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On Discrete Inequalities of Wirtinger's Type

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Discrete inequalities of Wirtinger's type are considered. Constants in the obtained inequalities are the best ones. In the special case the inequalities (1) and (2) are obtained. They are proved by K. Fan, O. Taussky, and J. Todd: Discrete analogs of inequalities of Wirtinger, *Montash. Math.* **59** (1955), 73–79.

Fan *et al.* proved [1] (see also [2–4]) the following results:

THEOREM A. *If $x_1, x_2,..., x_n$ are n real numbers and $x_1 = 0$, then*

$$\sum_{k=1}^{n-1} (x_k - x_{k+1})^2 > 4 \sin^2 \frac{\pi}{2(2n-1)} \sum_{k=2}^{n} x_k^2, \tag{1}$$

unless $x_k = A\hat{x}_k$, where

$$\hat{x}_k = \sin \frac{(k-1)\pi}{2n-1} \qquad (k = 1, 2,..., n).$$

THEOREM B. *If $x_1, x_2,..., x_n$ are n real numbers, then*

$$\sum_{k=0}^{n} (x_k - x_{k+1})^2 > 4 \sin^2 \frac{\pi}{2(n+1)} \sum_{k=0}^{n} x_k^2 \tag{2}$$

(where $x_0 = x_{n+1} = 0$) unless $x_k = A\hat{x}_k$, where

$$\hat{x}_k = \sin \frac{k\pi}{n+1} \qquad (k = 1, 2,..., n).$$

A generalization of inequalities (1) and (2) will be given in this paper. In

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fact, we shall consider the determination of the best constants A_n and B_n in the inequalities

$$A_n \sum_{k=2}^{n} p_k x_k^2 \leqq \sum_{k=1}^{n-1} r_k (x_k - x_{k+1})^2 \leqq B_n \sum_{k=2}^{n} p_k x_k^2,$$

where $\mathrm{p} = (p_k)$ and $\mathrm{r} = (r_k)$ are given weight sequences.

Let $\mathrm{a} = (a_1, a_2, \ldots, a_n)$ and $\mathrm{b} = (b_1, b_2, \ldots, b_{n-1})$ be sequences of positive numbers.

A symmetric three-diagonal matrix of nth order will be denoted by $H_n(\mathrm{a}, \mathrm{b})$. The main diagonal of the matrix consists of elements of the sequence a, and lateral diagonals consist of elements of the sequence b, i.e.,

$$H_n(\mathrm{a}, \mathrm{b}) = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ b_1 & a_2 & b_2 & & 0 & 0 \\ \vdots & & . & & & \\ 0 & 0 & 0 & & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & & b_{n-1} & a_n \end{bmatrix}.$$

First, we shall prove two auxiliary results:

LEMMA 1. *Let $(r_k)_{k \in N}$ and $(p_k)_{k \in N}$ be positive sequences. Then the matrix $H_{n-1}(\mathrm{a}, \mathrm{b})$, with*

$$\mathrm{a} = \left(\frac{r_1 + r_2}{p_2}, \ldots, \frac{r_{n-2} + r_{n-1}}{p_{n-1}}, \frac{r_{n-1}}{p_n} \right),$$

$$\mathrm{b} = \left(-\frac{r_2}{\sqrt{p_2 p_3}}, \ldots, -\frac{r_{n-1}}{\sqrt{p_{n-1} p_n}} \right),$$

is positive definite.

Proof. Denote the main minors in each matrix $H_k(\mathrm{a}, \mathrm{b})$ $(k = 1, \ldots, n-1)$ with $D_{k1}, D_{k2}, \ldots, D_{kk}$ $(= \det H_k(\mathrm{a}, \mathrm{b}))$. We should prove that $D_{ij} > 0$ $(i = 1, \ldots, n-1;\ j = 1, \ldots, n-1;\ i \geqq j)$ (see [5, 6]). As $D_{km} = D_{k-1,m} = \cdots = D_{m+1,m}$ for $m = 2, \ldots, k-1$, we shall prove that

$$D_{km} = \left(\sum C^{(m-2)}_{r_3 r_4 \cdots r_{m+1}} \Big/ \prod_{i=2}^{m+1} p_i \right) D_{22} + \left(\prod_{i=3}^{m+1} r_i \Big/ \prod_{i=3}^{m+1} p_i \right) D_{21}, \tag{3}$$

where $\sum C^{(k)}_{r_1 r_2 \cdots r_i}$ is a sum of products of all combinations without repetition of elements $r_1, r_2, \ldots, r_i$ of kth class. Since $D_{22} = r_1 r_2 / p_2 p_3 > 0$ and $D_{21} = (r_1 + r_2)/p_2 > 0$, it can be checked that (3) is correct for $m = 2$.

Assume that (3) is correct for a fixed m $(<k)$ and let us then show that it is correct for $m+1$ $(<k)$ too.

Since

$$D_{m+2,m+1} = -\frac{r_{m+1}^2}{p_{m+1}p_{m+2}} D_{m+1,m-1} + \frac{r_{m+1}+r_{m+2}}{p_{m+2}} D_{m+1,m},$$

we have, based on (3), that

$$\begin{aligned} D_{k,m+1} &= \cdots = D_{m+2,m+1} \\ &= -(r_{m+1}^2/p_{m+1}p_{m+2}) \left(\left(\sum C_{r_3\cdots r_m}^{(m-3)} \Big/ \prod_{i=4}^{m} p_i \right) D_{22} \right. \\ &\quad \left. + \left(\prod_{i=3}^{m} r_i \Big/ \prod_{i=3}^{m} p_i \right) D_{21} \right) + ((r_{m+1}+r_{m+2})/p_{m+2}) \\ &\quad \times \left(\left(\sum C_{r_3\cdots r_{m+1}}^{(m-2)} \Big/ \prod_{i=4}^{m+1} p_i \right) D_{22} + \left(\prod_{i=3}^{m+1} r_i \Big/ \prod_{i=3}^{m+1} p_i \right) D_{21} \right), \end{aligned}$$

i.e.,

$$D_{k,m+1} = \left(\sum C_{r_3\cdots r_{m+2}}^{(m-1)} \Big/ \prod_{i=4}^{m+2} p_i \right) D_{22} + \left(\prod_{i=3}^{m+2} r_i \Big/ \prod_{i=3}^{m+2} p_i \right) D_{21},$$

which have to be proved. Since $D_{21} > 0$ and $D_{22} > 0$, we conclude from (3) that $D_{km} > 0$ for $k = 1,\ldots, n-1$ and $m = 2, 3,\ldots, k-1$.

It should be proved that

$$D_{kk} = \sum_{i=1}^{k} r_i \Big/ \sum_{i=2}^{k+1} p_i, \tag{4}$$

wherefrom we have $D_{kk} > 0$.

As $D_{kk} = (r_k/p_{k+1}) D_{k,k-1} - (r_k^2/p_k p_{k+1}) D_{k-1,k-2}$, on the basis of (3), Eq. (4) is obtained by arranging the right-hand side of the preceeding equality. Now, the proof of Lemma 1 is completed.

Using the same procedure, the following statement can be proved:

LEMMA 2. *Let $(r_k)_{k\in N_0}$ and $(p_k)_{k\in N}$ be positive sequences. Then the matrix $H_n(\mathsf{a}, \mathsf{b})$, with*

$$\mathsf{a} = \left(\frac{r_0+r_1}{p_1},\ldots, \frac{r_{n-1}+r_n}{p_n} \right), \qquad \mathsf{b} = \left(-\frac{r_1}{\sqrt{p_1p_2}},\ldots, -\frac{r_{n-1}}{\sqrt{p_{n-1}p_n}} \right),$$

is positive definite.

THEOREM 1. *Let $(r_k)_{k\in N_0}$ and $(p_k)_{k\in N}$ be two positive sequences and let $(Q_k(x))_{k\in N_0}$ be the sequence of polynomials defined with*

$$\frac{r_{k+2}}{\sqrt{p_{k+2}p_{k+3}}}Q_{k+1}(x)=\left(\frac{r_{k+1}+r_{k+2}}{p_{k+2}}-x\right)Q_k(x)-\frac{r_{k+1}}{\sqrt{p_{k+1}p_{k+2}}}Q_{k-1}(x),$$
$$Q_0(x)=1,\qquad Q_{-1}(x)=0. \tag{5}$$

Then for any sequence of real numbers x_1 (=0), $x_2,..., x_n$, the following inequalities are valid:

$$A_n\sum_{k=2}^{n}p_kx_k^2\leqq\sum_{k=1}^{n-1}r_k(x_k-x_{k+1})^2\leqq B_n\sum_{k=2}^{n}p_kx_k^2, \tag{6}$$

where A_n and B_n are minimal and maximal zeros of polynomial $x\mapsto R_{n-1}(x)$, defined with

$$R_{n-1}(x)=\left(\frac{r_{n-1}}{p_n}-x\right)Q_{n-2}(x)-\frac{r_{n-1}}{\sqrt{p_{n-1}p_n}}Q_{n-3}(x). \tag{7}$$

Equality in the left (right) inequality (6) exists if and only if

$$x_1=0,\qquad x_k=(C/\sqrt{p_k})\,Q_{k-2}(\lambda)\qquad(k=2,...,n), \tag{8}$$

where $\lambda=A_n$ ($\lambda=B_n$) and C is an arbitrary constant.

Proof. Let X be Euklid's space of an $(n-1)$-dimensional vector with scalar product $(\mathbf{z},\mathbf{w})=\sum_{k=1}^{n-1}z_kw_k$, where $\mathbf{z}=[z_1\cdots z_{n-1}]^T$ and $\mathbf{w}=[w_1\cdots w_{n-1}]^T$.

Let

$$F=\sum_{k=1}^{n-1}r_k(x_k-x_{k+1})^2\qquad\text{and}\qquad G=\sum_{k=2}^{n}p_kx_k^2.$$

If we put $\sqrt{p_k}x_k=y_k$ $(k=1,..., n)$, F and G are transformed in

$$F=\sum_{k=1}^{n-1}\frac{r_k}{p_kp_{k+1}}(\sqrt{p_{k+1}}\,y_k-\sqrt{p_k}\,y_{k+1})^2=(H_{n-1}(\mathrm{a},\mathrm{b})\mathbf{y},\mathbf{y})$$

and

$$G=\sum_{k=2}^{n}y_k^2=(\mathbf{y},\mathbf{y}),$$

where $\mathbf{y} = [y_2 \cdots y_n]^T$ and $H_{n-1}(a, b)$ is a three-diagonal matrix defined as in Lemma 1. On the other hand, let us consider a sequence of polynomials $(Q_k(x))_{k\in N_0}$ which is defined by (5). For $k = 0, 1, ..., n-1$, on the basis of (5), we obtained the equality

$$H_{n-1}(a, b)\mathbf{z} = x\mathbf{z} + \left(\left(\frac{r_{n-1}}{p_n} - x \right) Q_{n-2}(x) - \frac{r_{n-1}}{\sqrt{p_{n-1} p_n}} Q_{n-3}(x) \right) \mathbf{e}, \quad (9)$$

where $\mathbf{e} = [0 \cdots 0 1]^T$ and $\mathbf{z} = (Q_0(x) \cdots Q_{n-2}(x)]^T$.

If we put $x = \lambda$, Eq. (9) reduces to

$$H_{n-1}(a, b)\mathbf{z} = \lambda \mathbf{z} + R_{n-1}(\lambda)\mathbf{e}, \quad (10)$$

where $x \mapsto R_{n-1}(x)$ is a polynomial defined by (7). According to (10) we conclude:

If λ is such that $R_{n-1}(\lambda) = 0$, then λ is an eigenvalue of matrix $H_{n-1}(a, b)$ and $\mathbf{z}$ is an eigenvector. Contrarily, if λ is an eigenvalue and $\mathbf{z}$ is an eigenvector of matrix $H_{n-1}(a, b)$, then $R_{n-1}(\lambda) = 0$, i.e., λ is a zero of polynomial $x \mapsto R_{n-1}(x)$.

Thus, eigenvalues of matrix $H_{n-1}(a, b)$ are zeros of polynomial $x \mapsto R_{n-1}(x)$ at the same time. Since on the basis of Lemma 1, $H_{n-1}(a, b)$ is a positive definite matrix, its eigenvalues are real and nonnegative (see, e.g., [5, 6]) so the zeros of the polynomials $x \mapsto R_{n-1}(x)$ are also real and nonnegative. On the basis of the inequality

$$\lambda_{\min}(A)(\mathbf{x}, \mathbf{x}) \leqq (A\mathbf{x}, \mathbf{x}) \leqq \lambda_{\max}(A)(\mathbf{x}, \mathbf{x}),$$

which is valid for any Hermitian matrix A, where we have the equality for eigenvectors corresponding to eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$, the conclusions of Theorem 1 follows, i.e., $B_n = \max_{1 \leqq i \leqq n-1} \lambda_i$ and $A_n = \min_{1 \leqq i \leqq n-1} \lambda_i (R_{n-1}(\lambda_i) = 0)$, as do the conditions of equality (8).

COROLLARY 1. *For any sequence of real numbers* $x_1(=0), x_2, ..., x_n$, *the inequality*

$$\sum_{k=1}^{n-1} (k-1)(x_k - x_{k+1})^2 \leqq B_n \sum_{k=2}^{n} x_k^2 \quad (11)$$

is valid. B_n *is a maximal zero of the generalized Laguerre polynomial* $x \mapsto L_{n-1}^{(-1)}(x) = \sum_{k=1}^{n} \binom{n-1}{n-k}((-x)^k/k!)$.

We have the equality in (11) *if and only if* $x_1 = 0$, $x_k = CL_{k-2}(B_n)$ $(k = 2, ..., n)$, *where* C *is arbitrary constant, and* $x \mapsto L_k(x)$ *a Laguerre polynomial.*

Proof. For $p_k = 1$ and $r_k = k-1$ $(k = 1,..., n)$, recursive relation (5) becomes

$$(k+1)\, Q_{k+1}(x) = (2k+1-x)\, Q_k(x) - kQ_{k-1}(x),$$

wherefrom we conclude that $Q_k(x) = L_k^*(x)(L_k^*(x) = L_k(x)/\|L_k\|)$, where $x \mapsto L_k(x)$ is a Laguerre polynomial.

According to the relation valid for generalized Laguerre polynomials (see [7, 8]), we obtain that $R_{n-1}(x) = (n-1)\, L_{n-1}^{(-1)}(x)$. Now the statement of Corollary 1 follows from Theorem 1.

It is not difficult to show that $B_3 = 2$, $B_4 = 3 + \sqrt{3}$.

COROLLARY 2. *For any sequence of real numbers* $x_1(=0), x_2,..., x_n$, *the following inequalities are valid*:

$$4 \sin^2 \frac{\pi}{2(2n-1)} \sum_{k=2}^{n} x_k^2 \leqslant \sum_{k=1}^{n-1} (x_k - x_{k+1})^2 \leqq 4 \cos^2 \frac{\pi}{2n-1} \sum_{k=2}^{n} x_k^2. \tag{12}$$

The equality in the left inequality (12) *holds if and only if*

$$x_k = C \sin \frac{(k-1)\pi}{2n-1} \qquad (k = 1,..., n),$$

where $C = \text{const}$.

The equality in the right inequality (12) *holds if and only if*

$$x_k = (-1)^k C \sin \frac{2(k-1)\pi}{2n-1} \qquad (k = 1,..., n),$$

where $C = \text{const}$.

Proof. For $p_k = r_k = 1$ $(k = 1,..., n)$, (5) becomes

$$Q_{k+1}(x) = (2-x)\, Q_k(x) - Q_{k-1}(x) \qquad (Q_0(x) = 1, \quad Q_1(x) = 2-x) \tag{13}$$

so

$$R_{n-1}(x) = Q_{n-1}(x) - Q_{n-2}(x). \tag{14}$$

If we put $y = \arctan(\sqrt{1+t^2}/t)$ and $t = \frac{1}{2}(2-x)$, one can easily obtain the solution of difference equation (13),

$$Q_n(x) = \sin(n+1)\theta/\sin\theta,$$

where $e^{i\theta} = t + i\sqrt{1-t^2}$.

Then

$$R_{n+1}(x) = \cos \frac{2n-1}{2} \theta / \cos \frac{\theta}{2},$$

wherefrom we have

$$\lambda_k = 4 \sin^2 \frac{(2k-1)\pi}{2(2n-1)} \qquad (k = 1, \ldots, n-1).$$

Thus

$$A_n = \min_{1 \leq k \leq n-1} \lambda_k = \lambda_1 = 4 \sin^2 \frac{\pi}{2(2n-1)}$$

and

$$B_n = \max_{1 \leq k \leq n-1} \lambda_k = \lambda_{n-1} = 4 \cos^2 \frac{\pi}{2n-1}.$$

The left equality in (12) occurs if and only if

$$x_1 = 0, \qquad x_k = C_1 Q_{k-2}(A_n) \qquad (k = 2, \ldots, n),$$

i.e.,

$$x_k = C \sin \frac{(k-1)}{2n-1} \pi \qquad (k = 1, \ldots, n).$$

where C is arbitrary constant.

Similarity is obtained when equality occurs in the right inequality (12).

Remark. Theorem A is contained in Corollary 2.

Using Lemma 2, similarly to Theorem 1, the following statement can be proved:

THEOREM 2. *Let $(r_k)_{k \in N_0}$ and $(p_k)_{k \in N_0}$ be two positive sequences and let $(Q_k(x))_{k \in N_0}$ be the sequence of polynomials defined with*

$$\frac{r_{k+1}}{\sqrt{p_{k+1} p_{k+2}}} Q_{k+1}(x) = \left(\frac{r_k + r_{k+1}}{p_{k+1}} - x \right) Q_k(x) - \frac{r_k}{\sqrt{p_k p_{k+1}}} Q_{k-1}(x), \tag{15}$$
$$Q_0(x) = 1, \qquad Q_{-1}(x) = 0.$$

Then for any sequence of real numbers x_0 (=0), $x_1,\dots,x_n, x_{n+1}$(=0), *inequalities*

$$A_n \sum_{k=0}^{n} p_k x_k^2 \leqq \sum_{k=0}^{n} r_k(x_k - x_{k+1})^2 \leqq B_n \sum_{k=0}^{n} p_k x_k^2, \tag{16}$$

hold, where A_n *and* B_n *are minimal and maximal zeros of polynomial* $x \mapsto Q_n(x)$.

Equality in the left (right) inequality (16) *holds if and only if*

$$x_0 = 0, \qquad x_k = \frac{C}{\sqrt{p_k}} Q_{k-1}(\lambda) \qquad (k = 1,\dots, n),$$

where $\lambda = A_n$ ($\lambda = B_n$) *and* C *is arbitrary constant.*

COROLLARY 3. *For each sequence of real numbers* $x_0(=0)$, $x_1,\dots,x_n$, $x_{n+1}(=0)$, *the inequality*

$$\sum_{k=0}^{n} k(x_k - x_{k+1})^2 \leqq B_n \sum_{k=0}^{n} x_k^2 \tag{17}$$

holds, where B_n *is a maximal zero of the Laguerre polynomial* $x \mapsto L_n(x)$.

We have the equality in (17) *if and only if* $x_0 = 0$, $x_k = CL_{k-1}(B_n)$ $(k = 1,\dots, n)$, *where* C *is arbitrary constant and* $x \mapsto L_{k-1}(x)$ *a Laguerre polynomial.*

Proof. For $p_k = 1$ and $r_k = k$ $(k = 0, 1,\dots, n)$, recursive relation (15) becomes

$$(k+1)\, Q_{k+1}(x) = (2k+1-x)\, Q_k(x) - kQ_{k-1}(x),$$

wherefrom we conclude that $x \mapsto Q_k(x)$ are Laguerre polynomials. Now the conclusion of Corollary 3 follows from Theorem 2.

COROLLARY 4. *For a sequence of real numbers* x_0 (=0), $x_1,\dots,x_n$, x_{n+1} (=0), *the following inequalities hold*:

$$4\sin^2\frac{\pi}{2(n+1)} \sum_{k=0}^{n} k^2 x_k^2 \leq \sum_{k=0}^{n} k(k+1)(x_k - x_{k+1})^2 \leqq 4\cos^2\frac{\pi}{2(n+1)} \sum_{k=0}^{n} k^2 x_k^2. \tag{18}$$

Equality in the left inequality (18) *holds if and only if*

$$x_k = \frac{C}{k} \sin \frac{k\pi}{k+1} \qquad (k = 1,..., n),$$

where $C = \text{const.}$

Equality in the right inequality (18) *holds if and only if*

$$x_k = \frac{(-1)^{k-1}}{k} C \sin \frac{k\pi}{n+1} \qquad (k = 1,..., n),$$

where $C = \text{const.}$

Proof. For $p_k = k^2$ *and* $r_k = k(k+1)$ $(k = 0, 1,..., n)$, (15) becomes (13). Now the conclusions of Corollary 4 follow from Theorem 2.

COROLLARY 5. *For each sequence of real numbers* $x_0(=0), x_1,..., x_n, x_{n+1}$ $(=0)$, *the following inequalities hold*:

$$4 \sin^2 \frac{\pi}{2(n+1)} \sum_{k=0}^{n} x_k^2 \leqq \sum_{k=0}^{n} (x_k - x_{k+1})^2 \leqq 4 \cos^2 \frac{\pi}{2(n+1)} \sum_{k=0}^{n} x_k^2. \tag{19}$$

Equality in the left inequality (19) *holds if and only if*

$$x_k = C \sin \frac{k\pi}{n+1} \qquad (k = 0,..., n),$$

where $C = \text{const.}$

Equality in the right inequality (19) *holds if and only if*

$$x_k = (-1)^{k-1} C \sin \frac{k\pi}{n+1} \qquad (k = 0, 1,..., n).$$

where $C = \text{const.}$

Remark. Theorem B is contained in Corollary 5.

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