Existence and construction of nonnegative matrices with complex spectrum

Oscar Rojo, Ricardo L. Soto *

Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280, Antofagasta, Chile

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Abstract

The following inverse spectrum problem for nonnegative matrices is considered: given a set of complex numbers \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), find necessary and sufficient conditions for the existence of an \( n \times n \) nonnegative matrix \( A \) with spectrum \( \sigma \). Our work is motivated by a relevant theoretical result of Guo Wuwen [Linear Algebra Appl. 266 (1997) 261, Theorem 2.1]: there exists a real parameter \( \lambda_0 \geq \max_{2 \leq j \leq n} |\lambda_j| \) such that the problem has a solution if and only if \( \lambda_1 \geq \lambda_0 \). In particular, we discuss how to compute \( \lambda_0 \) and the solution matrix \( A \) for certain class of matrices. A sufficient condition for the problem to have a solution is also derived.

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1. Introduction

Definition 1. A matrix \( A = (a_{ij}) \) of order \( n \) is said to be nonnegative if \( a_{ij} \geq 0 \), \( i, j = 1, 2, \ldots, n \).

The inverse spectrum problem for nonnegative matrices has a very simple formulation: given a set \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) of complex numbers, find necessary and
sufficient conditions for the existence of an $n \times n$ nonnegative matrix $A$ with spectrum $\sigma$. If there exists a nonnegative matrix $A$ with spectrum $\sigma$, we say that $\sigma$ is realized by $A$. This problem is a difficult one and it remains unsolved. Important progresses have been done in the case of a real prescribed spectrum by Suleimanova [16], Perfect [12], Kellogg [8], Salzmann [15], Fiedler [6], Borobia [3] and others, while in the complex prescribed spectrum case the problem was solved for $n = 3$ by Loewy and London [10] and for matrices with trace 0 of order $n = 4$ and $n = 5$ by Reams [14] and Laffey and Meehan [9], respectively.

It is interesting to notice that the sufficient condition given by Laffey and Meehan [9, Theorem 3.1] also holds for matrices with positive trace in the following sense: If $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\sum \lambda_i > 0$ is given, then we may consider $\sigma' = \{\mu_1, \mu_2, \ldots, \mu_n\}$, where $\mu_i = \lambda_i - \alpha$ with $\alpha = \frac{1}{n} \sum \lambda_i$. Thus, if $\sigma'$ satisfies the sufficient condition in [9], there exists a nonnegative matrix $B$ with spectrum $\sigma'$ and a nonnegative matrix $A = B + \alpha I$ with spectrum $\sigma$.

Sufficient conditions have also been obtained for a normal nonnegative matrix of order $n$ by Xu [18] and Radwan [13] in the complex case. A relevant contribution to the study of the problem is the fundamental paper by Guo Wuwen [17]. There, Guo Wuwen states the following result:

**Theorem 2** (Guo Wuwen). If $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a set of complex numbers with $\sigma = \overline{\sigma}$, then there exist a real number $\lambda_0$, $\max_{2 \leq j \leq n} |\lambda_j| \leq \lambda_0 \leq 2n \max_{2 \leq j \leq n} |\lambda_j|$, (1)

such that $\sigma$ is realized by a nonnegative matrix $A$ of order $n$ if and only if $\lambda_1 \geq \lambda_0$.

This theorem guarantees the existence of a nonnegative matrix $A$ with spectrum $\sigma$ for all $\lambda_1 \geq 2n \max_{2 \leq j \leq n} |\lambda_j|$. (2)

However, for $\lambda_1 < 2n \max_{2 \leq j \leq n} |\lambda_j|$ we may have no solution. Besides, although a solution may exist ($\lambda_1 \geq \lambda_0$), we can not compute it from Theorem 2. The problem of finding the real number $\lambda_0$ is a difficult one and unsolved in [17]. In this work, we do some progress in order to compute $\lambda_0$ and the solution matrix $A$.

The following theorem, due to Brauer [4, Theorem 33], plays an important role in the study of the inverse spectrum problem for nonnegative matrices. In particular, the theorem is important not only to derive sufficient conditions given in this work, but also to compute a solution matrix. A fairly standard proof of this theorem is given in [14].

**Theorem 3** (Brauer). Let $A$ be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v = (v_1, v_2, \ldots, v_n)^T$ an eigenvector of $A$ associated with the eigenvalue $\lambda_k$ and let $q$ be any $n$-dimensional vector. Then the matrix $A + vq^T$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k-1, \lambda_k + v^T q, \lambda_k+1, \ldots, \lambda_n$, $k = 1, 2, \ldots, n$. 


After this paper was submitted, the authors in [1] proved, by using the Brauer theorem 3, that the upper bound in (1) may be reduced to

$$\lambda_0 \leq (n + \sqrt{2} - 1) \max_{2 \leq j \leq n} |\lambda_j|$$ (3)

In fact, let \(\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) with \(\varpi = \sigma\). Let \(m = \max_{2 \leq j \leq n} |\lambda_j|\) and let \(\mu_i = \lambda_i / m, i = 2, 3, \ldots, n\). Then, \(\sigma' = \{\lambda_1, \mu_2, \ldots, \mu_n\}\) is a set of complex numbers such that \(|\mu_i| \leq 1, i = 2, 3, \ldots, n\). Consider the initial matrix

$$B = \begin{pmatrix}
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-\mu_2 & \mu_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
-\mu_p & \cdots & \mu_p & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\beta_{p+1} & \cdots & x_t & -y_t & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
\beta_n & 0 & \cdots & \cdots & \cdots & 0 & y_t & x_t & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},$$

where \(\mu_2, \mu_3, \ldots, \mu_p\) are real, \(x_j = \text{Re} \mu_j, y_j = \text{Im} \mu_j, j = p + 1, \ldots, n\), and the real numbers \(\beta_{p+1}, \ldots, \beta_n\) are such that \(Be = 0\), that is, all rows of \(B\) sum to zero. Since \(|\mu_j| \leq 1, j = 2, 3, \ldots, n\), then

$$\max_{x_j^2 + y_j^2 \leq 1} (x_j \pm y_j) = \sqrt{2}.$$

Let \(q = (\sqrt{2}, 1, 1, \ldots, 1)^T\). Then the matrix \(A' = B + eq^T\) is nonnegative with eigenvalues \(\lambda_1 = \sum_{k=1}^n q_k = n + \sqrt{2} - 1\) and \(\mu_2, \mu_3, \ldots, \mu_n\). The matrix \(A = mA'\) is the desired nonnegative matrix with spectrum \(\sigma = (n + \sqrt{2} - 1)m, \lambda_2, \lambda_3, \ldots, \lambda_n\).

It was also shown in [1] that if \(\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) is a set of real numbers such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0 \geq \lambda_{p+1} \geq \cdots \geq \lambda_n,$$

then the Guo Wuwen upper bound becomes

$$\lambda_0 \leq (n - 1) \max_{2 \leq j \leq n} |\lambda_j|.$$ (4)

Thus, the Guo Wuwen result guarantees the existence of a nonnegative matrix \(A\) with spectrum \(\sigma\) for all

$$\lambda_1 \geq (n + \sqrt{2} - 1) \max_{2 \leq j \leq n} |\lambda_j|$$ (5)

in the complex case and for all

$$\lambda_1 \geq (n - 1) \max_{2 \leq j \leq n} |\lambda_j|$$ (6)

in the real case.
Let \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be a set of complex numbers, which is realizable by a nonnegative matrix \( A \), with \( \lambda_1 \) being the Perron root of \( A \). Then it is clear that:

(i) If \( \sum_{i=1}^n \lambda_i = 0 \), then \( \lambda_1 \) is the Guo Wuwen index.

(ii) It is also clear, as Guo Wuwen shows in [17], that if \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is realized by a nonnegative matrix, then \( \sigma_e = \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\} \) is also realized by a nonnegative matrix.

(iii) Consider the case \( n = 3 \) with \( \sigma = \{\lambda_1, -a + ib, -a - ib\} \), where \( a > 0, b > 0 \). The Loewy and London result in [10] yields that \( \sigma \) is realizable by a nonnegative matrix if and only if it is realizable by \( B + sI \), where \( s \geq 0 \) and \( B \) is the nonnegative companion matrix with eigenvalues \( 2(a + s), -(a + s) + ib, -(a + s) - ib \), for some nonnegative \( s \). That is:

\[
B = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
c_3 & -c_2 & 0
\end{pmatrix},
\]

with \( c_2 = -3(a + s)^2 + b^2 \) and \( c_3 = 2(a + s)[(a + s)^2 + b^2] \).

In order that \( B \) be nonnegative, we required that \( 3(a + s)^2 - b^2 \geq 0 \). That is, \( a + s \geq b/\sqrt{3} \).

Then,

\[
s \geq \max \left\{ 0, \frac{b}{\sqrt{3}} - a \right\}
\]

and \( B + sI \) is the solution matrix. Therefore, \( \sigma \) is realizable by a nonnegative matrix if and only if \( \lambda_1 \geq 2a + 3s \).

Then we have

**Proposition 4.** Let \( \sigma = \{\lambda_1, -a + ib, -a - ib\} \), with \( a > 0, b > 0 \) and \( \lambda_1 \geq \sqrt{a^2 + b^2} \), then \( \sigma \) is realizable by a nonnegative matrix \( A \) if and only if

\[
\lambda_1 \geq 2a + 3 \max \left\{ 0, \frac{b}{\sqrt{3}} - a \right\}.
\]

(7)

If \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is a set of complex numbers with \( \text{Re} \lambda_k < 0, k = 2, 3, \ldots, n \), which is realized by a nonnegative matrix \( A \) of trace zero, then

\[
\lambda_1 = -\sum_{i=2}^n \lambda_i \geq \sqrt{\left(\text{Re} \lambda_k\right)^2 + \left(\text{Im} \lambda_k\right)^2}, \quad k = 2, 3, \ldots, n.
\]

yields to

\[
\max_k (\text{Im} \lambda_k) \leq \sqrt{\left( -\sum_{i=2}^n \lambda_i \right)^2 - \min_k (\text{Re} \lambda_k)^2}.
\]

\[
\max_k (\text{Im} \lambda_k) \leq \sqrt{\left( -\sum_{i=2}^n \lambda_i \right)^2 - \min_k (\text{Re} \lambda_k)^2}.
\]
We observe that (7) is also a necessary and sufficient condition for the existence of a circulant nonnegative matrix
\[
C = \begin{pmatrix}
\frac{1}{3}(\lambda_1 - 2a) & \frac{1}{3}(\lambda_1 + a + \sqrt{3}b) & \frac{1}{3}(\lambda_1 + a - \sqrt{3}b) \\
\frac{1}{3}(\lambda_1 + a - \sqrt{3}b) & \frac{1}{3}(\lambda_1 - 2a) & \frac{1}{3}(\lambda_1 + a + \sqrt{3}b) \\
\frac{1}{3}(\lambda_1 + a + \sqrt{3}b) & \frac{1}{3}(\lambda_1 + a - \sqrt{3}b) & \frac{1}{3}(\lambda_1 - 2a)
\end{pmatrix}
\]
with spectrum \(\sigma = \{\lambda_1, -a + ib, -a - ib\}\). In fact,
\[
C = \begin{pmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
c_{n-2} & c_{n-1} & \cdots & \cdots & c_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_2 & \cdots & \cdots & \cdots & c_1 \\
c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & c_0
\end{pmatrix}
\]
where \(C = \circ(c_0, c_1, \ldots, c_{n-1})\).

**Definition 5.** A nonnegative \(n \times n\) matrix \(A\) is called row stochastic, or simply stochastic, if all its rows sums to 1.

An immediate consequence of the definition of a stochastic matrix is:

(i) A nonnegative \(n \times n\) matrix \(A\) is stochastic if and only if \(AJ = J\), where \(J\) is the \(n \times n\) matrix of 1s.

(ii) A nonnegative \(n \times n\) matrix \(A\) is stochastic if and only if \(e = (1, 1, \ldots, 1)^T\) is an eigenvector corresponding to the maximal eigenvalue 1 of \(A\).

(iii) If \(\lambda_k\) is an eigenvalue of \(A\), then \(|\lambda_k| \leq 1\).

**Definition 6.** An \(n \times n\) matrix \(C\) of the form
\[
C = \begin{pmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\
c_{n-2} & c_{n-1} & \cdots & \cdots & c_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_2 & \cdots & \cdots & \cdots & c_1 \\
c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & c_0
\end{pmatrix}
\]
is called a circulant matrix.

We observe that a circulant matrix is uniquely determined by the entries of its first row. This fact allows us to denote the circulant matrix \(C\) by \(C = \circ(c_0, c_1, \ldots, c_{n-1})\).

In the inverse spectrum problem for nonnegative matrices of order \(n\), we have three obvious necessary conditions. If \(\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) is the spectrum of a nonnegative matrix \(A\), then:

(i) \(\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \sigma\);

(ii) \(\max_j |\lambda_j| \in \sigma\);

(iii) \(s_m(\sigma) = \sum_{j=1}^n \lambda_j^m \geq 0, m = 1, 2, \ldots\)
The most important necessary condition for the inverse spectrum problem for nonnegative matrices is given by the following theorem, due to Loewy and London [10]:

**Theorem 7.** Let \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be the spectrum of a nonnegative matrix of order \( n \). Then

\[
(s_k(\sigma))^m \leq n^{m-1}s_{km}(\sigma),
\]

for any positive integers \( k \) and \( m \).

**Definition 8.** A matrix \( A = (a_{kj}) \) of order \( n \) is said to be generalized stochastic if

\[
\sum_{j=1}^n a_{kj} = s, \quad k = 1, 2, \ldots, n.
\]

Given a set \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), the problem of finding a nonnegative matrix \( A \) with spectrum \( \sigma \) is equivalent to the problem of finding a nonnegative generalized stochastic matrix \( B = D^{-1}AD \) with spectrum \( \sigma \). In fact, let \( A \) be an arbitrary \( n \times n \) irreducible nonnegative matrix with spectrum \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \). Let \( \lambda_1 \) be the maximal eigenvalue of \( A \) and let \( x = (x_1, x_2, \ldots, x_n)^T \) be its corresponding maximal eigenvector. Let \( D = \text{diag}(x_1, x_2, \ldots, x_n) \), \( x_i > 0 \). Then \( D^{-1}Axe = \lambda_1e \) with \( e = (1, 1, \ldots, 1)^T \) and the matrix \( D^{-1}AD \) is a nonnegative generalized stochastic matrix with spectrum \( \sigma \).

If \( A \) is reducible (nonnegative), then following Johnson [7] there exists a permutation matrix \( P \) such that \( P^TAP \) is a block triangular matrix, where each block \( A_k \) is irreducible nonnegative of size \( n_k \times n_k \), \( k = 1, 2, \ldots, m \). Then the matrix \( D_k^{-1}A_kD_k \) is nonnegative generalized stochastic with row sums \( \lambda_k \). We may assume, without loss of generality, that \( \lambda_1 = \max_k \lambda_k \). Next we choose the matrices \( A_{kj} \) of size \( n_k \times n_1 \), with nonnegative entries and row sums \( \lambda_1 - \lambda_k \), \( k = 2, 3, \ldots, m \), in such a way that the matrix

\[
\tilde{A} = \begin{pmatrix}
D_1^{-1}A_1D_1 & 0 & \cdots & 0 \\
A_{21} & D_2^{-1}A_2D_2 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
A_{m1} & 0 & \cdots & D_m^{-1}A_mD_m
\end{pmatrix}
\]

is nonnegative generalized stochastic with row sums \( \lambda_1 \).

A necessary and sufficient condition for the existence of a circulant nonnegative matrix \( A \) with spectrum \( \sigma \) is derived in Section 2. Here we find the Guo Wuwen index \( \lambda_0 \). Sufficient conditions for a solution to the problem, which always allow us to compute a solution matrix are presented in Section 3.
2. On the inverse spectrum problem for circulant nonnegative matrices

In this section we find the index $\lambda_0$ of Guo Wuwen in the case of the inverse spectrum problem for circulant nonnegative matrices, that is, we find a necessary and sufficient condition for $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ to be the spectrum of some circulant nonnegative matrix.

Firstly, we recall some basic spectral properties of circulant matrices. The proofs can be found in [5]:

1. If $\omega = \exp(2\pi i/n)$, $i^2 = -1$, then the vectors
   \[
   \mathbf{1}_n = (1, 1, \ldots, 1)^T, \\
   \mathbf{v}_j = (1, \omega^{j-1}, \omega^{2(j-1)}, \ldots, \omega^{(n-1)(j-1)})^T, \quad j = 2, 3, \ldots, n,
   \]
   form an orthogonal basis of eigenvectors for any complex circulant matrix. Moreover,
   \[
   \mathbf{v}_{n-j+2} = \overline{\mathbf{v}_j}, \quad j = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor, \tag{10}
   \]
   where $\overline{\mathbf{v}_j}$ denotes the vector whose components are the complex conjugates of the components of $\mathbf{v}_j$ and $\left\lfloor (n+1)/2 \right\rfloor$ is the greatest integer not exceeding $(n + 1)/2$.

2. The $n \times n$ circulant matrix $\mathbf{C} = \text{circ}(c_0, c_1, \ldots, c_{n-1})$ has eigenvalues
   \[
   \lambda_1 = c_0 + c_1 + c_2 + \cdots + c_{n-1}, \\
   \lambda_j = c_0 + c_1 \omega^{j-1} + c_2 \omega^{2(j-1)} + \cdots + c_{n-1} \omega^{(n-1)(j-1)}, \quad j = 2, 3, \ldots, n,
   \]
   with corresponding eigenvectors $\mathbf{1}_n, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$, respectively. If $\mathbf{C}$ is a real circulant matrix, then
   \[
   \lambda_{n-j+2} = \overline{\lambda_j}, \quad j = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{12}
   \]

3. Let
   \[
   \mathbf{F} = (f_{kj}) = [\mathbf{1}_n \mid \mathbf{v}_2 \mid \mathbf{v}_3 \mid \cdots \mid \mathbf{v}_n]. \tag{13}
   \]
   Then,
   \[
   f_{kj} = \omega^{(k-1)(j-1)}, \quad 1 \leq k, \quad j \leq n, \tag{14}
   \]
   and
   \[
   \mathbf{F}^\top \mathbf{F} = \mathbf{F} \mathbf{F} = n \mathbf{I}_n, \tag{15}
   \]
   where $\mathbf{I}_n$ denotes the identity matrix of order $n$.

Let $\mathbf{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ and $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})^T$. From (11)
   \[
   \mathbf{F} \mathbf{c} = \mathbf{\lambda}. \tag{16}
   \]
Then, 
\[ c = \frac{1}{n} \mathcal{F} \lambda. \]  
(17)

The relation given in (12) motivates the following definition.

**Definition 9.** A vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T \) is said to be a conjugate-even vector if and only if 
\[ \lambda_1 \in \mathbb{R} \quad \text{and} \quad \lambda_{n-j+2} = \overline{\lambda_j}, \quad j = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor. \]

We observe that if \( \lambda \) is a conjugate-even vector and \( n = 2m + 2 \), then \( \lambda_{m+2} \in \mathbb{R} \).

From (17), the problem of finding a circulant matrix with a prescribed spectrum may be easily solved. An important point to observe is that the matrix–vector multiplication in either (16) or (17) can be computed in a fast and stable way by the use of the fast Fourier transform algorithm. Suppose that all numbers \( \lambda_i \) are distinct. Let \( P_1 \neq P_2 \) be permutation matrices of order \( n \). Then \( P_1 \lambda \neq P_2 \lambda \) and 
\[ \frac{1}{n} \mathcal{F} P_1 \lambda \neq \frac{1}{n} \mathcal{F} P_2 \lambda. \]

Therefore, if all the numbers \( \lambda_i \) are distinct, there are exactly \( n! \) many distinct circulant matrices with the prescribed spectrum \( \{\lambda_i\}_n^\pi \). However, because of the conjugate-even property of the set \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), the number of real circulant matrices becomes \( 2^m m! \), where \( n = 2m + 1 \) or \( n = 2m + 2 \).

The following theorem shows how to compute the index \( \lambda_0 \) of Guo Wuwen [17, Theorem 2.1], giving a necessary and sufficient condition for the existence of a circulant nonnegative matrix with prescribed spectrum \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \).

**Theorem 10.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T \) be a conjugate-even vector and let \( \lambda_1 \geq \max_{2 \leq k \leq n} |\lambda_k| \). A necessary and sufficient condition for \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) to be the spectrum of some circulant nonnegative matrix is

\[ \lambda_1 \geq \min_{\pi \in \pi} \max_{0 \leq k \leq 2m} \left\{ \left\{ \begin{array}{l} -2 \sum_{j=2}^{m+1} \text{Re} \; p(\lambda_j) \cos \frac{2k(j-1)\pi}{2m+1} \\ -2 \sum_{j=2}^{m+1} \text{Im} \; p(\lambda_j) \sin \frac{2k(j-1)\pi}{2m+1} \end{array} \right\} \right\} \]  
(18)

if \( n = 2m + 1 \), or

\[ \lambda_1 \geq \min_{\pi \in \pi} \max_{0 \leq k \leq 2m+1} \left\{ \left\{ \begin{array}{l} -2 \sum_{j=2}^{m+1} \text{Re} \; p(\lambda_j) \cos \frac{k(j-1)\pi}{m+1} - (-1)^k \lambda_{m+2} \\ -2 \sum_{j=2}^{m+1} \text{Im} \; p(\lambda_j) \sin \frac{k(j-1)\pi}{m+1} \end{array} \right\} \right\} \]  
(19)
if \( n = 2m + 2 \), where

\[ \mathcal{P} = \{ p : p \text{ permutation on } \{\lambda_2, \lambda_3, \ldots, \lambda_n\}, \]

\[ p(\lambda_j) = \lambda_k \text{ iff } p(\lambda_{n-j+2}) = \lambda_k, \quad j = 2, 3, \ldots, n \}. \]

**Proof.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T \) be a conjugate-even vector and \( \lambda_1 \geq \max_{2 \leq k \leq n} |\lambda_k| \). The condition on \( \mathcal{P} \) guarantees that \( p(\lambda) = (\lambda_1, p(\lambda_2), \ldots, p(\lambda_n))^T \) is a conjugate-even vector for each \( p \in \mathcal{P} \). That is,

\[ p(\lambda) = (\lambda_1, p(\lambda_2), p(\lambda_3), \ldots, p(\lambda_{n+1}), p(\lambda_2))^T. \]

Suppose that \( \sigma \) is realized by a circulant nonnegative matrix

\[ C = \text{circ}(c_0, c_1, c_2, \ldots, c_{n-1}). \]

From (11), we have that the eigenvalues of \( C \) are

\[ c_0 + c_1 + c_2 + \cdots + c_{n-1}, \quad c_0 + c_1 \omega^{j-1} + c_2 \omega^{2(j-1)} + \cdots + c_{n-1} \omega^{(n-1)(j-1)}, \]

for \( j = 2, 3, \ldots, n \). Since

\[ |c_0 + c_1 \omega^{j-1} + c_2 \omega^{2(j-1)} + \cdots + c_{n-1} \omega^{(n-1)(j-1)}| \leq c_0 + c_1 + c_2 + \cdots + c_{n-1}, \]

we have

\[ c_0 + c_1 + c_2 + \cdots + c_{n-1} = \lambda_1. \]

For the other eigenvalues, we have

\[ c_0 + c_1 \omega^2 + c_2 + \cdots + c_{n-1} = p(\lambda_2), \quad c_0 + c_1 \omega^4 + c_2 + \cdots + c_{n-1} = p(\lambda_3), \quad c_0 + c_1 \omega^6 + c_2 + \cdots + c_{n-1} = p(\lambda_4), \]

\[ \vdots \]

\[ c_0 + c_1 \omega^{n-2} + c_2 + \cdots + c_{n-1} = p(\lambda_{n-1}) \]

\[ c_0 + c_1 \omega^{n-1} + c_2 + \cdots + c_{n-1} = p(\lambda_n) \]

for some \( p \in \mathcal{P} \). Thus, the real vector \( c = (c_0, c_1, \ldots, c_{n-1})^T \) is given by

\[ c = \frac{1}{n} \mathcal{F} p(\lambda). \]
From (20), the explicit formulas for the $c_k$ are:

$$c_k = \frac{1}{2m+1}\left(\lambda_1 + 2 \sum_{j=2}^{m+1} \text{Re } p(\lambda_j) \cos \frac{2k(j-1)\pi}{2m+1}\right)$$

$$+ 2 \sum_{j=2}^{m+1} \text{Im } p(\lambda_j) \sin \frac{2k(j-1)\pi}{2m+1}\right)$$

$$= \frac{1}{2m+1}\left(\lambda_1 + 2 \sum_{j=2}^{m+1} \text{Re } p(\lambda_j) \cos \frac{2k(j-1)\pi}{2m+1}\right)$$

$$+ 2 \sum_{j=2}^{m+1} \text{Im } p(\lambda_j) \sin \frac{2k(j-1)\pi}{2m+1}\right)$$

(21)

$k = 0, 1, \ldots, 2m$, if $n = 2m + 1$ and

$$c_k = \frac{1}{n}\left(\lambda_1 + 2 \sum_{j=2}^{m+1} \text{Re } p(\lambda_j) \cos \frac{k(j-1)\pi}{m+1}\right)$$

$$+ (-1)^k \lambda_{m+2} + 2 \sum_{j=2}^{m+1} \text{Im } p(\lambda_j) \sin \frac{k(j-1)\pi}{m+1}\right)$$

(22)

$k = 0, 1, \ldots, 2m + 1$, if $n = 2m + 2$.

Let $n = 2m + 1$. Since $c_k \geq 0$ for all $k$, we have that

$$\lambda_1 \geq -2 \sum_{j=2}^{m+1} \text{Re } p(\lambda_j) \cos \frac{2k(j-1)\pi}{2m+1} - 2 \sum_{j=2}^{m+1} \text{Im } p(\lambda_j) \sin \frac{2k(j-1)\pi}{2m+1}$$

(23)

for all $k$ and for some permutation $p \in \mathcal{P}$. From (23),

$$\lambda_1 \geq \max_{0 \leq k \leq 2m} \left\{-2 \sum_{j=2}^{m+1} \text{Re } p(\lambda_j) \cos \frac{2k(j-1)\pi}{2m+1} \right\}$$

(24)

for some $p \in \mathcal{P}$. Hence

$$\lambda_1 \geq \min_{p \in \mathcal{P}} \max_{0 \leq k \leq 2m} \left\{-2 \sum_{j=2}^{m+1} \text{Re } p(\lambda_j) \cos \frac{2k(j-1)\pi}{2m+1} \right\}$$

(25)

We have proved that if $\sigma = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ is realized by a circulant nonnegative matrix, then (25) holds. Conversely, assume that (25) holds. Then, (24) holds for some $p \in \mathcal{P}$. This implies (23) and thus the right hand side in (21) is nonnegative for all $k$. Therefore, $\sigma = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ is realized by the circulant nonnegative matrix
\(C = \text{circ}(c_0, c_1, c_2, \ldots, c_{n-1})\) with the \(c_k\) given by (21). The proof for \(n = 2m + 2\) is similar. □

**Example 11.** Let \(\sigma = \{\lambda_1, -6 + 2i, -6 - 2i, 5 - i, 5 + i, 1 + 2i, 1 - 2i\}\). Then, for \(\lambda_1 \in \mathbb{R}\), the vector \((\lambda_1, -6 + 2i, 5 - i, 1 + 2i, 1 - 2i)^T\) is a conjugate-even vector. To four decimal places the right hand side of (18) is \(\lambda_0 = 12.9873\). Therefore, the set \(\{\lambda_1, -6 + 2i, 5 - i, 1 + 2i, 1 - 2i\}\) is realized by a circulant nonnegative matrix if and only if \(\lambda_1 \geq 12.9873\). In particular, for \(\lambda_1 = 13\), \(\sigma\) is realized by the positive circulant matrix

\[C = \text{circ}(1.8571, 0.1333, 2.2575, 4.1430, 4.3146, 0.0018, 0.2927).\]

### 3. Sufficient conditions for the nonnegative inverse spectrum problem

By the use of Theorem 3 of Brauer, a sufficient condition for \(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) to be realized by a circulant nonnegative matrix can be computed very easily, as we show below. This is important because the cardinality of \(\mathcal{P}\) in Theorem 10 is \(2^m m!\) for \(n = 2m + 1\) or \(n = 2m + 2\) and then, to compute \(\lambda_0\) requires a lot of work which for a large \(n\) becomes a prohibitive task.

**Theorem 12.** Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T\) be a conjugate-even vector. Let

\[\mu = -\sum_{j=2}^{n} \lambda_j,\]

\[c = (c_0, c_1, \ldots, c_{n-1})^T = \frac{1}{n} F(\mu, \lambda_2, \lambda_3, \ldots, \lambda_n)^T\]

and

\[c = \min\{c_0, c_1, \ldots, c_{n-1}\}\]

If

\[\lambda_1 \geq \mu - nc\]

then there exists a circulant nonnegative matrix with eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\).

**Proof.** Let \(\mu\) and \(c\) as in (26) and (27). Since \((\mu, \lambda_2, \lambda_3, \ldots, \lambda_n)^T\) is a conjugate-even vector, we have that \(c\) is a real vector. Let \(c\) be as in (28). Let \(C = \text{circ}(c_0, c_1, \ldots, c_{n-1})\). For \(\lambda \in \mathbb{R}\), we define the matrix

\[B_\lambda = C + \frac{\lambda - \mu}{n} ee^T.\]

We see that \(B_\lambda = \text{circ}(b_0, b_1, \ldots, b_{n-1})\) is a real circulant matrix with

\[b_k = c_k + \frac{\lambda - \mu}{n}.\]
From Theorem 3, it follows that the eigenvalues of $B_\lambda$ are $\lambda, \lambda^2, \lambda^3, \ldots, \lambda^n$. Moreover,

$$b_k \geq c + \frac{\lambda - \mu}{n}.$$ 

The requirement

$$c + \frac{\lambda - \mu}{n} = 0$$

gives the smallest value of $\lambda$ for which $B_\lambda$ is a nonnegative matrix. Therefore, if $\lambda_1 \geq \mu - cn$ then

$$A = C + \frac{\lambda_1 - \mu}{n} e e^T$$

is a nonnegative circulant matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

We observe that the vector $c$ in (27) can be computed in a fast and stable way by using the fast Fourier transform. \hfill \Box

**Corollary 13.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be a real vector such that

$$\lambda_{n-j+2} = \lambda_j, \quad j = 2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor.$$

If

$$\lambda_j \leq 0, \quad j = 2, 3, \ldots, n$$

then a necessary and sufficient condition for the existence of a circulant nonnegative with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is

$$\lambda_1 \geq - \sum_{j=2}^n \lambda_j.$$  \hspace{1cm} (31)

**Proof.** It is clear that (31) is a necessary condition. Suppose that (31) holds. From (30), we see that the real vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is a conjugate-even vector. Let $\mu$ and $c$ as in Theorem 12. That is,

$$\mu = - \sum_{j=2}^n \lambda_j \quad \text{and} \quad c = \frac{1}{n} F \lambda.$$

Since $c$ is a real vector, for $k = 1, 2, \ldots, n$, we have

$$c_{k-1} = \frac{1}{n} \left( \mu + \sum_{j=2}^n \lambda_j \cos^{k-1}(j-1) \right)$$

$$= \frac{1}{n} \left( \mu + \sum_{j=2}^n \lambda_j \cos \frac{2(k-1)(j-1)\pi}{n} \right)$$

$$= \frac{1}{n} \sum_{j=2}^n \left( \cos \frac{2(k-1)(j-1)\pi}{n} - 1 \right) \lambda_j.$$
Now, we use the hypotheses \( \lambda_j \leq 0 \) for \( j = 2, 3, \ldots, n \) to conclude that \( c_{k-1} \geq 0 \) for \( k = 1, 2, \ldots, n \). In particular, \( c_0 = 0 \). Then, \( c = \min\{c_0, c_1, \ldots, c_{n-1}\} = 0 \). We conclude from Theorem 12 that if \( \lambda_1 \geq \mu = -\sum_{j=2}^{n} \lambda_j \) then there exists a circulant nonnegative matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). □

Now, we recall that the matrix
\[
B = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & \cdots \\
\cdots & \cdots & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}
\]
is the companion matrix of the polynomial
\[
f(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j) = \lambda^n - \sum_{k=1}^{n} c_k \lambda^{n-k}.
\] (32)

We shall say that the matrix \( B \) is the companion matrix of the set \( \sigma = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\} \).

**Lemma 14.** Let \( \sigma = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\} \). Let \( c = (c_n, \ldots, c_2, c_1) \) be the last row of the companion matrix \( B \) of the set \( \sigma \). Let \( \lambda \in \sigma \). Then
\[
\sum_{k=1}^{n} c_k \lambda^{n-k} = \lambda^n
\]
and \( v = (1, \lambda, \lambda^2, \ldots, \lambda^{n-1})^T \) is an eigenvector of \( B \) corresponding to the eigenvalue \( \lambda \).

**Proof.** Let \( f \) as defined in (32). Then,
\[
f(\lambda) = 0 = \lambda^n - \sum_{k=1}^{n} c_k \lambda^{n-k}
\]
and \( Bv = \lambda v \). □

Let \( \sigma = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\} \) be a set of complex numbers such that \( \sigma = \overline{\sigma} \). The following result, which may be seen as a partial alternative presentation of the Guo Wuwen Theorem 2.1 in [17], has the merit of allowing us to compute a nonnegative matrix \( A \) of order \( (n + 1) \), whose spectrum contains \( \sigma \). The Perron root of \( A \), \( \lambda + v^T q \), may be obtained as less than \( (n + \sqrt{2} - 1) \max_{2 \leq j \leq n} |\lambda_j| \).

**Theorem 15.** Let \( \sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be a given set of complex numbers such that \( \sigma = \overline{\sigma} \). Then, there exists a nonnegative matrix \( A \) of order \( (n + 1) \), whose spectrum contains \( \sigma \).
Proof. Let \( \sigma' = \{ \lambda, \lambda_1, \lambda_2, \ldots, \lambda_n \} \), \( \lambda > 0 \) and let \( \mathbf{c} = (c_{n+1}, c_n, \ldots, c_2, c_1) \) be the last row of the companion matrix \( B \) of the set \( \sigma' \). Let \( \mathbf{v} = (1, \lambda, \lambda^2, \ldots, \lambda^n)^T \) and let \( \mathbf{q} = (q_1, q_2, \ldots, q_{n+1})^T \), where \( q_k = 0 \) if \( c_{n+2-k} \geq 0 \) or \( q_k \geq -\frac{1}{\lambda^k}c_{n+2-k} \) if \( c_{n+2-k} < 0 \). Then,
\[
\mathbf{v}^T \mathbf{q} = \sum_{k=1}^{n+1} \lambda^{k-1} q_k
\]
and
\[
\mathbf{v} \mathbf{q}^T = \begin{pmatrix}
q_1 & q_2 & \cdots & q_{n+1} \\
\lambda q_1 & \lambda q_2 & \cdots & \lambda q_{n+1} \\
\lambda^2 q_1 & \lambda^2 q_2 & \cdots & \lambda^2 q_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda^n q_1 & \lambda^n q_2 & \cdots & \lambda^n q_{n+1}
\end{pmatrix}.
\]
Thus, the matrix \( A = B + \mathbf{v} \mathbf{q}^T \) is nonnegative of order \((n+1)\) with eigenvalues \( \lambda + \mathbf{v}^T \mathbf{q}, \lambda_1, \lambda_2, \ldots, \lambda_n \). \( \square \)

It is clear that taking \( q_k > 0 \) if \( c_{n+2-k} \geq 0 \) or \( q_k \geq -\lambda^{k-1}c_{n+2-k} \) if \( c_{n+2-k} < 0 \) yields to a positive matrix \( A = B + \mathbf{v} \mathbf{q}^T \).

The following result gives a simple sufficient condition for the inverse spectrum problem for nonnegative matrices have a solution in the case \( \sum_{i=2}^n \lambda_i < 0 \). The result is always useful in practice in the sense that if the sufficient condition is satisfied, we may compute an explicit solution matrix \( A \) with prescribed spectrum \( \sigma \) from \( \sigma \) itself.

**Theorem 16.** Let \( \sigma = \{ \lambda, \lambda_2, \ldots, \lambda_n \} \) be a set of complex numbers such that \( \overline{\sigma} = \sigma \) and \( s = \sum_{i=2}^n \lambda_i < 0 \). If
\[
\lambda_1 \geq -s - (-s)^{1-n} \sum_{c_k < 0} (-s)^{n-k} c_k,
\]
where \( \mathbf{c} = (c_n, \ldots, c_2, c_1) \) is the last row of the companion matrix of the auxiliary set \( \sigma' = \{-s, \lambda_2, \ldots, \lambda_n \} \), then there exists a nonnegative matrix \( A \) of order \( n \) with spectrum \( \sigma \).

**Proof.** Let \( s = \sum_{i=2}^n \lambda_i < 0 \) and \( \sigma' = \{ \lambda, \lambda_2, \ldots, \lambda_n \} \) with \( \lambda = -s \). Let
\[
B = \begin{pmatrix}
0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
c_n & c_{n-1} & \cdots & 1 \\
c_1 & c_2 & \cdots & 0
\end{pmatrix}
\]
the companion matrix of \( \sigma' \) and \( \mathbf{v} = (1, \lambda, \lambda^2, \ldots, \lambda^{n-1})^T \) an eigenvector of \( B \) corresponding to the eigenvalue \( \lambda \). Let \( \mathbf{q} = (q_1, q_2, \ldots, q_n)^T \) with
\[
q_k \geq 0 \quad \text{if} \quad c_{n+1-k} \geq 0 \quad \text{or} \quad q_k \geq -\lambda^{1-k}c_{n+1-k} \quad \text{if} \quad c_{n+1-k} < 0, 
\]
\( k = 1, 2, \ldots, n \).
Then $v^Tq = \sum_{k=1}^{n} \lambda_k^{k-1} q_k$ and the matrix $A = B + vq^T$ is nonnegative with eigenvalues $\lambda_2, \ldots, \lambda_n$ and

$$\lambda_1 = \lambda + v^Tq = -s - (-s)^{1-n} \sum_{\epsilon_k < 0} (-s)^{n-k} e_k.$$  □

The nonnegative matrix $A$ of Theorem 16 may be obtained positive. Moreover, if $\Re \lambda_k$ and $\Im \lambda_k$, $k = 1, 2, \ldots, n$, are integer numbers, then $A$ may be obtained nonnegative (positive) with integer entries.

**Example 17.** Let $\sigma = \{\lambda_1, 3, -4, -2 + i, -2 - i\}$. Applying Theorem 16 we may compute a nonnegative matrix $A$ with spectrum $\sigma$ for all $\lambda_1 \geq 6.72$. In particular for $\lambda_1 = 6.72$ we have

$$A = \begin{pmatrix}
12 & 156 & 0 & 0 & 0 \\
12 & 31 & 1 & 0 & 0 \\
60 & 31 & 0 & 0 & 0 \\
0 & 0 & 28 & 28 & 0
\end{pmatrix}.$$  

Since $\sigma$ is realized by the above nonnegative matrix $A$, then Theorem 1 of Borobia and Moro [2] guarantees that $\sigma' = \{\lambda_1, 3t, -4t, -2t + ti, -2t - ti\}$ is realized by the positive matrix

$$tS + (1 - t)\lambda_1^n ee^T,$$

for every $0 \leq t < 1$, where $S = D^{-1}AD$, with $D = \text{diag}(1, 5, 25, 125, 625)$, is a generalized stochastic matrix with spectrum $\sigma$ and $e = (1, 1, \ldots, 1)^T$.

**Example 18.** This example shows that the sufficient condition given by Theorem 16 is not contained in the sufficient condition of Xu [18] and Radwan [13, Theorems 3.2 and 3.3] for normal nonnegative matrices. Consider the set $\sigma = \{2, -1, i, -i\}$. Theorem 16 gives

$$A = \frac{1}{4} \begin{pmatrix}
1 & 5 & 1 & 1 \\
1 & 1 & 5 & 1 \\
1 & 1 & 1 & 5 \\
5 & 1 & 1 & 1
\end{pmatrix},$$

which is normal and nonnegative with spectrum $\sigma$. The conditions of Xu and Radwan are not satisfied. This is a particular case of the more general example where $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, with $\lambda_2, \ldots, \lambda_n$ being the $n$th roots of the unity excluding 1, is realized for all $\lambda_1 \geq 1$ by the normal circulant nonnegative matrix $A$. 

\[ A = \begin{pmatrix} a & a+1 & a & \cdots & a \\ a & a & a+1 & a & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & \cdots & \cdots & \cdots & a \\ a+1 & \cdots & a & a & a \end{pmatrix}, \]

where \( a = (\lambda_1 - 1)/n \). For this kind of matrices the Guo Wuwen parameter \( \lambda_0 \) is
\[ \lambda_0 = \max_{2 \leq k \leq n} |\lambda_k| = 1. \]

**Example 19.** Let \( \sigma = \{1, \frac{1}{14} + \frac{i}{14}, \frac{1}{14} - \frac{i}{14}\} \) be given. Consider the set \( 14\sigma = \{14, 1 + i, 1 - i\} \). Applying Theorem 16 on \( 14\sigma \) with \( \lambda_0 = 1 \), and \( q = (4, 5, 4)^T \) we obtain the matrix
\[ A = \begin{pmatrix} 4 & 6 & 4 \\ 4 & 5 & 5 \\ 6 & 1 & 7 \end{pmatrix}, \]
which is nonnegative generalized stochastic with spectrum \( 14\sigma \), while \( \frac{1}{14} A \) is a stochastic matrix with spectrum \( \sigma \).

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