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On the Analytic Setting of C-Calculus

MASSIMO SQUILLANTE AND ALDO G. S. VENTRE

Istituto di Matematica, Facoltà di Architettura, Università di Napoli, Via Monteoliveto, 3, I-80134 Naples, Italy

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1. INTRODUCTION

The observation of some common characteristics of several facts related to measurement, evaluation, or perception (e.g., visual perception) has led E. R. Caianiello [2] to the formulation of a principle and a procedure, that are comprehensively indicated as C-calculus.

It is quite natural to admit that a better knowledge of a single object can be induced by a search of its context. The aim of C-calculus is to formalize this point of view.

Pattern recognition is a field where first C-calculus was mainly applied. The results in this area can enlighten us about further extensions. Let us describe briefly the procedure that, accompanied by the development of ad hoc technical devices, has given rise to a number of interesting results in pattern recognition (see, e.g., [1, 3, 4]).

If a black and white picture P to be processed is viewed through a rectangular (regular) grid, it appears decomposed into elements or cells w_i, $i = 1, ..., n$, for each of which it is possible to state the values of the minimum m_i and maximum M_i of the grey levels in w_i, where the levels are referred to a suitable scale. Suppose that m_i's and M_i's are the unique information about picture analysis. This information could not be satisfactory for us and we would like to attain a better approximation in such a way that suitable smaller intervals of the grey scale correspond to parts in w_i. Let us translate the grid so that another decomposition of P results: let now w'_j, $j = 1, ..., m$, be the general cell, $[m'_j, M'_j]$ the related grey interval. Consider now two cells w_i and w'_j with non empty intersection, $w = w_i \cap w'_j$. We associate the interval L whose endpoints are the maximum among the minima, and the minimum among the maxima, $L = [\max(m_i, m'_j), \min(M_i, M'_j)]$, to the cell w. So the grey levels of a single element in $w_i \cap w'_j$ lie in the interval L.

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The procedure can continue and, on every element e of the picture P it assumes one of the two characters: (i) the sequence of intervals L related to the sequence of cells containing e tends to a single real number, or (ii) the width of L tends to a positive value (in particular it is a constant) with the shiftings of the grid. In the first case the procedure of picture processing is convergent and provides, in every detail of P, a reading as precise as one prefers. In the second case, the process works as a filter by eliminating from the reading the regions of P where L does not degenerate [3–5] (see Section 4). From this one can understand how C-calculus be a method for dealing with situations in which increasing degrees of precision in measurement or prediction are desired. We shall see (Section 3) that the procedure induced by C-calculus includes, in a suitable framework, the concept of integral. A mathematical model of C-calculus was proposed in [5] in a measure theoretical context. Our present aim is to continue the investigation about the analytic formalization of C-calculus.

2. Preliminaries

Let X be a non empty set and $(X^0) = (X_1^0, ..., X_p^0)$ a dissection of X, i.e., a finite pairwise disjoint family of non empty subsets of X such that $\bigcup_{i=1}^p X_i^0 = X$.

Let us define the simple functions,

$$c(x) = \sum_{i=1}^{p} c_i \operatorname{ch}_i(x), \qquad d(x) = \sum_{i=1}^{p} d_i \operatorname{ch}_i(x),$$

where $0 \leqslant c_i \leqslant d_i \leqslant 1$ and $\operatorname{ch}_i(x)$ is the characteristic function of the subset X_i^0, for every $x \in X$. The triple $((X^0), c, d)$ is said to be a *composite set*, or briefly, a *C-set* in X.

Given a function $g: X \to [0, 1] = I$ and a dissection $(X^0) = (X_1^0, ..., X_p^0)$ of X a C-set in X is induced as follows. Let us consider

$$m_i^0 = \inf_{x \in X_i^0} g(x), \qquad M_i^0 = \sup_{x \in X_i^0} g(x)$$

and the simple functions

$$m^0(x) = \sum_{i=1}^{p} m_i^0 \operatorname{ch}_i(x), \qquad M^0(x) = \sum_{i=1}^{p} M_i^0 \operatorname{ch}_i(x);$$

the C-set $((X^0, m^0, M^0)$ is said to be induced by the function g and the dissection (X^0).

Let us consider the further C-set $((X'), m', M')$ induced by g and the dissection $(X') = (X'_1, ..., X'_n)$, where

$$m'(x) = \sum_{j=1}^{n} m'_j \operatorname{ch}_j(x), \qquad M'(x) = \sum_{j=1}^{n} M'_j \operatorname{ch}_j(x).$$

Let us define, for $X_i^0 \cap X'_j \neq \varnothing$,

$$m_{ij}^1 = \max(m_i^0, m'_j), \qquad M_{ij}^1 = \min(M_i^0, M'_j)$$
$$\bar{m}_{ij} = \inf_{X_i^0 \cap X'_j} g(x), \qquad \bar{M}_{ij} = \sup_{X_i^0 \cap X'_j} g(x)$$

and, for $X_i^0 \cap X'_j = \varnothing$,

$$m_{ij}^1 = M_{ij}^1 = \bar{m}_{ij} = \bar{M}_{ij} = 0.$$

The set (X^1) of all non empty entries of the matrix $(X_i^0 \cap X'_j)$ is a dissection of X. If ch_{ij}^1 denotes the characteristic function of $X_i^0 \cap X'_j \neq \varnothing$, the following composition $*$ defines a C-set

$$((X^1), m^1, M^1) = ((X^0), m^0, M^0) * ((X'), m', M'),$$

where $m^1(x) = \sum_{ij} m_{ij}^1 \operatorname{ch}_{ij}^1$, $M^1(x) = \sum_{ij} M_{ij}^1 \operatorname{ch}_{ij}^1(x)$, because $m_{ij}^1 \leqslant \bar{m}_{ij} \leqslant \bar{M}_{ij} \leqslant M_{ij}^1$.

The C-set $((X^1), m^1, M^1)$ is indicated as the C-product of $((X^0), m^0, M^0)$ and $((X'), m', M')$. The operation $*$ is commutative [5]. Observe that the C-product of two C-sets induced by g is not, in general, a C-set induced by g.

The validity of $*$ can be extended. In fact, let $((X''), m'', M'')$ be a C-set in X induced by g and the dissection $(X'') = (X''_1, ..., X''_q)$, with ch''_h the characteristic function of X''_h, $h = 1, ..., q$. Let, analogously,

$$m_{ijh}^2 = \max(m_{ij}^1, m''_h)$$
$$M_{ijh}^2 = \min(M_{ij}^1, M''_h).$$

If $(X^2) = (X_{ijh}^2)$, the triple $((X^2), m^2, M^2)$, with $m^2(x) = \sum_{ijh} m_{ijh}^2 \operatorname{ch}_{ijh}^2(x)$, $M^2(x) = \sum_{ijh} M_{ijh}^2 \operatorname{ch}_{ijh}^2(x)$ is a C-set, because $m^2(x) \leqslant M^2(x)$. Assuming this extension, C-product is associative and the validity of $*$ can be extended to a finite number of C-sets.

Let Cg denote the subset of all C-sets in X whose elements are C-sets induced by g or their C-products. $(Cg, *)$ is a commutative monoid whose identity is $(X, \mathbf{0}, \mathbf{1})$, the C-set having the one element dissection, where $\mathbf{0}$ and $\mathbf{1}$ are the constant functions taking values 0 and 1 in X, respectively.

Consider now the sequence,

$$\begin{aligned} C_0 &= ((X^0), m^0, M^0) \\ C_1 &= ((X^1), m^1, M^1) = ((X^0), m^0, M^0) * ((X^{(1)}), m^{(1)}, M^{(1)}) \\ &= C_0 * ((X^{(1)}, m^{(1)}, M^{(1)}) \\ &\vdots \\ C_k &= ((X^k), m^k, M^k) \\ &= ((X^{k-1}), m^{k-1}, M^{k-1}) * (X^{(k)}, m^{(k)}, M^{(k)}) \\ &= C_{k-1} * ((X^{(k)}), m^{(k)}, M^{(k)}) \\ &\vdots \end{aligned} \tag{1}$$

The sequence $\{C_k\}$ is said to be induced by g. If X_s^k is the general non empty element of the dissection (X^k), then for every $x \in X_s^k$ it is

$$m^0(x) \leqslant m^1(x) \leqslant \cdots \leqslant m^k(x) \leqslant g(x) \leqslant M^k(x) \leqslant \cdots \leqslant M^1(x) \leqslant M^0(x) \tag{2}$$

with $m^k(x) = \sum_s m_s^k \operatorname{ch}_s^k(x)$, $M^k(x) = \sum_s M_s^k \operatorname{ch}_s^k(x)$.

Therefore the sequences $\{m^k(x)\}$ and $\{M^k(x)\}$ converge in every point of X. Let us set

$$m(x) = \lim_k m^k(x), \qquad M(x) = \lim_k M^k(x).$$

It is, by (2),

$$m(x) \leqslant g(x) \leqslant M(x), \qquad x \in X.$$

It is worth sketching some interpretation. The sequence $\{C_k\}$ represents a reading of the set X, depending on the function g. Equation (1) determines the transition from the state $(k-1)$th to the kth.

The reading at the kth state is "not worse" than all previous readings because the gap $M^i(x) - m^i(x)$ in every x does not increase with i.

3. Convergence

For every $x \in X$ there is a unique element X_{sk}^k of the dissection (X^k) that contains x, and a sequence of nested intervals

$$I_{so}^0 \supseteq \cdots \supseteq I_{sk}^k \supseteq \cdots, \qquad I_{sk}^k = [m_{sk}^k, M_{sk}^k]$$

is determined. It is known [8] that given the real intervals $[a, b]$ and $[a', b']$, the function d, defined by $d([a, b], [a', b']) = \max(|a' - a|, |b' - b|)$

is a distance defined in the family of the real intervals. If the sequence $\{I_{sk}^k\}$ converges to a non degenerate interval, then we say that $\{C_k\}$ is of the *first kind* in x. If $\{I_{sk}^k\}$ converges to a real number r, then $\{C_k\}$ is said to be of the *second kind* in x or the related procedure of C-products is said to be a *convergence case.*

Of course, one of the two cases necessarily occurs. If $\{C_k\}$ is of the first (second) kind for every x in a subset A of X, then $\{C_k\}$ is said to be of the first (second) kind in A. Furthermore $\{C_k\}$ is, by definition, uniformly of the second kind in $A \subseteq X$, if

$$\lim_k d(I_{sk}(x), g(x)) = 0$$

holds uniformly with respect to $x \in A$.

The following result is related to the convergence case in a dense subset of X.

PROPOSITION 1. *Let X be a topological space and $\{C_k\} = \{((X^k), m^k, M^k)\}$ a sequence induced by g, where all elements in the dissections are open sets in X. If $\{C_k\}$ is of the second kind uniformly in a dense subset Y of X, then $\{C_k\}$ is of the second kind in X.*

Proof. If Y is dense in X, then there is a common point to Y and the non empty open set X_s^k. Then it is, for x and x' in a given element of the dissection X_{sk}^k,

$$M_{sk}^k(x) - m_{sk}^k(x) = M_{sk}^k(x') - m_{sk}^k(x'). \tag{3}$$

As $\{C_k\}$ is of the second kind uniformly in Y, for every $\varepsilon > 0$, then there exists $\bar{k} \in N$ such that

$$k > \bar{k} \text{ implies } M^k(\bar{x}) - m^k(\bar{x}) < \varepsilon, \qquad \forall \bar{x} \in Y.$$

So, by (3), $\{C_k\}$ is of the second kind also in x.

In order to study sequences $\{C_k\}$ of first or second kind (in suitable subsets of X), let us introduce some measure theoretical hypotheses. Throughout the present paper (X, A, μ) will denote a measure space with A a σ-algebra of subsets of X and μ a totally finite positive measure. A C-set induced by a measurable dissection is indicated as a measurable C-set. Let us consider measurable C-sets and put

$$L(g, X^k) = \sum_s \mu(X_s^k)\, m_s^k, \qquad U(g, X^k) = \sum_s \mu(X_s^k)\, M_s^k$$

$$\bar{L}(g, X^k) = \sum_s \mu(X_s^k)\, \bar{m}_s^k, \qquad \bar{U}(g, X^k) = \sum_s \mu(X_s^k)\, \bar{M}_s^k,$$

where $\bar{m}_s^k = \inf_{X_s^k} g(x)$, $\bar{M}_s^k = \sup_{X_s^k} g(x)$. (The index s_k will be replaced with s, whenever possible.) The following inequalities hold for every k

$$\begin{gathered} L(g, X^k) \leqslant \bar{L}(g, X^k) \leqslant \bar{U}(g, X^k) \leqslant U(g, X^k) \\ \bar{U}(g, X^k) - \bar{L}(g, X^k) \leqslant U(g, X^k) - L(g, X^k) = \sum_s \mu(X_s^k)\, d_s^k \leqslant \mu(X), \end{gathered} \tag{4}$$

with $d_s^k = M_s^k - m_s^k$. For every dissection (X^k) each of the couples $(\bar{L}(g, X^k), \bar{U}(g, X^k))_k$ and $(L(g, X^k), U(g, X^k))_k$ defines separated classes. Moreover if $S = \text{g.l.b.}_{X_k}(U(g, X^k)) = \text{l.u.b.}_{X^k}(L(g, X^k))$, then, by (4), $S = \int g\, d\mu$.

Let us consider some examples of sequences of the second kind. We observe first that if g is continuous on a complete metric space X and the diameter of (X^k) tends to zero as $k \to \infty$, then the sequence $\{C_k\}$ is of the second kind in X.

PROPOSITION 2 (Compare [5]). *Let $\{C_k\}$ be a sequence of measurable C-sets. If for every $\varepsilon > 0$ there exists $k' \in N$ such that*

$$k > k' \qquad \textit{implies} \qquad \sum_s \mu(X_s^k)\, d_s^k < \varepsilon$$

then

(i) *g is integrable and the sequences $\{m^k(x)\}$ and $\{M^k(x)\}$ converge in the mean to g in X;*

(ii) *the sequence $\{C_k\}$ is of the second kind almost everywhere in X.*

The following proposition provides for a preliminary information on the kind of given special sequences.

PROPOSITION 3. *Let $g: X \to I$ a measurable function and $((Y^k), m^k, M^k)$ a sequence of measurable C-sets induced by g and $(Y^{(0)})$, with $(Y^{(k)}) = (Y_1^{(k)}, ..., Y_{pk}^{(k)})$.*

If, for every $j = 1, ..., p_k$, there exists $i_j = 1, ..., 2^k + 1$ such that

$$Y_j^{(k)} \subseteq X_{i_j}^k = \left\{ x \in X: \frac{i_j - 1}{2^k} \leqslant g(x) < \frac{i_j}{2^k} \right\}, \tag{5}$$

then the sequence of C-sets

$$\begin{aligned} C_0 &= (Y^{(0)}, m_y^{(0)}, M_y^{(0)}) \\ C_1 &= ((Y^1), m_y^1, M_y^1) = C_0 * ((Y^{(1)}), m_y^{(1)}, M_y^{(1)}) \\ &\vdots \\ C_k &= ((Y^k), m_y^k, M_y^k)) = C_{k-1} * ((Y^{(k)}), m_y^{(k)}, M_y^{(k)}), \end{aligned}$$

(*where* $m_y^{(k)} = \sum_j \inf g(x)\,\mathrm{ch}_j(x)$, $M_y^{(k)} = \sum_j \sup g(x)\,\mathrm{ch}_j(x)$, *the infima and suprema are taken over the* $Y_j^{(k)}$*'s and* ch_j *is the characteristic function of* $Y_j^{(k)}$) *is uniformly of the second kind in* X.

In order to prove Proposition 3, let us mention the following lemmas:

LEMMA 1 (Compare [7]). *Let f be a measurable non negative function defined in X. Then there exists a non decreasing sequence $\{f_n(x)\}$ of non negative simple functions such that*

$$\lim_n f_n(x) = f(x), \qquad \forall x \in X. \tag{6}$$

If f is bounded, then (6) *holds uniformly in X. The sequence $\{f_n\}$ is defined by*

$$f_n(x) = \begin{cases} \dfrac{i-1}{2^n} & \text{for } \dfrac{i-1}{2^n} \leqslant f(x) < \dfrac{i}{2^n},\ i = 1, ..., n \cdot 2^n \\ n & \text{for } f(x) > n. \end{cases}$$

LEMMA 2. *Let f be a non negative measurable function defined in X. Then there exists a non increasing sequence of simple functions $\{h_n\}$ such that*

$$\lim_n h_n(x) = f(x), \qquad \text{uniformly in } X. \tag{7}$$

Indeed, let us set $M = \sup_X f(x)$ and

$$h_n(x) = \begin{cases} \dfrac{i}{2^n} & \text{for } \dfrac{i-1}{2^n} \leqslant f(x) < \dfrac{i}{2^n},\ i = 1, ..., n \cdot 2^n \\ M + \dfrac{1}{2^n} & \text{for } f(x) \geqslant n. \end{cases}$$

It is easy to see that the sequence $\{h_n(x)\}$ is decreasing and as it holds $0 \leqslant g_n(x) - g(x) \leqslant 1/2^n$, (7) follows by choosing $n > M$. It is worth observing, from Lemma 2, that:

A. If

$$0 \leqslant f(x) \leqslant 1 \tag{8}$$

then

$$0 \leqslant h_n(x) \leqslant 1, \qquad n > 1.$$

B. Under hypothesis (8) the following representations are allowed for f_n and h_n

$$f_n(x) = \sum_{i=1}^{2^n} \frac{i-1}{2^n} \mathrm{ch}_{i,n}(x) \tag{9}$$

$$h_n(x) = \sum_{i=1}^{2^n} \frac{i}{2^n} \mathrm{ch}_{i,n}(x) \tag{10}$$

$n > 1$, where $\mathrm{ch}_{i,n}$ is the characteristic function of the ith element in the dissection (X^n) defined as in (5).

We are now able to prove Proposition 3.

Observe first that, by hypothesis (5), every element of dissection (X^{k+1}) has the representation

$$X_i^{k+1} = \bigcup_{m=1}^{q_i} Y_{jm}^{(k+1)}. \tag{11}$$

From (9), (10), and (11) we obtain for $n > 1$,

$$m_y^k \geqslant m_y^{(k+1)}(x) = \sum_{j=1}^{p_{k+1}} m_{j(y)}^{(k+1)} \mathrm{ch}_{j,k+1}^{y}(x) \geqslant f_{k+1}(x).$$

Analogously,

$$M_y^k(x) \leqslant h_{k+1}(x).$$

Then Proposition 3 follows.

4. Filtering

1. Our present concern is to analyze sequences of C-products of the first kind in a non empty subset of X. For a reason that will be illustrated in the next section by an application, this case is indicated as filtering case.

Let us observe that if the sequence $\{C_k\}$ is of the first kind for some $x \in X$, then a multifunction is defined

$$G: x \in X \to I(x) = \lim_k I_s^k = [m(x), M(x)]$$

depending on g and the sequence of C-products, where $m(x)$, $M(x)$, and I_s^k are defined in Sections 2 and 3. (For definitions and results on multifunctions see [6, 9].)

PROPOSITION 1. *Let $g: X \to [0, 1]$ be a function and $\{C_k\}$ a sequence of C-products. Then G is measurable.*

Proof. It is sufficient to show that for every $[a, b] \subset \mathbb{R}$, $G^{-1}([a, b])$ is measurable, where

$$G^{-1}([a, b]) = \{x \in X: \quad [m(x), M(x)] \cap [a, b] \neq \emptyset\}.$$

Indeed, let us observe first that

$$[m(x), M(x)] \cap [a, b] = \emptyset \qquad \text{iff} \quad M(x) < a \text{ or } m(x) > b;$$

therefore

$$G^{-1}([a, b]) = X - (\{x \in X: M(x) < a\} \cup \{x \in X: m(x) > b\}). \qquad (12)$$

Because

$$m(x) = \lim_k m^k(x) \qquad \text{and} \qquad M(x) = \lim_k M^k(x)$$

are measurable functions, then the set (12) is measurable.

Then the measurability of G follows. Let us observe that for every $x \in X$, from the proposition above it follows the existence of a Castaing representation for G

$$G(x) = \text{cl} \bigcup_{n \in N} s_n(x),$$

i.e., the set $\{s_n(x)\}_{n \in N}$ is dense in $G(x)$, where $\{s_n\}$ is a sequence of measurable selections of G.

2. It is worth stating some regularity properties of G under suitable more restrictive hypotheses.

Before we do this let us recall the following result.

PROPOSITION 2 (See [9]). *Let $G: X \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a uniformly bounded compact valued multifunction. Then G is continuous if and only if for every $\bar{x} \in X$ it is*

$$\lim_{x \to \bar{x}} \text{haus}(G(x), G(\bar{x})) = 0$$

PROPOSITION 3. *Let X be a compact in $\mathbb{R}^m$, $g: X \to [0, 1]$ a measurable function and $\{C_k\}$ a sequence of C-products induced by g, with all the elements of the dissection (X^k) open on X. If $m(x)$ is upper semicontinuous and $M(x)$ is lower semicontinuous, then G is continuous in X.*

Proof. Let now $\bar{x}$ be an element of X. By definition

$$\text{haus}(G(x), G(\bar{x})) = \max(|m(x) - m(\bar{x})|, |M(x) - M(\bar{x})|).$$

If all (X^k) are open dissections, then the functions $m^k(x)$ and $M^k(x)$ are continuous.

Then, as the sequences $\{m^k(x)\}$ and $\{M^k(x)\}$ are monotone and the related limits m and M are semicontinuous their convergence is uniform and both functions m and M are continuous in X.

Thus

$$\lim_{x \to \bar{x}} \text{haus}(G(x), G(\bar{x})) = 0$$

and therefore, by Proposition 2, the continuity of G follows.

5. A Filtering Case in Pattern Recognition

Let a digitized picture in the plane xy be represented by a matrix $A = (a_{ij})$, whose entries yield the grey levels $g = g(x, y)$ (see [1, 5]).

Furthermore a window is provided such that the maximum M and the minimum m of the grey levels are observed over the rectangular area w, viewed through the window. If the picture is scanned, so that the matrix A is completely covered, all the addends in the sum in (4), for $k = 1$, are known.

The picture is approximated by a grid superimposed to it, which shows, for each area w, the bounds of the grey levels. Translate now the grid so that a different partition of the picture is read. The C-product of the two readings takes into account all pieces of information of both of them: a refinement of the distribution of the grey levels in the picture is obtained or not depending on the width of the window, the width of the translation of the grid, and the differences d_s^k between the grey levels.

Let us mention the typical filtering procedure of "the saucer on the chessboard [1]." If the region viewed through the window includes strictly the single square, the sequence $\{C_k\}$ induced by the translations of the grid, acts as a filter because the values M_s^k and m_s^k over each region X_s^k remain constant—say 1 and 0, respectively—for every k and s. Thus only the saucer can be seen because the sequence of intervals I_{sk}^k is the sequence of real unit intervals $[0, 1]$, for every x out of the saucer and on the chessboard.

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