About a condition for starlikeness

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Abstract

A condition for starlikeness will be improved given by the inequality \( \Re \left( f'(x) + \alpha zf''(z) \right) > 0, \ z \in U, \) concerning analytic functions of the form \( f(z) = z + a_2 z^2 + \cdots \) which are defined on the unit disk \( U = \{ z \in C: |z| < 1 \} \).

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1. Introduction

Let \( \mathcal{A} \) be the class of analytic functions defined on the unit disk \( U = \{ z \in C: |z| < 1 \} \) and having the form \( f(z) = z + a_2 z^2 + \cdots, z \in U \). Let \( S^* \) denote the subclass of \( \mathcal{A} \) which consists of functions for which \( f(U) \) is a starlike domain in \( C \) with respect to 0.

An analytic description of \( S^* \) is given by

\[
S^* = \left\{ f \in \mathcal{A} \mid \Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \ z \in C \right\}.
\]

For \( \alpha \geq 0 \) the class \( R_\alpha \) is defined by the equality

\[
R_\alpha = \left\{ f \in \mathcal{A}: \Re \left( f'(z) + \alpha zf''(z) \right) > 0, \ z \in U \right\}.
\]

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The integral versions of the results which we are going to improve in this paper involve the integral operators of Alexander and Libera defined by

\[ A(f)(z) = \frac{z}{t} \int_{0}^{z} f(t) \, dt \quad \text{and} \quad L(f)(z) = \frac{2}{z} \int_{0}^{z} f(t) \, dt. \]

J. Krzyz in [1] has proved that \( R_0 \not\subset S^* \). On the other hand R. Singh and S. Singh have shown in [4] that \( A(R_0) \subset S^* \), where \( A \) denotes the operator of Alexander. This result is equivalent to \( R_1 \subset S^* \).

It is simple to show that \( R_\alpha \subset R_\beta \), if \( \alpha > \beta \geq 0 \).

In [2] P.T. Mocanu improved the result of R. Singh and S. Singh [4] by proving that \( L(R_0) \subset S^* \) which can be rewritten in the form: \( R_{1/2} \subset S^* \). He put in this article the problem to determine

\[ m = \inf \left\{ \alpha \in (0, \infty): R_\alpha \subset S^* \right\}. \] (1)

Up to now this question has not been solved. In [3] the author has proved that if \( \alpha \geq 0.348 \ldots \) then \( R_\alpha \subset S^* \).

In this paper we will prove that \( m \leq \frac{1}{17} = 0.1428 \ldots \).

In the mentioned papers the authors used the method of differential subordinations. The method of convolution seems to be a better tool in the study of this particular case so we will use it in this paper.

We need the following definitions and theorems to prove the main result.

2. Preliminaries

**Definition 2.1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) be two analytic functions in \( U \). The convolution of the functions \( f \) and \( g \) is defined by the equality

\[ (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \]

**Definition 2.2.** Let \( A_0 \) be the class of analytic functions in \( U \) which satisfy \( f(0) = 1 \). If \( V \subset A_0 \) then the dual of \( V \) denoted by \( V^d \) consists of functions \( g \) which satisfy \( g \in A_0 \) and \( (f * g)(z) \neq 0 \) for every \( f \in V \) and every \( z \in U \).

The class \( \mathcal{P} \) is the subset of \( A_0 \) defined by

\[ \mathcal{P} = \left\{ f \in A_0 : \Re(f(z)) > 0, \ z \in U \right\}. \]

**Lemma 1** (Duality theorem). (See [5, p. 23].) The dual of the class \( \mathcal{P} \) is

\[ \mathcal{P}^d = \left\{ f \in A_0 \left| \Re(f(z)) > \frac{1}{2}, \ z \in U \right. \right\}. \]
Let \( h_T \) be the function defined by the equality
\[
h_T(z) = \frac{1}{1 + i T} \left[ i T \frac{z}{1 - z} + \frac{z}{(1 - z)^2} \right], \quad T \in \mathbb{R}.
\]
It is simple to observe that \( h_T \) is an element of the class \( A \).

**Lemma 2.** (See [5, p. 94].) The function \( f \in A \) belongs to \( S^* \), the class of the starlike functions if and only if \( \frac{f(z)}{z} * \frac{h_T(z)}{z} \neq 0 \) for all \( T \in \mathbb{R} \) and for all \( z \in U \).

**Lemma 3** *(The Herglotz formula).* For all \( f \in P \) there exists a probability measure \( \mu \) on the interval \([0, 2\pi]\) so that
\[
f(z) = \frac{2\pi}{2} \int_0^1 \left( \sum_{n=1}^\infty z^n e^{-int} \right) d\mu(t).
\]

The converse of the theorem is also valid.

**Lemma 4.** If \( \theta \in (0, 2\pi) \) and \( p \in (0, \infty) \) then we have the following integral representations
\[
\frac{1}{2p} + \sum_{n=1}^\infty \frac{\cos n\theta}{n + p} = \int_0^\infty \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(p^2 + x^2)(e^{2\pi x} - 1)} \, dx,
\]
\[
\frac{1}{2p} + \sum_{n=1}^\infty \frac{\cos n\theta}{(n + p)(n + 1)} = (p + 1) \int_0^\infty \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(1 + x^2)(p^2 + x^2)(e^{2\pi x} - 1)} \, dx,
\]
\[
\sum_{n=1}^\infty \frac{n \sin n\theta}{(n + p)(n + 1)} = (p + 1) \int_0^\infty \frac{x^2(e^{(2\pi-\theta)x} - e^{\theta x})}{(p^2 + x^2)(1 + x^2)(e^{2\pi x} - 1)} \, dx.
\]

**Proof.** Let \( \Gamma = \gamma_1 \cup \gamma_3 \cup \gamma_2 \cup \gamma_4 \) denote the contour constructed by the following curves
\( \gamma_1(t) = R_m e^{it} \), \( \gamma_2(t) = r e^{-it} \), \( t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) and \( \gamma_3(t) = i R_m + t(\pi - i R_m) \), \( \gamma_4(t) = -ir + t(\pi - i R_m) \), \( t \in [0, 1] \), \( 0 < r < 1 < R_m \), where \( R_m = m + \frac{1}{2}, m \in \mathbb{N} \). We calculate the \( \int_\Gamma f(z) \, dz \) integral where \( f(z) = \frac{e^{it\theta}}{(p+z)(e^{2\pi i z} - 1)} \), \( p > 0 \) using the residue theory.

We observe that
\[
\lim_{R_m \to \infty} \int_{\gamma_1} f(z) \, dz = 0, \quad \text{Res}(f, n) = \frac{e^{i\theta n}}{2\pi i (n + p)}, \quad \forall n \in \mathbb{N},
\]
\[
\lim_{r \to 0} \int_{\gamma_2} f(z) \, dz = -i \pi \cdot \text{Res}(f, 0) = -\frac{1}{2p}
\]
and we get the equality
\[
\lim_{R_m \to \infty} \int_{\gamma_4} f(z) \, dz + \lim_{R_m \to \infty} \int_{\gamma_5} f(z) \, dz = \frac{1}{2p} + \sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n + p}.
\]
So for \( \theta \in (0, 2\pi) \) and \( p \in (0, \infty) \) follows that
\[
\int_0^\infty \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(p^2 + x^2)(e^{2\pi x} - 1)} \, dx + ip \int_0^\infty \frac{(e^{(2\pi-\theta)x} - e^{\theta x})}{(p^2 + x^2)(e^{2\pi x} - 1)} \, dx
\]
\[
= \frac{1}{2p} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n + p} + i \sum_{n=1}^{\infty} \frac{\sin n\theta}{n + p}.
\]
A simple calculation shows that this identity implies the assertion of the lemma. \( \square \)

3. Main result

**Theorem 1.** If \( m \) is defined by (1) then \( m \in \left(\frac{1}{8}, \frac{1}{7}\right) \).

**Proof.** The condition of the theorem can be rewritten in the form \( f'(z) + \alpha z f''(z) \in \mathcal{P} \) and if the development of \( f \) in Maclaurin series is \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), then we get from the Herglotz formula that
\[
f'(z) + \alpha z f''(z) = 1 + \sum_{n=1}^{\infty} a_{n+1}(n + 1)(an + 1)z^n = 1 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} \, d\mu(t).
\]
This implies the equalities
\[
a_n = \frac{2}{n(1 + \alpha(n - 1))} \int_0^{2\pi} e^{-i(n-1)t} \, d\mu(t), \quad n \in N, \ n \geq 2,
\]
and
\[
\frac{f(z)}{z} = 1 + 2\sum_{n=1}^{\infty} \frac{z^n}{(n + 1)(an + 1)} \int_0^{2\pi} e^{-int} \, d\mu(t).
\]
According to Lemma 2 the function \( f \) is starlike if and only if
\[
\frac{f(z)}{z} \ast \frac{h_T(z)}{z} \neq 0, \quad \forall z \in U, \ \forall T \in \mathbb{R},
\]
where \( h_T \) is
\[
h_T(z) = z + \sum_{n=1}^{\infty} \frac{n + 1 + iT}{1 + iT} z^{n+1}.
\]
The properties of the convolution imply that
\[ \frac{f(z)}{z} \ast \frac{h_T(z)}{z} = \left( 1 + 2 \sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} \, d\mu(t) \right) \ast \left( 1 + \sum_{n=1}^{\infty} \frac{n + 1 + iT}{(1 + iT)(n + 1)(an + 1)} z^n \right). \] (3)

According to Lemma 3 we have
\[ 1 + 2 \sum_{n=1}^{\infty} z^n \int_0^{2\pi} e^{-int} \, d\mu(t) \in \mathcal{P}, \]
and thus the equality (3) and Lemma 1 imply that the condition (2) holds true if and only if
\[ u(r, \theta, T) = \text{Re} \left( 1 + \sum_{n=1}^{\infty} \frac{n + 1 + iT}{(1 + iT)(n + 1)(an + 1)} z^n \right) \geq \frac{1}{2}, \]
\[ \forall z = re^{i\theta} \in U, \ \forall T \in \mathbb{R}. \] (4)

The minimum principle for armonic functions implies that the condition (4) is equivalent to
\[ u(1, \theta, T) \geq \frac{1}{2}, \ \forall \theta \in (0, 2\pi), \ \forall T \in \mathbb{R}. \]

If we put \( p = \frac{1}{a} \) then the above condition becomes
\[ \frac{1}{2p} + \frac{1}{1 + T^2} \left( \sum_{n=1}^{\infty} \frac{\cos n\theta}{n + p} + T \sum_{n=1}^{\infty} \frac{n \sin n\theta}{(n + 1)(n + p)} + T^2 \sum_{n=1}^{\infty} \frac{\cos n\theta}{(n + 1)(n + p)} \right) \geq 0, \]
\[ \forall \theta \in (0, 2\pi), \ \forall T \in \mathbb{R}. \] (5)

Let introduce the notation:
\[ M(\theta, T) = \frac{1}{2p} + \frac{1}{1 + T^2} \left( \sum_{n=1}^{\infty} \frac{\cos n\theta}{n + p} + T \sum_{n=1}^{\infty} \frac{n \sin n\theta}{(n + 1)(n + p)} + T^2 \sum_{n=1}^{\infty} \frac{\cos n\theta}{(n + 1)(n + p)} \right). \]

From Lemma 4 we get:
\[ M(\theta, T) = \frac{1}{1 + T^2} \left( \int_0^{\infty} \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(p^2 + x^2)(e^{2\pi x} - 1)} \, dx \right. \]
\[ + T(p + 1) \int_0^{\infty} \frac{x^2(e^{(2\pi-\theta)x} - e^{\theta x})}{(p^2 + x^2)(1 + x^2)(e^{2\pi x} - 1)} \, dx + T^2(p + 1) \int_0^{\infty} \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(1 + x^2)(p^2 + x^2)(e^{2\pi x} - 1)} \, dx \right), \ \theta \in (0, 2\pi), \ \forall T \in \mathbb{R}. \]
In the expression of $M(\theta, T)$ there is in the bracket a polynomial of degree two with respect to $T$, so the condition (5) for starlikeness of the function $f$, holds true if and only if

$$\Delta(\theta) \leq 0, \quad \forall \theta \in (0, 2\pi),$$

where $\Delta(\theta)$ is the discriminant of the mentioned polynomial:

$$\Delta(\theta) = \left( (p + 1) \int_{0}^{\infty} \frac{x^2(e^{(2\pi-\theta)x} - e^{\theta x})}{(p^2 + x^2)(1 + x^2)(e^{2\pi x} - 1)} \, dx \right)^2 - 4(p + 1) \int_{0}^{\infty} \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(1 + x^2)(p^2 + x^2)(e^{2\pi x} - 1)} \, dx \int_{0}^{\infty} \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(p^2 + x^2)(e^{2\pi x} - 1)} \, dx.$$

It is simple to observe that $\Delta(2\pi - \theta) = \Delta(\theta)$, $\forall \theta \in (0, 2\pi)$ and so to prove $\Delta(\theta) \leq 0, \forall \theta \in (0, 2\pi)$ it is enough to show that

$$\Delta(\theta) \leq 0, \quad \forall \theta \in (0, \pi).$$

(6)

Let $f_1, f_2 : (0, \pi) \to (0, \infty)$, be the functions defined by the equalities:

$$f_1(\theta) = (p + 1) \int_{0}^{\infty} \frac{x^2(e^{(2\pi-\theta)x} - e^{\theta x})}{(p^2 + x^2)(1 + x^2)(e^{2\pi x} - 1)} \, dx,$$

$$f_2(\theta) = 4(p + 1) \int_{0}^{\infty} \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(1 + x^2)(p^2 + x^2)(e^{2\pi x} - 1)} \, dx \int_{0}^{\infty} \frac{x(e^{(2\pi-\theta)x} + e^{\theta x})}{(p^2 + x^2)(e^{2\pi x} - 1)} \, dx.$$ 

(7)

It is simple to prove that if $\theta_1, \theta_2 \in (0, \pi), \theta_1 > \theta_2$ then the inequalities

$$0 < e^{(2\pi-\theta_1)x} + e^{\theta_1 x} < e^{(2\pi-\theta_2)x} + e^{\theta_2 x},$$

$$0 < e^{(2\pi-\theta_1)x} - e^{\theta_1 x} < e^{(2\pi-\theta_2)x} - e^{\theta_2 x}$$

hold true for every $x \in (0, \infty)$. So we get from (7) that the functions $f_1$ and $f_2$ both are decreasing on the interval $(0, \pi)$. Since $\Delta(\theta) = f_1(\theta) - f_2(\theta)$, if we check that

$$f_1(\theta_k) < f_2(\theta_{k+1}), \quad \theta_k = \frac{k\pi}{1000}, \quad k = 0.999 \quad \left( f_1(0) = \lim_{\theta \searrow 0} f_1(\theta) = \frac{\pi^2}{4} \right)$$

(8)

then from the monotony of $f_1$ and $f_2$ follow the inequalities:

$$f_1(\theta) < f_1(\theta_k) < f_2(\theta_{k+1}) < f_2(\theta), \quad \theta \in [\theta_k, \theta_{k+1}], \quad k = 0.999$$

and (6) holds true. From Lemma 4 after some calculations results that:

$$f_1(\theta) = \left( \sum_{n=1}^{\infty} \frac{n \sin n\theta}{(n + 1)(n + p)} \right)^2$$

$$= \frac{1}{(p - 1)^2} \left[ p \left( \frac{\pi - \theta}{2} \cos p\theta + \sin p\theta \ln \left( 2 \sin \frac{\theta}{2} \right) + \sum_{k=1}^{p} \frac{\sin(p - k)\theta}{k} \right) \right]$$

$$- \frac{\pi - \theta}{2} \cos \theta - \sin \theta \ln \left( 2 \sin \frac{\theta}{2} \right)^2$$
and
\[
f_2(\theta) = 4 \left( \frac{1}{2p} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n + p} \right) \left( \frac{1}{2p} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{(n + p)(n + 1)} \right)
\]
\[
= 4 \left( \frac{1}{2p} - \cos p\theta \ln 2 \sin \frac{\theta}{2} + \frac{\pi - \theta}{2} \sin p\theta - \sum_{k=1}^{p} \frac{\cos (p-k)\theta}{k} \right)
\times \left[ \frac{1}{2p} + \frac{1}{p-1} \left( \frac{\pi - \theta}{2} \sin \theta - \cos \theta \ln 2 \sin \frac{\theta}{2} - 1 \right.ight.
\]
\[
+ \cos p\theta \ln 2 \sin \frac{\theta}{2} - \frac{\pi - \theta}{2} \sin p\theta + \sum_{k=1}^{p} \frac{\cos (p-k)\theta}{k} \left. \right) \right].
\]

If we put \( p = 7 \), then the inequalities (8) can be checked very easily using a computer.

For \( p = 8 \) and \( \theta_{125} = \frac{125\pi}{1000} \) we have
\[
f_1(\theta_{125}) > f_2(\theta_{125}) \iff \Delta(\theta_{125}) > 0.
\]
This means that the condition for starlikeness (2) does not hold true for every function \( f \), which satisfies a weaker condition \( (f'(z) + \frac{1}{8}zf''(z) > 0, z \in U) \). So we get that
\[
m \in \left( \frac{1}{8}, \frac{1}{7} \right).
\]

We observe that, if it is necessary, the used method offers a more precise calculation of the \( m \).

References