{a}{\mu_j} + \frac{\epsilon}{\lambda_1} + \frac{\epsilon}{\mu_j} < 0$, provided $\epsilon > 0$ is sufficiently small. Therefore, by going to the limit as $\|u\|_c \to \infty$, one gets

$$I(u) \to -\infty.$$  \hspace{1cm} (32)
On the other hand, for every \( u \in X \), it follows from (26) that \( u = u^0 + \tilde{u} \), where \( u^0 \in H^1_0(\Omega) \) and \( \tilde{u} \in Y \). Taking into account the \( \epsilon \)-orthogonality of \( \tilde{u} \) and \( u^0 \) in \( H^1(\Omega) \), and assuming without loss of generality from now on that \( \beta \geq 0 \), one has

\[
I(u) = \frac{1}{2} \| u^0 \|_c^2 + \frac{1}{2} \| \tilde{u} \|_c^2 - \int F(x, u) - \oint G(x, u)
\]

\[
\geq \frac{1}{2} \| u^0 \|_c^2 + \frac{1}{2} \| \tilde{u} \|_c^2 - \frac{1}{2} (\beta + \epsilon) \int u^2 - \frac{1}{2} (b + \epsilon) \oint u^2 - \tilde{c}.
\]

Therefore, using the Parseval identities obtained in [4, p. 331] it follows that

\[
I(u) \geq \frac{1}{2} \left[ 1 - \frac{\beta}{\lambda_1} - \frac{\epsilon}{\lambda_1} \right] \| u^0 \|_c^2 + \frac{1}{2} \left( 1 - \frac{\beta}{\lambda_1} - \frac{b}{\mu_{j+1}} - \frac{\epsilon}{\lambda_1} \right) \| \tilde{u} \|_c^2 - \tilde{c}.
\]

Since by the second inequality in (11) one has that \( \lambda_1 > \beta \) and \( 1 - \frac{\beta}{\lambda_1} - \frac{b}{\mu_{j+1}} > 0 \), it follows that for \( \epsilon > 0 \) sufficiently small, \( 1 - \frac{\beta}{\lambda_1} - \frac{\epsilon}{\lambda_1} > 0 \) and \( 1 - \frac{\beta}{\lambda_1} - \frac{b}{\mu_{j+1}} - \frac{\epsilon}{\lambda_1} - \frac{\epsilon}{\mu_{j+1}} > 0 \). Therefore

\[
I(u) \geq \frac{1}{2} \min \left( 1 - \frac{\beta}{\lambda_1} - \frac{\epsilon}{\lambda_1}, 1 - \frac{\beta}{\lambda_1} - \frac{b}{\mu_{j+1}} - \frac{\epsilon}{\lambda_1} - \frac{\epsilon}{\mu_{j+1}} \right) \| u \|_c^2 - \tilde{c}.
\]

By going to the limit as \( \| u \|_c \to \infty \), on gets

\[
I(u) \to \infty \quad \text{as} \quad \| u \|_c \to \infty.
\]

Thus, \( I \) is coercive on \( X \). Furthermore, it follows from the coercivity of \( I \) on \( X \) and condition (C3) that \( I \) is bounded below by a constant on \( X \). Therefore, using (32) we obtain the assertion (28) for some constant \( r > 0 \).

It remains to prove that the functional \( I \) satisfies the Palais–Smale condition. It suffices, according to Proposition 4.3, to show that for any sequence \( \{ u_m \} \) in \( H^1(\Omega) \) such that \( \| I(u_m) \| \) is bounded and \( \lim_{m \to \infty} I'(u_m) = 0 \), it follows that \( \{ u_m \} \) is bounded.

Notice that condition (C5) implies that for every \( \epsilon > 0 \) there exists \( r > 0 \) such that for \( |u| \geq r \),

\[
a - \epsilon \leq \frac{g(x, u)}{u} \leq b + \epsilon \quad \text{for all} \quad x \in \overline{\Omega}.
\]

Let us define \( \gamma : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) by

\[
\gamma(x, u) = \begin{cases} 
\frac{g(x, u)}{u} & \text{for } |u| \geq r, \\
\frac{g(x, r) + g(x, -r)}{2r^2} u + \frac{g(x, r) - g(x, -r)}{2r} & \text{for } |u| < r.
\end{cases}
\]

The function \( \gamma \) is continuous in \( \overline{\Omega} \times \mathbb{R} \) since \( g \) is, moreover by (33) one has

\[
a - \epsilon \leq \gamma(x, u) \leq b + \epsilon \quad \text{for all } u \in \mathbb{R} \text{ and for all } x \in \overline{\Omega}.
\]

Define \( h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) by

\[
h(x, u) = g(x, u) - \gamma(x, u) u,
\]

\[
(35)
\]
then it follows from the continuity of $g$ and $\gamma$ that
\begin{equation}
|h(x, u)| \leq K, \tag{36}
\end{equation}
for all $(x, u) \in \overline{\Omega} \times \mathbb{R}$, where $K > 0$ is a constant.

Using a similar decomposition for the function $f$, we get that
\begin{equation}
l(x, u) = f(x, u) - \tau(x, u)u, \tag{37}
\end{equation}
where $\tau$ and $l(x, u)$ satisfy
\begin{equation}
\alpha - \epsilon \leq \tau(x, u) \leq \beta + \epsilon \tag{38}
\end{equation}
and
\begin{equation}
\|l(x, u)\| \leq K, \tag{39}
\end{equation}
for all $(x, u) \in \overline{\Omega} \times \mathbb{R}$, where $K > 0$ is a constant as above.

Now, let $\{u_m\} \subset H^1(\Omega)$ be such that $\{l(u_m)\}$ is bounded and $\lim_{m \to \infty} l'(u_m) = 0$. By (27) one has $u_m = v_m + x_m$, where $v_m \in V$ and $x_m \in X$. Moreover, by (26) $x_m = x_m^0 + \bar{x}_m$, where $x_m^0 \in H^1_0(\Omega)$ and $\bar{x}_m \in Y$.

Since $\lim_{m \to \infty} l'(u_m) = 0$, it follows that for every $\epsilon > 0$, there exists $N > 0$ such that for all $m \geq N$,
\[
\sup_{\psi \neq 0} \frac{|l'(u_m)\psi|}{\|\psi\|_c} < \epsilon.
\]

Set $\psi = x_m - v_m$ for $m$ large. Then, $l'(u_m)(x_m - v_m) < \epsilon\|x_m - v_m\|_c$. Taking into account the $c$-orthogonality of $x_m$ and $v_m$ in $H^1(\Omega)$, (35) and (37), one gets from the definition of $l'$ that
\[
\begin{aligned}
\|x_m\|_c^2 - \|v_m\|_c^2 &\leq \int \tau(x, u_m)x_m^2 + \int \tau(x, u_m)v_m^2 - \int \gamma(x, u_m)x_m^2 + \int \gamma(x, u_m)v_m^2 \\
&< \epsilon (\|x_m\|_c + \|v_m\|_c) + \int l(x, u_m)x_m - \int l(x, u_m)v_m + \int h(x, u_m)x_m - \int h(x, u_m)v_m.
\end{aligned}
\]

By using (34)–(39), (9) and the continuity of the trace operator, one obtains
\[
\begin{aligned}
\left(1 - \frac{\beta}{\lambda_1} + \frac{\epsilon}{\lambda_1}\right) \left(\|x_m\|_c^2 + \|\bar{x}_m\|_c^2\right) - \|x_m\|_c^2 - \|v_m\|_c^2 &+ \frac{(\alpha - \epsilon)}{\lambda_1} \|v_m\|_c^2 + a \|v_m\|_c^2 \\
&< \epsilon (\|x_m\|_c + \|v_m\|_c) + \bar{K} \|\bar{x}_m\|_c + \bar{K} \|v_m\|_c,
\end{aligned}
\]
provided it is assumed without loss of generality, as before, that $\alpha - \epsilon \leq 0$.

Now, using the Parseval identities obtained in [4, p. 331] it follows that
\[
\begin{aligned}
\left(1 - \frac{\beta}{\lambda_1} + \frac{\epsilon}{\lambda_1}\right) \|x_m\|_c^2 + \left(1 - \frac{\beta}{\lambda_1} - \frac{b}{\mu_{j+1}} + \frac{\epsilon}{\lambda_1} - \frac{\epsilon}{\mu_{j+1}}\right) \|\bar{x}_m\|_c^2 \\
+ \left(\frac{\alpha}{\lambda_1} + \frac{a}{\mu_j} - 1 - \frac{\epsilon}{\lambda_1} - \frac{\epsilon}{\mu_j}\right) \|v_m\|_c^2 &< K_0 (\|x_m\|_c + \|v_m\|_c).
\end{aligned}
\]
Since \( \lambda_1 b + \mu_{j+1} \beta < \lambda_1 \mu_{j+1} \) and \( \lambda_1 \mu_j < \lambda_1 a + \mu_j \alpha \), one has that, for \( \epsilon > 0 \) sufficiently small,

\[
\delta \left( \| x_m^0 \|_c^2 + \| \tilde{x}_m \|_c^2 + \| v_m \|_c^2 \right) < K_0 \left( \| x_m \|_c + \| v_m \|_c \right),
\]

where \( 0 < \delta < \min\{1 - \frac{b}{\lambda_1}, \frac{a}{\mu_j} + \frac{\alpha}{\lambda_j} - 1\} \). Hence,

\[
\| u_m \|_c^2 < K_0 \| u_m \|_c,
\]

which implies that \( \{u_m\} \) is bounded in \( H^1(\Omega) \). Therefore, by Proposition 4.3, \( I \) satisfies the Palais–Smale condition. The proof is complete. \( \square \)

**Proof of Theorem 3.3.** As before we will show that under the assumptions of Theorem 3.3 the conditions of the Saddle Point Theorem are fulfilled. Using the eigenfunction expansion in terms of eigenfunctions of the Neumann problem with linear homogeneous boundary condition, we write the space \( L^2(\Omega) \) in the following way:

\[
L^2(\Omega) = V \oplus V^\perp
\]

where \( V \) is the finite-dimensional subspace of \( L^2(\Omega) \) spanned by the eigenfunctions associated with the eigenvalues \( \lambda_1, \ldots, \lambda_j \), and \( X = V^\perp \) is its infinite-dimensional orthogonal in \( L^2(\Omega) \). Thus, \( X = V^\perp \) is spanned by the eigenfunctions associated with the eigenvalues \( \lambda_{j+1}, \lambda_{j+2}, \ldots \), and hence

\[
H^1(\Omega) = (V \cap H^1(\Omega)) \oplus (V^\perp \cap H^1(\Omega)). \tag{40}
\]

Assuming without loss of generality that \( a \leq 0 \), it follows from inequalities (7), (31) and the eigenfunction expansion that for every \( u \in V \cap H^1(\Omega) \) one has that

\[
I(u) = \frac{1}{2} \| u \|_c^2 - \int F(x, u) - \int G(x, u)
\leq \frac{1}{2} \| u \|_c^2 - \frac{1}{2} (\alpha - \epsilon) \int u^2 - \frac{1}{2} (a - \epsilon) \int u^2 + \tilde{C}
\leq \frac{1}{2} \left[ 1 - \frac{a}{\mu_1} + \frac{\epsilon}{\mu_1} \right] \| u \|_c^2 - \frac{1}{2} (\alpha - \epsilon) \int u^2 + \tilde{C}
\leq \frac{1}{2} \left[ 1 - \frac{a}{\mu_1} - \frac{\alpha}{\lambda_j} + \frac{\epsilon}{\mu_1} + \frac{\epsilon}{\lambda_j} \right] \| u \|_c^2 + \tilde{C}.
\]

By the first inequality in (12) it follows that \( 1 - \frac{a}{\mu_1} - \frac{\alpha}{\lambda_j} + \frac{\epsilon}{\mu_1} + \frac{\epsilon}{\lambda_j} < 0 \), provided \( \epsilon > 0 \) is sufficiently small. Therefore, by going to the limit as \( \| u \|_c \to \infty \), one gets \( I(u) \to -\infty \). Thus, \( -I \) is coercive on \( V \cap H^1(\Omega) \).

Now, assuming without loss of generality that \( b \geq 0 \), it follows from inequalities (7), (31) and the eigenfunction expansion that for every \( u \in X \cap H^1(\Omega) := V^\perp \cap H^1(\Omega) \) one has that

\[
I(u) = \frac{1}{2} \| u \|_c^2 - \int F(x, u) - \int G(x, u)
\geq \frac{1}{2} \| u \|_c^2 - \frac{1}{2} (\beta + \epsilon) \int u^2 - \frac{1}{2} (b + \epsilon) \int u^2 + \tilde{C}
\]
\[ \geq \frac{1}{2} \left[ 1 - \frac{b}{\mu_1} - \frac{\epsilon}{\mu_1} \right] \|u\|_2^2 - \frac{1}{2} (\beta + \epsilon) \int u^2 + \tilde{C} \]

\[ \geq \frac{1}{2} \left[ 1 - \frac{b}{\mu_1} - \frac{\beta}{\lambda_{j+1}} - \frac{\epsilon}{\mu_1} - \frac{\epsilon}{\lambda_{j+1}} \right] \|u\|_C^2 + \tilde{C}. \]

By the second inequality in (12) it follows that \( 1 - \frac{b}{\mu_1} - \frac{\beta}{\lambda_{j+1}} - \frac{\epsilon}{\mu_1} - \frac{\epsilon}{\lambda_{j+1}} > 0 \), provided \( \epsilon > 0 \) is sufficiently small. Therefore, by going to the limit as \( \|u\|_C \to \infty \), one gets \( I(u) \to \infty \). Thus, \( I \) is coercive on \( X \cap H^1(\Omega) \).

Finally, following the steps similar to those used in the proof of Theorem 3.2, one concludes that the Palais–Smale condition is fulfilled. The proof is complete. \( \square \)

Remark 1. If the reaction term \( f \equiv 0 \), then we obtain results comparing the boundary nonlinearity \( g \) with both the first as well as higher Steklov eigenvalues; the latter are not in general simple eigenvalues. This appears to be the first time that the boundary nonlinearity \( g \) is compared with higher Steklov eigenvalues (also see e.g. [13]). Even at the first Steklov eigenvalue, we impose conditions on the potential of the boundary nonlinearity \( g \) rather than on \( g \) itself as was done in previous papers. Notice that, in this case, we do not require a (one-sided) linear growth on \( g \) as was done in [2,10,11] nor do we require monotonicity conditions as was done in [10,11].

Remark 2. Our approach can (slightly) be modified to accommodate the case when \( c \equiv 0 \). The first Steklov and Neumann eigenvalues are both equal to zero in this case (see e.g. [4,7,16]). We require that both \( \mu \) and \( \lambda \) be nonpositive with \( \mu + \lambda < 0 \) in Theorem 3.1, and that the inequality (10) be omitted. Moreover, we ask that \( \alpha = \beta = 0 \) in Theorem 3.2, and that \( a = b = 0 \) in Theorem 3.3; that is, the (nonlinear) reaction term \( f \) is sublinear in Theorem 3.2, whereas the (nonlinear) boundary term \( g \) is sublinear in Theorem 3.3. Both inequalities (11) and (12) must be omitted in this case.

Remark 3. Our results remain valid if one considers nonlinear equations with a more general linear part (in divergence form) with variable coefficients.

\[
\begin{cases}
-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + c(x)u &= f(x,u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial v} + \sigma(x)u &= g(x,u) \quad \text{on } \partial \Omega,
\end{cases}
\]  

(41)

where \( \sigma \in L^\infty(\partial \Omega) \) with \( \sigma(x) \geq 0 \) a.e. on \( \partial \Omega \), and \( \partial/\partial v := v \cdot A\nabla \) is the outward (unit) conormal derivative. The matrix \( A(x) := (a_{ij}(x)) \) is symmetric with \( a_{ij} \in L^\infty(\Omega) \) such that there is a constant \( \gamma > 0 \) such that for all \( \xi \in \mathbb{R}^n \),

\[ \langle A(x)\xi, \xi \rangle \geq \gamma |\xi|^2 \quad \text{a.e. on } \Omega. \]

This includes the so-called regular oblique derivative or Robin boundary conditions. In this case, by using the "eigenvalue lines," the (nonlinear) reaction term \( f \) is compared with the spectrum of the linear equation with a homogeneous Robin boundary condition, whereas the (nonlinear) boundary term \( g \) is compared with the Steklov spectrum.

References