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J. Math. Anal. Appl. 339 (2008) 1451–1467

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Extensions of operator algebras I

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Received 12 April 2007

Available online 7 August 2007

Submitted by Richard M. Aron

Abstract

We transcribe a portion of the theory of extensions of C^* -algebras to general operator algebras. We also include several new general facts about approximately unital ideals in operator algebras and the C^* -algebras which they generate.

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Keywords: Extensions; Nonselfadjoint operator algebras; Busby invariant; Ideals in operator algebras; C^* -envelope; C^* -algebra generated by an operator algebra; Ext group

1. Introduction

By an operator algebra, we mean a closed, not necessarily selfadjoint, algebra of operators on a Hilbert space. Our purpose here is to transcribe as much as possible of the powerful and important C^* -algebraic theory of extensions, to general operator algebras, where hopefully it will also play a role. Although there is no requirement on our algebras to have any kind of identity or approximate identity, we assume for specificity that all *ideals* of operator algebras in this paper, and therefore also the ‘first term’ in any extension, have an approximate identity. This is often not the most interesting case (for example, nontrivial ideals in the free semigroup operator algebras which have been studied extensively recently, will generically have no kind of identity). However, it is the case that is closest to the rich C^* -algebra theory of extensions (see e.g. [2,6,7,10]); it is of course no restriction at all in the case of extensions of operator algebras by C^* -algebras. We remark that some of our results have variants valid for Banach algebras which we also have not seen in the literature, which may lend further justification for our endeavour. In addition to the theory of extensions, we include several new general facts of interest.

We now describe the contents of the paper. In Section 2 we give several new results about ideals in operator algebras, and about generated C^* -algebras, which will be used in later sections, and which are independently interesting. In Section 3, we describe the basic theory of extensions of operator algebras, following in large part the diagrammatic approach of Eilers, Loring, and Pedersen [13]. In parts of this section, aspects which are very similar to the C^* -algebra or Banach algebra case are described quite hastily; a more thorough exposition is given in the second author’s thesis [28] (actually much of the contents of the paper is amplified there, together with additional results). In Section 4, we discuss how extensions of operator algebras are related to extensions of containing C^* -algebras.

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In a sequel paper we will apply the contents of this paper to notions such as *semisplit* extensions, variants of the *Ext* semigroup (or group) bivariate functor, and exactness of operator algebras. Some further developments will also be contained in [28], for example nonselfadjoint variants of some other results from [13,25,26].

An operator algebra may be thought of as a closed subalgebra of a C^* -algebra. We refer the reader to [9, Chapter 2] for the basic facts and notations which we shall need concerning operator algebras, such as the ones below. A few of these may also be found in [24]. An operator algebra is *unital* if it has an identity of norm 1, and is *approximately unital* if it has a contractive two-sided approximate identity (cai). All ideals are assumed to be two-sided and closed. By a *morphism* we mean a linear completely contractive homomorphism $\theta : A \rightarrow B$ between operator algebras. If $\theta(1) = 1$ we say that θ is *unital*. If θ takes some cai for A to a cai for B then we say that θ is *proper*. We write A^1 for Meyer's unitization of a nonunital operator algebra (see [21] or [9, Section 2.1]). The multiplier algebra of A will be denoted as $\mathcal{M}(A)$ (see [9, Section 2.6]), and $\mathcal{Q}(A)$ will denote the corona algebra $\mathcal{M}(A)/A$ of A , which is a unital operator algebra (it is (0) if A is unital). We write π_A for the canonical map $\mathcal{M}(A) \rightarrow \mathcal{Q}(A)$. If A is an ideal in B , then the canonical morphism from B into $\mathcal{M}(A)$ will be denoted by σ . To say that A is an *essential ideal* in B is to say that σ is one-to-one. A proper morphism $\alpha : A \rightarrow B$ extends canonically to a unital morphism $\tilde{\alpha} : \mathcal{M}(A) \rightarrow \mathcal{M}(B)$, and this induces a unital morphism $\tilde{\alpha} : \mathcal{Q}(A) \rightarrow \mathcal{Q}(B)$. Moreover $\tilde{\alpha}$ is completely isometric if and only if α is completely isometric, and implies that $\tilde{\alpha}$ is completely isometric (see Corollary 2.3). A C^* -cover of an operator algebra B is a pair (\mathcal{E}, j) consisting of a completely isometric homomorphism j from B into a C^* -algebra \mathcal{E} , such that \mathcal{E} is generated as a C^* -algebra by $j(B)$. If B has a cai (e_t) , then it follows that $(j(e_t))$ is a cai for \mathcal{E} . There exists a natural ordering and equivalence of C^* -covers of B , and a maximal and minimal equivalence class, $C^*_{\max}(B)$ and $C^*_e(B)$, respectively. The latter is the C^* -envelope of B (see [14]), and it is a quotient of any other C^* -cover of B . The former, $C^*_{\max}(B)$, is characterized by its universal property: every completely contractive homomorphism $\pi : B \rightarrow \mathcal{D}$ into a C^* -algebra \mathcal{D} extends to a $*$ -homomorphism $C^*_{\max}(B) \rightarrow \mathcal{D}$. We note that the maximal and minimal C^* -covers of B may be defined to be the C^* -algebra generated by B inside the same C^* -cover of the unitization B^1 . Although these are only needed in the sequel paper, we will also consider the minimal and maximal tensor products, \otimes_{\min} (which we sometimes write as \otimes) and \otimes_{\max} , of operator algebras (see e.g. [9, Chapter 6]).

In this paper we will work with two main categories. The first is the category **OA** of all operator algebras, with morphisms the completely contractive homomorphisms. The second is the subcategory **AUOA** of approximately unital operator algebras, with the same morphisms. On one or two occasions one needs stronger hypotheses for the smaller category (for example the pullback construction in **AUOA** needs additional restrictions in order to reside in the same category). In any case, in both categories we will need to assume that the first term in any *extension* (defined below) is approximately unital. Most parts of the paper can be read twice, once for each of the above two categories. Often we will only state the **AUOA** case of our results; and leave the other case to the reader. Actually, many of the results of this paper have variants in eight categories, the other six being: the variants of **OA** and **AUOA** where the morphisms are contractive homomorphisms, and then the 'up to constants' variants of the last four categories, where in the description of the morphisms we change the word 'contractive' to 'bounded,' and replace 'cais' by 'bais' (bounded approximate identities). Indeed, many of our results have obvious variants valid for (preferably, Arens regular) Banach algebras, which we have not seen in the Banach algebras literature (which focuses on quite different directions, see e.g. [4]). We will not usually take the time to state these other cases and variants, we leave this to the reader (see also [28]), and for specificity restrict our attention to the one closest to the C^* -algebra case. We will use the terms morphism, subobject, quotient, etc., in the obvious way. Thus, for example, a subobject in **OA** is a closed subalgebra, whereas in **AUOA** it is an approximately unital closed subalgebra.

We write \mathbb{K}, \mathbb{B} for the compact and bounded operators on a separable Hilbert space. We say that an operator algebra B is *stable* if $B \cong B \otimes \mathbb{K}$ completely isometrically isomorphically.

2. Ideals and C^* -covers

It is of enormous importance in C^* -algebra theory, that a $*$ -homomorphism (or equivalently, a contractive homomorphism) on a C^* -algebra has closed range, and is a complete quotient map onto its range. For nonselfadjoint algebras, this is not at all the case. To see this, recall that contractive Banach space maps need not have closed range, and even if they do, they need not induce isometries after quotienting by their kernel. These facts, combined with the $\mathcal{U}(X)$ construction from 2.2.10–2.2.11 of [9], then implies the same facts for contractive or completely contractive homomorphisms between operator algebras. Nonetheless, the following results in this direction will be useful:

Lemma 2.1. *Suppose that $\theta : \mathcal{E} \rightarrow \mathcal{F}$ is a morphism between operator algebras (that is, a morphism in **OA**), which is also a complete quotient map. Suppose that B is a closed subalgebra of \mathcal{E} , and there is a cai for $\text{Ker}(\theta)$ which either (a) lies in B , or (b) has a weak* limit point in $B^{\perp\perp}$. Then $\theta(B)$ is closed, and $\theta|_B$ is a complete quotient map onto $\theta(B)$. Indeed, $\theta(\text{Ball}(B)) = \text{Ball}(\theta(B))$, and similarly at all matrix levels. We have $\theta(B) \cong B / \text{Ker}(\theta|_B) \subset \mathcal{E} / \text{Ker}(\theta)$ completely isometrically isomorphically.*

Proof. Hypotheses (a) and (b) are equivalent (this follows from a seemingly deep result in [8,17]). Let $C = \theta(B)$, $\mathcal{D} = \text{Ker}(\theta)$ and $A = \text{Ker}(\theta|_B)$, these are ideals in \mathcal{E} and B respectively, and they have a common cai. Let p be a weak* limit of the cai in the second dual; clearly $p \in B^{\perp\perp}$ and p is the (central) support projection of \mathcal{D} in \mathcal{E}^{**} , and also is the support projection of A in $B^{**} \cong B^{\perp\perp} \subset \mathcal{E}^{**}$. Writing 1 for the identity of a unital operator algebra containing \mathcal{E} , we have $\mathcal{E}^{**}(1 - p) \subset \mathcal{E}^{**}$. Indeed the map $\eta \mapsto \eta(1 - p) = \eta - \eta p$ is a completely contractive projection on \mathcal{E}^{**} , which is a homomorphism, and its kernel is $\mathcal{D}^{\perp\perp}$. We deduce that

$$\mathcal{D}^{**} \cong \mathcal{D}^{\perp\perp} = \mathcal{E}^{**} p, \quad \mathcal{F}^{**} \cong \mathcal{E}^{**} / \mathcal{D}^{\perp\perp} \cong \mathcal{E}^{**}(1 - p),$$

and so

$$\mathcal{E}^{**} = \mathcal{E}^{**} p \oplus^\infty \mathcal{E}^{**}(1 - p) \cong \mathcal{D}^{**} \oplus^\infty \mathcal{F}^{**}.$$

Similarly,

$$A^{**} \cong A^{\perp\perp} = B^{\perp\perp} p, \quad (B/A)^{**} \cong B^{\perp\perp} / B^{\perp\perp} p \cong B^{\perp\perp}(1 - p).$$

The composition of the canonical complete contractions $B/A \rightarrow \mathcal{E}/\mathcal{D} \rightarrow \mathcal{E}^{**}(1 - p)$, agrees with the composition of the canonical complete isometries $B/A \rightarrow B^{\perp\perp}(1 - p) \rightarrow \mathcal{E}^{**}(1 - p)$. Thus the map from B/A to \mathcal{E}/\mathcal{D} is a complete isometry. Composing it with the complete isometry $\mathcal{E}/\mathcal{D} \rightarrow \mathcal{F}$, we obtain a complete isometry $B/A \rightarrow \mathcal{F}$. It is easy to see that this coincides with the composition of the canonical map $\widetilde{\theta|_B} : B/A \rightarrow C$ induced by $\theta|_B$, and the inclusion map $C \hookrightarrow \mathcal{F}$. It follows that $\widetilde{\theta|_B}$ is a complete isometry and has closed range, so that $\theta|_B$ is a complete quotient map with closed range. The assertion that $\theta(\text{Ball}(B)) = \text{Ball}(\text{Ran}(\theta))$ follows from the fact that approximately unital ideals in (we may assume, unital) operator algebras are M -ideals, and hence are proximal (see e.g. [9, Section 4.8] and [16]). A similar assertion holds at all matrix levels. \square

Lemma 2.2. *Suppose that B is a closed subalgebra of an operator algebra \mathcal{E} , that \mathcal{D}, A are ideals in \mathcal{E}, B respectively, with $A \subset \mathcal{D}$, and suppose that there is a common cai for \mathcal{D} and A . Then $B/A \subset \mathcal{E}/\mathcal{D}$ completely isometrically isomorphically.*

Proof. Just as in the proof of the last result. \square

Remark. There is a ‘one-sided’ version of the last results and their proofs, where e.g. we have right ideals and left cai. Again we get $B/A \hookrightarrow \mathcal{E}/\mathcal{D}$ completely isometrically. Similarly, $B/A \hookrightarrow \mathcal{E}/\mathcal{D}$ in the Banach algebra variant.

Corollary 2.3. *If A is a closed subalgebra of an operator algebra B , and if they have a common cai, then $\mathcal{Q}(A) \subset \mathcal{Q}(B)$ completely isometrically, via the map $\tilde{\alpha}$ described in the introduction, taking $\alpha : A \rightarrow B$ to be the inclusion.*

Lemma 2.4. *If A is a closed approximately unital ideal in a closed subalgebra B of a C^* -algebra \mathcal{E} , then the C^* -subalgebra of \mathcal{E} generated by A is a two-sided ideal in the C^* -subalgebra of \mathcal{E} generated by B .*

Proof. By 2.1.6 in [9], if (e_t) is a cai in A then for $b \in B, a \in A$ we have $ba^* = \lim_t be_t a^* \in \overline{AA^*}$. Similarly, b^*a, ab^*, a^*b lie in the C^* -subalgebra generated by A , from which the result is clear. \square

Lemma 2.5. *If A is a closed approximately unital ideal in an operator algebra B , then $C_e^*(A)$ is a closed approximately unital ideal in $C_e^*(B)$. More specifically, the C^* -subalgebra of $C_e^*(B)$ generated by A is a C^* -envelope of A .*

Proof. We can assume that B is unital, since if not, $C_e^*(B)$ is the C^* -algebra generated by B in $C_e^*(B^1)$ (see [9, Section 4.3]). First suppose that $A = Bp$ for a central projection $p \in B$, then it is easy to see that $C_e^*(B)p$ is a C^* -envelope of A . In the general case we go to the second duals. If (D, j) is a C^* -envelope of B^{**} , then its C^* -subalgebra C generated by $j(B)$ is a C^* -envelope of B by [8, Lemma 5.3]. If p is the support projection of A in B^{**} , then by the first line of the proof, $Dj(p)$ is a C^* -envelope of $A^{**} \cong A^{\perp\perp} = B^{**}p$. Thus by [8, Lemma 5.3] again, the C^* -subalgebra J of $Dj(p)$ generated by $j(A)$ is a C^* -envelope of A . Just as in the proof of the last result, J is clearly an ideal in C . \square

Remark. Although we shall not use this, the following interesting fact follows from the last result. If A, B are as in that result, with A, B both approximately unital, then the injective envelope $I(A)$ (whose theory may be found in e.g. [11,24]) may be viewed as a subalgebra of $I(B)$. In fact there is a projection $p \in I(B)$ with $I(A) = pI(B)p$. To see this, apply the last result, the fact that $I(C_e^*(\cdot)) = I(\cdot)$, and [15, Theorem 6.5].

A two-sided ideal A in B is *essential* if the canonical map $\sigma : B \rightarrow \mathcal{M}(A)$ is one-to-one. We say that the ideal is *completely essential* if σ is completely isometric. Later we will characterize these properties in terms of the Busby invariant. Here we give the following characterization along the lines of [18]:

Proposition 2.6. *If A is a closed approximately unital two-sided ideal in an operator algebra B , then the following are equivalent:*

- (i) A is a completely essential ideal in B .
- (ii) Any complete contraction with domain B is completely isometric iff its restriction to A is completely isometric.
- (iii) There is a C^* -cover \mathcal{E} of B such that the C^* -subalgebra J of \mathcal{E} generated by A is an essential ideal in \mathcal{E} .
- (iv) Same as (iii), but with $\mathcal{E} = C_e^*(B)$.
- (v) If $j : B \rightarrow I(B)$ is the canonical map into the injective envelope of B , then $(I(B), j|_A)$ is an injective envelope of A .

If B is nonunital, these are equivalent to

- (vi) A is a completely essential ideal in the unitization B^1 .

Proof. We begin by showing that (i) is equivalent to (vi) if B is not unital. In this case, (vi) \Rightarrow (i) is trivial. Conversely, if $\sigma : B \rightarrow \mathcal{M}(A)$ is completely isometric, then by Meyer’s unitization theorem (see [9, Corollary 2.1.15]) it follows that the canonical map $B^1 \rightarrow \mathcal{M}(A)$ is completely isometric, giving (vi).

(ii) \Rightarrow (i) This follows from the fact that the restriction of $\sigma : B \rightarrow \mathcal{M}(A)$ to A is completely isometric. That (iv) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i) The canonical map $\mathcal{E} \rightarrow \mathcal{M}(J)$ is a one-to-one $*$ -homomorphism, and hence completely isometric. Thus the restriction ρ to B is completely isometric. Now J and A have a common cai (e_t) . By the proof of (2.23) in [9], we have

$$\|[b_{ij}]\| = \|\rho(b_{ij})\| = \sup_t \|[b_{ij}e_t]\| = \|\sigma(b_{ij})\|, \quad [b_{ij}] \in M_n(B).$$

(i) \Rightarrow (iv) We are supposing $\sigma : B \rightarrow \mathcal{M}(A)$ is completely isometric. View $C_e^*(A) \subset C_e^*(B)$ as in Lemma 2.5, and consider the canonical $*$ -homomorphism $\sigma' : C_e^*(B) \rightarrow \mathcal{M}(C_e^*(A))$. Since $C_e^*(A)$ and A have a common cai (e_t) , the last centered equation in the last paragraph, shows in the current setting that the restriction ρ of σ' to B is completely isometric. By the ‘essential property’ of the C^* -envelope (see e.g. 4.3.6 in [9]), σ' is completely isometric.

(v) \Rightarrow (ii) Given a complete contraction $T : B \rightarrow B(H)$ whose restriction to A is completely isometric, extend T to a complete contraction $\hat{T} : I(B) = I(A) \rightarrow B(H)$. By the ‘essential property’ of $I(A)$ (see e.g. [9, Section 4.2]), \hat{T} is completely isometric, and hence also T .

(iv) \Rightarrow (v) We may assume that B is approximately unital, since in the contrary case one may appeal to (vi), and also use the fact that $C_e^*(B)$ is the C^* -algebra generated by B in $C_e^*(B^1)$ (see [9, Section 4.3]). Then this follows from the C^* -algebraic case of (i) \Rightarrow (v) from [18], together with the fact that $I(A) = I(C_e^*(A))$ (and similarly for B). \square

Remark. The proof shows that the conditions are also equivalent to B being A – A -essential (respectively A – \mathbb{C} -essential) in the sense of [18].

Lemma 2.7. *If A is approximately unital, and is an ideal in an operator algebra B , define \mathcal{D} to be the C^* -subalgebra of $C_{\max}^*(B)$ generated by A . Then \mathcal{D} (respectively $C_{\max}^*(B)/\mathcal{D}$) is a maximal C^* -cover of A (respectively of B/A). That is, $C_{\max}^*(A) = \mathcal{D}$ and $C_{\max}^*(B/A) = C_{\max}^*(B)/\mathcal{D}$.*

Proof. We first prove that \mathcal{D} above has the universal property characterizing $C_{\max}^*(A)$. If $\pi : A \rightarrow B(H)$ is a non-degenerate completely contractive homomorphism, then by 2.6.13 of [9], π extends to a completely contractive homomorphism $B^1 \rightarrow B(H)$, and hence to a $*$ -homomorphism from $C_{\max}^*(B^1) \rightarrow B(H)$. The restriction of the latter to \mathcal{D} is a nondegenerate $*$ -homomorphism extending π . Thus $\mathcal{D} = C_{\max}^*(A)$. By Lemma 2.2, $B/A \subset C_{\max}^*(B)/\mathcal{D}$ completely isometrically, and it is easy to see that $C = B/A$ generates the latter C^* -algebra, and so this C^* -algebra is a C^* -cover of C . The fact that it is the maximal one follows by showing that it has the universal property characterizing $C_{\max}^*(C)$: a nondegenerate completely contractive homomorphism $\pi : C \rightarrow B(H)$ induces a homomorphism $B \rightarrow B(H)$ which annihilates A , which in turn extends to a $*$ -homomorphism $C_{\max}^*(B) \rightarrow B(H)$ which annihilates \mathcal{D} . This induces a $*$ -homomorphism on $C_{\max}^*(B)/\mathcal{D}$, and it is easy to check that this ‘extends’ π . \square

The following results, which are needed in the sequel paper, use the language of operator algebra tensor products (see e.g. [9, Section 6.1]):

Lemma 2.8. *If B is any C^* -algebra, and if A is any approximately unital operator algebra, then $C_{\max}^*(B \otimes_{\max} A) = B \otimes_{\max} C_{\max}^*(A)$. If B is in addition a nuclear C^* -algebra, then $C_{\max}^*(B \otimes_{\min} A) = B \otimes_{\min} C_{\max}^*(A)$.*

Proof. By (6.3) in [9], we have $B \otimes_{\max} A \subset B \otimes_{\max} C_{\max}^*(A)$. Clearly $B \otimes A$ generates the latter C^* -algebra. We show that $B \otimes_{\max} C_{\max}^*(A)$ has the universal property of $C_{\max}^*(B \otimes_{\max} A)$. Let $\theta : B \otimes_{\max} A \rightarrow B(H)$ be a completely contractive homomorphism. By [9, Corollary 6.1.7], there are two completely contractive homomorphisms $\pi : B \rightarrow B(H)$ and $\rho : A \rightarrow B(H)$ with commuting ranges such that $\theta(b \otimes a) = \pi(b)\rho(a)$. Now π is forced to be a $*$ -homomorphism by [9, Proposition 1.2.4], and hence the range of the canonical extension $\tilde{\rho}$ of ρ to $C_{\max}^*(A)$ commutes with $\pi(B)$. Hence we obtain a $*$ -homomorphism $\tilde{\theta} : B \otimes_{\max} C_{\max}^*(A) \rightarrow B(H)$ with

$$\tilde{\theta}(b \otimes a) = \pi(b)\tilde{\rho}(a) = \pi(b)\rho(a) = \theta(b \otimes a), \quad a \in A, b \in B,$$

proving the result. \square

Lemma 2.9. *If A, B are approximately unital operator algebras then $B \otimes_{\min} A$ is a completely essential ideal in $B^1 \otimes_{\min} A^1$. Here A^1 is the unitization, set equal to A if A is already unital, and similarly for B^1 .*

Proof. Let $\sigma : B^1 \otimes_{\min} A^1 \rightarrow \mathcal{M}(B \otimes_{\min} A)$ be the canonical morphism. Assume A and B are nondegenerately represented on Hilbert spaces K and H , respectively. Then $B^1 \otimes_{\min} A^1$ may be regarded as a unital subalgebra of $B(H \otimes K)$. For $u \in B^1 \otimes_{\min} A^1$ and $\zeta \in \text{Ball}(H \otimes K)$, and (e_t) and (f_s) cais for A and B respectively, we have

$$\|\sigma(u)\| \geq \|u(f_s \otimes e_t)\| \geq \|u(f_s \otimes e_t)\zeta\|.$$

Taking a limit gives $\|\sigma(u)\| \geq \|u\zeta\|$, so that $\|\sigma(u)\| \geq \|u\|$. So σ is an isometry, and similarly it is a complete isometry. \square

Theorem 2.10. *If A and B are two operator systems, or two approximately unital operator algebra, then $C_e^*(B \otimes_{\min} A) = C_e^*(B) \otimes_{\min} C_e^*(A)$.*

Proof. First assume that A, B are unital. In this case, one may assume below that A, B are operator systems if one likes, by replacing A by $A + A^*$, and similarly for B . Then the result is proved in [15, Theorem 6.8]. We include a more modern proof for the readers convenience. Let $\Phi : A \rightarrow B(H)$ be a completely isometric unital boundary representation in the sense of [12] (this paper is simplified in [3], where these maps are said to have the *unique extension property*). Then Φ extends to a unital $*$ -monomorphism from $C_e^*(A)$ into $B(H)$, by definition of a boundary

representation, and the ‘essential’ property of $C_e^*(A)$. So we may identify A as a subspace of $B(H)$ such that the C^* -algebra it generates inside $B(H)$ is $C_e^*(A)$. Similarly, we may assume that $B \subset C_e^*(B) \subset B(K)$ for a Hilbert space K , with the inclusion map being a boundary representation. We may view $B \otimes_{\min} A$ as a subspace of $B(K \otimes H)$, and $C_e^*(B) \otimes_{\min} C_e^*(A)$ is the C^* -subalgebra it generates, in $B(K \otimes H)$. The injective envelope $I(B \otimes_{\min} A)$ may be viewed as a subspace of $B(K \otimes H)$, and so the canonical map $\pi : C_e^*(B) \otimes_{\min} C_e^*(A) \rightarrow C_e^*(B \otimes_{\min} A)$ may be viewed as a completely positive unital map into $B(K \otimes H)$. It suffices to show that π is one-to-one. Let $\theta(y) = \pi(1 \otimes y)$, a completely positive unital map $C_e^*(A) \rightarrow B(K \otimes H)$ extending $I_K \otimes I_A$. The map $y \rightarrow I_K \otimes y$ on $C_e^*(A)$ is a boundary representation too, since any ‘multiple’ of a boundary representation is easily seen to be a boundary representation (using [9, Proposition 4.1.12] if necessary). It follows that $\theta(y) = I_K \otimes y$ for all $y \in C_e^*(A)$. Similarly, $\pi(x \otimes 1) = x \otimes I_H$ for all $x \in C_e^*(B)$. Because of the latter, it follows by 1.3.12 in [9] that $\pi(x \otimes y) = (x \otimes 1)\pi(1 \otimes y)$ for all $x \in C_e^*(B), y \in C_e^*(A)$. Thus π is the ‘identity map,’ and is completely isometric.

Next, suppose that A, B have cais $(e_r), (f_s)$, respectively. Let J be a boundary ideal (see e.g. [1] and [9, p. 99]) for $B \otimes_{\min} A$ in $C_e^*(B) \otimes_{\min} C_e^*(A)$. Then J is also an ideal in $C_e^*(B)^1 \otimes_{\min} C_e^*(A)^1$. Let $\theta : B^1 \otimes_{\min} A^1 \rightarrow (C_e^*(B)^1 \otimes_{\min} C_e^*(A)^1)/J$ be the canonical completely contractive morphism factoring through $C_e^*(B)^1 \otimes_{\min} C_e^*(A)^1$. The restriction of θ to $B \otimes_{\min} A$ is completely isometric, being the composition of the canonical morphism $B \otimes_{\min} A \rightarrow (C_e^*(B) \otimes_{\min} C_e^*(A))/J$, and the ‘inclusion’ $(C_e^*(B) \otimes_{\min} C_e^*(A))/J \rightarrow (C_e^*(B)^1 \otimes_{\min} C_e^*(A)^1)/J$. It follows from Proposition 2.6 and Lemma 2.9 that θ is completely isometric. Hence $J = (0)$, proving our result. \square

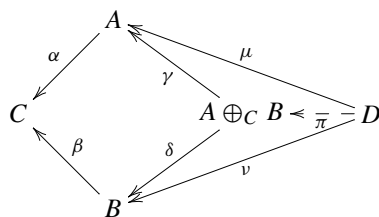
If we define the cone and suspension of a nonselfadjoint operator algebra just as one does in the C^* -literature (we will not take the time to review this; but remark there is a unitized and nonunitized version of these constructions, both of which work for our purposes in the sequel paper), it follows from the last two results that:

Corollary 2.11. *The cone and suspension operations both commute with both of C_e^* and C_{\max}^* , at least for approximately unital operator algebras.*

3. Theory of extensions

3.1. The pullback

Given three objects A, B, C in a category, and morphisms $\alpha : A \rightarrow C, \beta : B \rightarrow C$, the *pullback* $A \oplus_C B$ of A and B (along α and β), is the object which, together with two fixed morphisms γ, δ from this object to A and B respectively, satisfies the universal property/diagram



That is, for any object D and morphisms $\mu : D \rightarrow A, \nu : D \rightarrow B$ with $\alpha\mu = \beta\nu$, there exists a unique morphism $\pi : D \rightarrow A \oplus_C B$ such that the diagram commutes. By an obvious variant of the usual argument, the pullback is unique up to the appropriate completely isometric isomorphism. Indeed, in our setting, concretely

$$A \oplus_C B = \{(a, b) \in A \oplus^\infty B : \alpha(a) = \beta(b)\},$$

and the morphisms γ, δ are just the projections onto the two coordinates of the set in the last displayed equation. The map π takes $d \in D$ to $(\mu(d), \nu(d))$, of course.

It is easy to see that the pullback is closed (complete). In the category of approximately unital operator algebras, it is not true in general that the pullback is approximately unital. However, in Section 3.4 we will give a condition under which it always will be.

The *pushforward* construction in the category **AUOA** works just as in the C^* -algebra case. Many results in the rest of this paper can be phrased in terms of pushforwards, just as in [13,25], etc., and we leave this to the reader (see [28] for more details).

3.2. Extensions

If A, C are nontrivial operator algebras, with A approximately unital, then an *extension of C by A* is an exact sequence

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

where B is an operator algebra, and α, β are completely contractive homomorphisms, with α completely isometric, and β a complete quotient map. Some of our results will have variants valid with a weaker assumption on A (cf. the annihilator ideal assumptions in [23, Theorem 1.2.11]), but for specificity we will not consider this here. If we are working in the category **AUOA** we will also want to assume that B, C are approximately unital too, of course. Actually, it usually causes no trouble if we assume that $A \subset B$, and α is the inclusion map. Thus in future, we will sometimes silently be assuming this.

The canonical example of an extension is the *corona extension*

$$0 \longrightarrow A \longrightarrow \mathcal{M}(A) \longrightarrow \mathcal{Q}(A) \longrightarrow 0$$

This is the largest *essential extension* with first term A (see [25,28] for more details). It is also what we call a *unital extension*, namely an extension whose middle term is unital.

Proposition 3.1. *Given an extension*

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

in the above sense, C is approximately unital iff B is approximately unital.

Proof. The one direction is obvious. For the other, we may suppose without loss of generality that $A \subset B \subset D$, where D is a unital operator algebra with identity which is the 1 appearing below, and $C = B/A$. Suppose that p is the (central) support projection for A in $B^{**} = B^{\perp\perp}$. As in the proof of Lemma 2.1 we have

$$A^{\perp\perp} = B^{**}p, \quad B^{**}/A^{\perp\perp} \cong B^{**}(1 - p) \cong C^{**}.$$

Thus $B^{**}(1 - p)$ is unital, and hence so is $B^{**} = B^{**}p \oplus^{\infty} B^{**}(1 - p)$. Thus B is approximately unital (see [9, Proposition 2.5.8]). \square

Remarks. (1) Note that if B is a C^* -algebra then A, C are also C^* -algebras. Conversely, if A, C are C^* -algebras, then B is a C^* -algebra. To see this note that $B^{**} \cong A^{**} \oplus^{\infty} C^{**}$, and one may appeal to [9, Lemma 7.1.6].

(2) An extension of the complex numbers by A , with the middle algebra B unital, is exactly the same thing as a unitization of A .

3.3. Morphisms between extensions

Given two extensions of C by A , it is of interest to find a dotted arrow (namely, find a completely contractive homomorphism) making the following diagram commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & A & \longrightarrow & B_2 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

If such a morphism exists, then it is unique (as we shall say again later in a more general setting). We may regard the existence of this morphism as giving a partial ordering on the set of extensions of C by A . If the dotted arrow is also a surjective complete isometry then we say that the two extensions are *strongly isomorphic*. We write $\mathbf{Ext}(C, A)$ for the set of equivalence classes with respect to strong isomorphism of extensions of C by A . This is easily seen to be a ‘bi-functor.’ Indeed any morphism $\theta : C_1 \rightarrow C_2$ gives a map $\mathbf{Ext}(C_2, A) \rightarrow \mathbf{Ext}(C_1, A)$ by what we call ‘Diagram I’ below. We remark that Diagrams I–IV of [13] work in our setting essentially just as in that paper. Any

proper morphism $A_1 \rightarrow A_2$ gives a map $\mathbf{Ext}(C, A_1) \rightarrow \mathbf{Ext}(C, A_2)$, by ‘Diagram III.’ If the morphism is not proper then the functoriality is much deeper (see the sequel to this paper).

As usual there is a notion of *split extension*, namely that there exists a morphism, $\gamma : C \rightarrow B$ such that $\beta\gamma = I_C$. These include, but usually do not coincide with, the extensions strongly isomorphic to the trivial extension (the one with $B = A \oplus^\infty C$, and α, β the obvious maps). See [11]. A split extension is *strongly unital* if γ above is also unital. Note that the second dual of any extension is, by the proof of Lemma 2.1, an extension which is strongly isomorphic to the trivial extension. Along those lines we remark that a short exact sequence is an extension in the sense of our paper iff its second dual is an extension.

More generally, given two extensions in $\mathbf{Ext}(C_1, A_1)$ and $\mathbf{Ext}(C_2, A_2)$, a morphism from the first to the second is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

We will see in Theorem 3.4 that if the left vertical arrow is proper, then the vertical arrow in the middle is uniquely determined by the other two vertical arrows, and we also list there a criterion for its existence. Two extensions of C by A are *weakly equivalent* if there exists a morphism from one onto the other, in the latter sense, with all three vertical arrows surjective complete isometries. There are other notions of equivalence which we consider in the sequel to this paper and [28].

We state a variant of the ‘five lemma’ from algebra:

Lemma 3.2. *Given a morphism between extensions as in the above diagram: if the outer vertical arrows are complete isometries (respectively complete quotient maps), then so is the middle vertical arrow.*

Proof. Taking second duals of all algebras and morphisms in the diagram, we may assume that both rows in the diagram are trivial extensions. It then is easy algebra to see that the middle map is of the form $\rho((a, c)) = (\mu(a) + \delta(c), \nu(c))$, for a morphism δ , where μ, ν are the outer vertical arrows. Since $\rho(1, 0) = \rho(1, 0)\rho(1, c)$, it follows that the projection $\mu(1)$ is orthogonal to the range of δ . Thus if μ, ν are complete quotient maps, then $\delta = 0$, and ρ is then easily seen a complete quotient map. We also have $\|\rho((a, c))\| = \max\{\|\mu(a)\|, \|\delta(c)\|, \|\nu(c)\|\}$, and similarly for matrices, from which the complete isometry case follows. \square

3.4. Diagram I

The next tool we mention is what is called *Diagram I* in [13,25]:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \circ & \dashrightarrow & \circ & \dashrightarrow & C' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \gamma \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \longrightarrow 0 \end{array}$$

That is, given an extension of operator algebras (the bottom row), and a completely contractive homomorphism γ from an operator algebra C' as shown (C is approximately unital of course, if we are working in **AUOA**), then we can complete the first row to an extension of operator algebras, and find vertical morphisms so that the diagram commutes. In fact, one completion of the diagram is as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\tilde{\alpha}} & B \oplus_C C' & \xrightarrow{q_2} & C' \longrightarrow 0 \\ & & \parallel & & \downarrow q_1 & & \downarrow \gamma \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \longrightarrow 0 \end{array}$$

Here $B \oplus_C C' = \{(b, c') \in B \oplus^\infty C' : \beta(b) = \gamma(c')\}$ is a pullback. The maps q_1, q_2 in the diagram are the canonical projections onto B and C' , respectively. It is easy to see that q_2 is a complete quotient map, and further details are

just as in [23, Theorem 1.2.10]. Note that it follows from Proposition 3.1 that the pullback $B \oplus_C C'$ is approximately unital if C' is.

This ‘completion’ of Diagram I is the universal one. That is, given any other extension constituting the top row of a commuting Diagram I, this extension factors through the one in the last paragraph, just as in [13].

As we remarked earlier, Diagrams II–IV of [13] also work in our setting just as in that paper, and we will use these tools without comment. Similarly for their notion of *corona extendibility*, which we shall not study here.

3.5. The Busby invariant

Given any extension

$$E : 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

of C by A , there is a morphism $\tau : C \rightarrow \mathcal{Q}(A)$ defined by $\tau(c) = \sigma(b) + A$, where $\sigma : B \rightarrow \mathcal{M}(A)$ is the canonical map (namely, $\sigma(b)(a) = \alpha^{-1}(b\alpha(a))$, for $a \in A, b \in B$), and $\beta(b) = c$. We call τ the *Busby invariant* of the extension E . The Banach algebra variant may be found in [23, Theorem 1.2.11], which we now explain briefly in our context. If $\tilde{\beta} : B/\alpha(A) \rightarrow C$ is the surjective complete isometry induced by the complete quotient map β , and if $\tilde{\sigma} : B/\alpha(A) \rightarrow \mathcal{M}(A)/A$ is the complete contraction induced from $\sigma : B \rightarrow \mathcal{M}(A)$, then $\tau = \tilde{\sigma} \circ \tilde{\beta}^{-1}$. This shows that τ is well defined and completely contractive.

Given two operator algebras A, C , with at least A approximately unital, and a completely contractive homomorphism $\gamma : C \rightarrow \mathcal{Q}(A)$, the Diagram I tool above gives an extension of C by A :

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\tilde{\iota}} & \text{PB} & \xrightarrow{q_2} & C \longrightarrow 0 \\ & & \parallel & & \downarrow q_1 & & \downarrow \gamma \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & \mathcal{M}(A) & \xrightarrow{\pi_A} & \mathcal{Q}(A) \longrightarrow 0 \end{array}$$

Here PB is the pullback $\mathcal{M}(A) \oplus_{\mathcal{Q}(A)} C$. We call this the *pullback extension constructed from γ* . As usual, the Busby invariant of the latter extension is exactly γ . Conversely, any extension

$$E : 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

is strongly isomorphic to the pullback extension constructed from τ as in the last paragraph, taking τ to be the Busby invariant of the extension E . That is, there is a completely isometric surjective morphism φ making the following commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & A & \xrightarrow{\tilde{\iota}} & \text{PB} & \xrightarrow{q_2} & C \longrightarrow 0 \end{array}$$

where PB is the pullback $\mathcal{M}(A) \oplus_{\mathcal{Q}(A)} C$ along π_A and τ . Indeed, by definition of the pullback there is a canonical completely contractive morphism $\varphi : B \rightarrow \mathcal{M}(A) \oplus_{\mathcal{Q}(A)} C$, given by $\varphi(b) = (\sigma(b), \beta(b))$, for $b \in B$. That φ is completely isometric and surjective, follows from Lemma 3.2.

As usual, the *trivial extension* (the one with $B = A \oplus^\infty C$), has Busby invariant the zero map, and we also have (see [23, Theorem 1.2.11]):

Theorem 3.3. *There is a bijection between $\text{Ext}(C, A)$ and $\text{Mor}(C, \mathcal{Q}(A))$, the space of completely contractive homomorphisms $\tau : C \rightarrow \mathcal{Q}(A)$. There is a (non-bijective) correspondence between the equivalence classes of split extensions and $\text{Mor}(C, \mathcal{M}(A))$. In fact, an extension is split precisely when its Busby invariant equals $\pi_A \circ \eta$ for some $\eta \in \text{Mor}(C, \mathcal{M}(A))$. If C is unital the bijection above restricts to a bijection between the unital extensions (that is, those with middle term a unital algebra), and unital morphisms (those taking 1 to 1).*

Thus we often refer to an element of $\text{Mor}(C, \mathcal{Q}(A))$ as an extension of C by A .

Recall that given two extensions in $\mathbf{Ext}(C_1, A_1)$ and $\mathbf{Ext}(C_2, A_2)$, a morphism from the first to the second is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow \mu & & \downarrow \rho & & \downarrow \nu & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \end{array}$$

If μ is a proper morphism, then μ extends to a map $\mathcal{M}(A_1) \rightarrow \mathcal{M}(A_2)$, which in turn induces a map $\tilde{\mu} : \mathcal{Q}(A_1) \rightarrow \mathcal{Q}(A_2)$. This notation is used in the following:

Theorem 3.4. *Given two extensions as above, and morphisms $\mu : A_1 \rightarrow A_2$ and $\nu : C_1 \rightarrow C_2$, with μ proper, then there exists a (necessarily unique) morphism $\rho : B_1 \rightarrow B_2$ so that the diagram above commutes if and only if we have $\tilde{\mu} \circ \tau_1 = \tau_2 \circ \nu$, in the notation above. Here τ_1, τ_2 are the Busby invariants of the two sequences.*

Proof. See [13, Lemma 2.1 and Theorem 2.2] for details of the proof. \square

Remark. The result [13, Theorem 2.4], which is technically important in that paper, also easily carries over to non-selfadjoint algebras, as well as many of its consequences. See [28] for details.

An extension will be called *essential* if $\alpha(A)$ is an essential ideal in B in the sense that the canonical map $\sigma : B \rightarrow \mathcal{M}(A)$ is one-to-one; this turns out to be equivalent to the associated Busby invariant τ being one-to-one. We say that an extension is *completely essential* if σ is completely isometric.

Lemma 3.5. *An extension is completely essential iff the associated Busby invariant is completely isometric.*

Proof. The definition of the Busby invariant τ gives a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \parallel & & \downarrow \sigma & & \downarrow \tau & & \\ 0 & \longrightarrow & A & \longrightarrow & \mathcal{M}(A) & \xrightarrow{\pi} & \mathcal{Q}(A) & \longrightarrow & 0 \end{array}$$

By Lemma 3.2, if τ is completely isometric then so is σ . Conversely, if σ is completely isometric, then it is easy to see using Lemma 2.2 that so is the map $\tilde{\sigma}$ that we used in the definition of the Busby invariant, and hence τ is completely isometric by the formula $\tau = \tilde{\sigma} \circ \beta^{-1}$ given in that place. \square

3.6. Subextensions

A *subextension* of an extension

$$0 \rightarrow \mathcal{D} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \rightarrow 0,$$

consists of closed subalgebras A, B, C of \mathcal{D}, \mathcal{E} , and \mathcal{F} respectively, such that we have an extension

$$0 \rightarrow A \xrightarrow{\alpha|_A} B \xrightarrow{\beta|_B} C \rightarrow 0.$$

Of course this forces $C = \beta(B)$, and $A = \alpha^{-1}(B)$. To see this, note that clearly $A \subset \alpha^{-1}(B)$. On the other hand, the canonical isomorphism $B/\alpha(A) \rightarrow C$ is the composition of the canonical map $B/(B \cap \alpha(\mathcal{D})) \rightarrow C$, and the canonical map $B/\alpha(A) \rightarrow B/(B \cap \alpha(\mathcal{D}))$. Thus the latter map is one-to-one, which implies that $B \cap \alpha(\mathcal{D}) = \alpha(A)$. Hence $A = \alpha^{-1}(B)$.

Proposition 3.6. *Given an extension*

$$E : 0 \rightarrow \mathcal{D} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \rightarrow 0,$$

and a nontrivial subobject B of \mathcal{E} , then there exists a subextension of E with ‘middle term’ B if and only if $\alpha(\mathcal{D}) \cap B$ is approximately unital, and $\beta|_B$ is a complete quotient map. If B contains a cai for $\alpha(\mathcal{D})$ then the last condition (that $\beta|_B$ is a complete quotient map) is automatically true.

Proof. One direction is obvious from the discussion above. For the other, if $\alpha(\mathcal{D}) \cap B$ is approximately unital, let $C = \overline{\beta(B)}$ and $A = \alpha^{-1}(B) = \alpha^{-1}(B \cap \alpha(\mathcal{D}))$. If $\beta(b) = 0$ then $b = \alpha(d)$ for some $d \in \mathcal{D}$. Clearly, $d \in A$. Thus $\text{Ker}(\beta|_B) = \alpha(A)$, and so B is an extension of C by A if $\beta|_B$ is a complete quotient map. The last assertion follows from Lemma 2.1. \square

We just saw that the middle algebra B in a subextension as above determines A and C . We now discuss how A or C determines the others. An approximately unital closed subalgebra C of \mathcal{F} gives a subextension

$$0 \rightarrow \mathcal{D} \rightarrow \beta^{-1}(C) \rightarrow C \rightarrow 0$$

of the original extension. However there will in general be many other subextensions with last term C . In fact, the reader could check that the one just mentioned is just the ‘universal completion’ extension discussed in the section on ‘Diagram I.’ Indeed, there may be many different subalgebras of \mathcal{E} which give subextensions with first term A and last term C . So A and C do not determine B uniquely.

We next suppose that we are given an approximately unital closed subalgebra A of \mathcal{D} . By the previous result, the subextensions with first term A are in bijective correspondence with the subobjects B of \mathcal{E} such that $B \cap \mathcal{D} = A$. If one would prefer to characterize subextensions starting with A in terms of C rather than B then the situation seems much more complicated. However, in the case that A contains a cai for \mathcal{D} , there is such a characterization of subextensions which is fairly straightforward. We remark that this ‘common cai’ condition is probably the most interesting case of subextensions. For example, very often \mathcal{D} is a C^* -algebra generated by A , so that by [9, Lemma 2.1.7] they share a common cai.

Theorem 3.7. *Given an extension*

$$0 \rightarrow \mathcal{D} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \rightarrow 0,$$

and a closed subalgebra A of \mathcal{D} which contains a cai for \mathcal{D} , then the subextensions beginning with A are in bijective correspondence with the nontrivial subobjects C of

$$\mathcal{G} \stackrel{\text{def}}{=} \{c \in \mathcal{F} : \exists b \in \mathcal{E} \text{ with } bA + Ab \subset A \text{ such that } \beta(b) = c\}.$$

The middle term in the ensuing subextension is unique and given by the formula

$$B = \{b \in \beta^{-1}(C) : bA + Ab \subset A\}.$$

Moreover, the Busby invariant τ' for the subextension is related to the Busby invariant τ for the original extension by the formula $j(\tau'(c)) = \tau(c)$ for any $c \in C$, or equivalently $\tau' = j^{-1} \circ \tau|_C$, where j is the canonical completely isometric morphism $\mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{D})$ from Corollary 2.3.

Proof. Given any subextension with terms A, B, C , then C is clearly a closed subalgebra of \mathcal{G} , and $B \subset \{b \in \beta^{-1}(C) : bA + Ab \subset A\}$. Conversely, if $b \in \beta^{-1}(C)$ with $bA + Ab \subset A$, then there exists $b_1 \in B$ with $b - b_1 \in \text{Ker}(\beta)$, so that $b - b_1 \in \mathcal{D}$. If $d = b - b_1$ then $dA + Ad \subset A$. Thus $d = \lim e_i d \in A$, where (e_i) is the common cai, so that $b = d + b_1 \in B$.

Conversely, given a subobject C of \mathcal{G} , we will show that A, B, C constitute a subextension. Clearly B is closed. Suppose that $b \in \mathcal{D}$ satisfies $bA + Ab \subset A$. Then $be_i \in A$, where (e_i) is a common cai for A and \mathcal{D} , so that $b \in A$. This shows that A, B, C constitutes an exact sequence, and by Lemma 2.1 we have a subextension.

To see the last assertion, let $c \in C, b \in B, \beta(b) = c$. Then $\tau'(c) = \sigma'(b) + A, \tau(c) = \sigma(b) + \mathcal{D}$, and so the result boils down to showing that the canonical embedding $\iota : \mathcal{M}(A) \subset \mathcal{M}(\mathcal{D})$ takes $\sigma'(b)$ to $\sigma(b)$. Here σ, σ' are the canonical maps from B, \mathcal{E} into $\mathcal{M}(A), \mathcal{M}(\mathcal{D})$, respectively. However for $b \in B, d \in \mathcal{D}$,

$$\iota(\sigma'(b))(d) = \lim_i \sigma'(b)(e_i)d = \lim_i be_i d = bd,$$

and $\sigma(b)(d) = bd$. \square

Remarks. (1) The set \mathcal{G} above is a subalgebra of \mathcal{F} . If we are working in the category **OA** then clearly there is a largest subextension with first term A , namely

$$0 \rightarrow A \xrightarrow{\alpha} \{b \in \mathcal{E} : bA + Ab \subset A\} \xrightarrow{\beta} \mathcal{G} \rightarrow 0.$$

If we are working in the category **AUOA** then the same is true if \mathcal{G} above has a cai, which happens for example if \mathcal{E} is unital.

(2) If $A = \mathcal{D}$ then the above all follows from ‘Diagram I.’

If A is a C^* -algebra, and if we have an extension

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0,$$

then there is a canonical C^* -algebra subextension. Note that by 2.1.2 in [9] the diagonal $\Delta(B) = B \cap B^*$ contains $\alpha(A)$. If the diagonal $\Delta(C) = C \cap C^*$ is a nontrivial subobject of C , then by Theorem 3.7 we get a subextension with first term A , last term $\Delta(C)$, and middle term $\beta^{-1}(C)$. The latter contains $\Delta(B)$ clearly (by 2.1.2 in [9] again), and hence equals $\Delta(B)$ by the remark after Proposition 3.1. Thus we have a C^* -algebra subextension

$$0 \rightarrow A \xrightarrow{\alpha} \Delta(B) \xrightarrow{\beta|_{\Delta(B)}} \Delta(C) \rightarrow 0.$$

Examples. (1) Every extension of the upper triangular matrix algebra T_n by \mathbb{K} is split. To see this, suppose that we have an extension

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} T_n \rightarrow 0,$$

where A is a C^* -algebra. It follows that $\alpha(A) \subset \Delta(B) = B \cap B^*$. By the remark after Theorem 3.7 we get a subextension

$$0 \rightarrow A \xrightarrow{\alpha} \Delta(B) \xrightarrow{\beta} D_n \rightarrow 0,$$

where $D_n = \Delta(T_n)$. If $A = \mathbb{K}$, then this is just an extension of ℓ_n^∞ by \mathbb{K} , and we can lift the n minimal projections in D_n to n mutually orthogonal projections p_i in $\Delta(B)$ (see e.g. [10]). Pick contractions $T_i \in B$ with $\beta(T_i) = e_{i,i+1}$. By replacing T_i by $p_i T_i p_{i+1}$, we may assume that $T_i = p_i T_i p_{i+1}$. Let $b_{ij} = T_i T_{i+1} \cdots T_{j-1}$, for $i < j$, $b_{ii} = p_i$. The map $[\alpha_{ij}] \in T_n \mapsto \sum_{ij} \alpha_{ij} b_{ij} \in B$ is a completely contractive homomorphism, by a result of McAsey and Muhly (see [20]), and is clearly a splitting for the extension.

(2) Every unital extension of the disk algebra, or more generally of Popescu’s noncommutative disk algebra A_n (see [27]), has a strongly unital splitting. This follows by the associated noncommutative von Neumann inequality. For example, unital morphisms on the disk algebra are in bijective correspondence with contractions in the algebra that the morphism maps into; and if the latter algebra is a quotient algebra then we can lift the contraction and hence can lift the morphism too. By a similar argument, every nonunital extension of A_n by \mathbb{K} splits. For the bidisk algebra $A(\mathbb{D}^2)$, we can follow the obvious argument for the disk algebra to see that the splitting of unital extensions of $A(\mathbb{D}^2)$ by the compacts say, amounts to lifting commuting pairs of contractions in \mathbb{B}/\mathbb{K} to commuting pairs of contractions in \mathbb{B} , and Ando’s theorem for such pairs (see e.g. 2.4.13 in [9]). It is known that some such pairs do lift, while others do not (see e.g. [5]), and so there are quite nontrivial extensions in this case. In the sequel paper we will see that *Ext* in this simple case already brings up interesting operator theoretic topics. However for the tridisk algebra $A(\mathbb{D}^3)$, and for algebras of analytic functions on other classical domains, the argument above based on von Neumann inequalities fails, although it is clear that one will usually get non-split unital extensions.

We will not treat the subject of *corona extendibility* [13] here, but will give a very simple, but very common, example of it. Namely, given an extension

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in the category **OA**, suppose that C is nonunital, and C^1 is the Meyer unitization of C . By Meyer’s theorem [9,21], the Busby invariant τ extends to a unital morphism $\tau^1 : C^1 \rightarrow Q(A)$. We leave it as an exercise that this gives the ‘superextension’:

$$0 \rightarrow A \rightarrow B^1 \rightarrow C^1 \rightarrow 0.$$

We call this the *unitization extension*. The original extension is a subextension of this one. Conversely, any unital extension of C^1 by A is the unitization extension of an extension of C by A . We leave this as an exercise in diagram chasing, using Theorem 3.3 and Meyer’s theorem.

4. Covering extensions

In this section we start with an extension

$$E : 0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0,$$

and construct another extension containing E as a subextension. Note that given any operator algebra A' containing A with a common cai, by Diagram III we obtain a smallest (or universal) ‘superextension’ with first term A' , namely

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

By Lemma 3.2, the middle arrow is a complete isometry, so that the new extension contains E as a subextension.

We are, however, more interested in superextensions consisting of C^* -algebras. If each of these C^* -algebras is a C^* -cover of the operator algebra in the matching place in the sequence E , then we call the C^* -algebra extension a *covering extension* of E . We now discuss these. For simplicity of exposition we will occasionally assume that $A \subset B$ and $C = B/A$.

If we have a covering extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \mu \downarrow & & \downarrow j & & \downarrow v & & \\ 0 & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

where μ, j, v are the canonical maps from A, B, C respectively into the given C^* -covers, then the Busby invariant τ^* of the covering extension is related to the Busby invariant τ of the original extension by the formula

$$\tilde{\mu} \circ \tau = \tau^* \circ v = \tau^*|_C, \tag{1}$$

by Theorem 3.7, where we are viewing $C \subset \mathcal{F}$ via v . Since $\tilde{\mu}$ is completely isometric by Corollary 2.3, we have

$$\tau = \tilde{\mu}^{-1} \circ (\tau^*)|_C. \tag{2}$$

Remark. An extension is completely essential iff there is an essential covering extension with first two terms C^* -envelopes, and iff there exists some covering extension which is essential. This follows from Proposition 2.6 (and the proof of Lemma 2.5).

As was the case for subextensions (see Proposition 3.6), we have:

Proposition 4.1. *Given an approximately unital ideal in an operator algebra B , the equivalence classes (with respect to strong isomorphism) of covering extensions of an extension*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

are in a bijective correspondence with the equivalence classes of C^ -covers (\mathcal{E}, j) of B .*

Proof. Note that such \mathcal{E} gives a covering extension: set $\mathcal{D} = C^*(j(A))$, and set $\mathcal{F} = \mathcal{E}/\mathcal{D}$. By Lemma 2.4, \mathcal{D} is a two-sided ideal in \mathcal{E} . There is a canonical completely isometric homomorphism $\nu : C = B/A \rightarrow \mathcal{F}$ (see Lemma 2.2). It is easy to see that (\mathcal{F}, ν) is a C^* -cover of C . With $\mu = j|_A : A \rightarrow \mathcal{D}$ we have a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow \mu & & \downarrow j & & \downarrow \nu \\ 0 & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} \longrightarrow 0 \end{array}$$

with all vertical arrows completely isometric homomorphisms. \square

Remark. An interesting consequence of this, is that it follows that the equivalence classes of covering extensions of a given extension are in a bijective order reversing correspondence with the open sets in a certain topology, by [9, p. 99].

Lemma 4.2. *In the definition above of a covering extension, it is not necessary to assume that \mathcal{E} is a C^* -cover of B . This is automatically implied by the other hypotheses.*

Proof. Let \mathcal{G} be the C^* -subalgebra of \mathcal{E} generated by B . The image (respectively inverse image) of G in \mathcal{F} (respectively \mathcal{D}) is a C^* -algebra, and hence must equal \mathcal{F} (respectively \mathcal{D}) since the latter is generated by C (respectively A). By the five lemma the inclusion map $\mathcal{G} \rightarrow \mathcal{E}$ is surjective. That is, $\mathcal{G} = \mathcal{E}$. \square

Lemma 4.3. *Given an extension*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

as above, consider the two canonical maps from the proof of Proposition 4.1, from the set of C^ -covers of B to the set of C^* -covers of A , and to the set of C^* -covers of C . These two maps preserve the natural ordering of C^* -covers. The first of these maps is surjective.*

Proof. We leave the first assertion as an exercise in diagram chasing. To see the second, note that for any C^* -cover \mathcal{D} of A , we have $\mathcal{Q}(A) \subset \mathcal{Q}(\mathcal{D})$ completely isometrically via the map $\tilde{\mu}$ above, by Corollary 2.3. The map $\tilde{\mu} \circ \tau$ on C extends to a $*$ -homomorphism $\tau^* : C^*_{\max}(C) \rightarrow \mathcal{Q}(\mathcal{D})$, which is the Busby invariant for an extension of $C^*_{\max}(C)$ by \mathcal{D} . Since $\tilde{\mu} \circ \tau = \tau^*|_C$, we see by Theorem 3.4 that we have a morphism of extensions, the middle vertical arrow being completely isometric by Lemma 3.2. By Lemma 4.2 we have constructed a covering extension with first term \mathcal{D} . \square

Remarks. Neither of the two maps in the last lemma are one-to-one in general. The second map need not be onto, even if the extension is completely essential. An example showing this may easily be constructed in the case that $A = \mathbb{K}$ and C is the upper triangular 2×2 matrices. In this case one may easily construct (as in the next result) an extension with Busby invariant $\tau : C \rightarrow \mathbb{B}/\mathbb{K}$, such that τ has no extension to a $*$ -homomorphism from $C^*_c(A)$ into \mathbb{B}/\mathbb{K} . We can even ensure that the extension is completely essential and trivial. For example, choose in \mathbb{B} a projection p , and a partial isometry u with $u = pup^\perp$, but $uu^* - p \notin \mathbb{K}$ and $u^*u - p^\perp \notin \mathbb{K}$. Let \dot{p}, \dot{u} be the corresponding elements of \mathbb{B}/\mathbb{K} . Then it is easy to see that using e.g. [9, Corollary 2.2.12], that the map

$$\tau : C \rightarrow \mathbb{B}/\mathbb{K}: \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mapsto a\dot{p} + b\dot{u} + c(1 - \dot{p}),$$

corresponds to a completely essential trivial extension, but does not extend to a $*$ -homomorphism on M_n . Hence there is no covering extension of this extension with third term $C^*_c(A)$. Note this is also an example in which $C^*_c(B)/C^*_c(A)$ is not isomorphic to $C^*_c(B/A)$.

Proposition 4.4. *Given an extension*

$$E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

as above, and C^ -covers (\mathcal{D}, μ) and (\mathcal{F}, ν) of A and C respectively, then up to strong isomorphism there exists at most one covering extension of E with first and third terms \mathcal{D} and \mathcal{F} . In fact there will exist one of these if and only if the canonical map $\tilde{\mu} \circ \tau \circ \nu^{-1} : \nu(C) \rightarrow \mathcal{Q}(\mathcal{D})$ extends to a $*$ -homomorphism $\mathcal{F} \rightarrow \mathcal{Q}(\mathcal{D})$.*

Proof. This essentially follows from Eq. (1). If $\tilde{\mu} \circ \tau \circ \nu^{-1}$ extends to a $*$ -homomorphism $\tau^* : \mathcal{F} \rightarrow \mathcal{Q}(\mathcal{D})$, then the latter is the Busby invariant of a C^* -algebraic extension of \mathcal{F} by \mathcal{D} . By Theorem 3.4 and Lemmas 3.2 and 4.2, it is easy to see that this is a covering extension.

Because $\nu(C)$ generates \mathcal{F} , $*$ -homomorphisms on \mathcal{F} are determined uniquely by their restrictions to $\nu(C)$. Thus there is at most one $*$ -homomorphism $\tau^* : \mathcal{F} \rightarrow \mathcal{Q}(\mathcal{D})$ satisfying $\tilde{\mu} \circ \tau = \tau^*|_C$. \square

Remark. There may exist no covering extension of the type mentioned in the last result, as is pointed out in the last remark.

Proposition 4.5. *For any separable operator algebra C , and stable approximately unital operator algebra A , there exists an essential split extension of C by A . The middle term in this extension may be chosen to be nonunital if we wish, or to be unital if C is unital.*

Proof. Let $\pi : C_{\max}^*(C) \rightarrow B(\ell^2)$ be a faithful $*$ -representation: this is possible since $C_{\max}^*(C)$ is separable. Furthermore, we can assume that $\pi(C_{\max}^*(C)) \cap \mathbb{K} = (0)$, by replacing π by $\pi \oplus \pi \oplus \dots$ (unital case) or by $0 \oplus \pi \oplus \pi \oplus \dots$ (nonunital case). Now

$$B(\ell^2) = \mathcal{M}(\mathbb{K}) \subset \mathcal{M}(A \otimes \mathbb{K}) \cong \mathcal{M}(A),$$

and so we obtain a faithful completely contractive representation $\theta : C_{\max}^*(C) \rightarrow \mathcal{M}(A)$ for which it is easy to see that $\theta(C_{\max}^*(C)) \cap A = (0)$. Consider the maps

$$C_{\max}^*(C) \rightarrow \mathcal{M}(A) \rightarrow \mathcal{Q}(A) \rightarrow \mathcal{Q}(C_{\max}^*(A)).$$

These compose to a $*$ -homomorphism, by [9, Proposition 1.2.4]. Note that if $\theta(x) \in C_{\max}^*(A)$, then

$$\theta(x) \in \mathcal{M}(A) \cap C_{\max}^*(A) \subset A^{\perp\perp} \cap C_{\max}^*(A) = A,$$

by [9, Lemma A.2.3(4)], so that $\theta(x) = 0$. Thus the composition of the maps in the last centered sequence is faithful, and so completely isometric. Hence the associated morphisms $C_{\max}^*(C) \rightarrow \mathcal{Q}(A)$ and $C \rightarrow \mathcal{Q}(A)$ are completely isometric, and they factor through $\mathcal{M}(A)$. \square

Corollary 4.6. *If we have a completely essential extension*

$$E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

and a C^ -cover (\mathcal{D}, μ) of A , then there exists a ‘smallest’ or universal covering extension*

$$E_{\min} : 0 \rightarrow \mathcal{D} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

with first term \mathcal{D} . More particularly, E_{\min} is a quotient extension (that is, all three maps constituting the morphism are complete quotient maps) of any other covering extension of E with first term \mathcal{D} . Also, E_{\min} is essential.

Proof. We have a complete isometry $\kappa = \tilde{\mu} \circ \tau : C \rightarrow \mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{D})$. The C^* -algebra \mathcal{F} generated by $\kappa(C)$ is a C^* -cover of C . By Proposition 4.4, the inclusion map $\mathcal{F} \rightarrow \mathcal{Q}(\mathcal{D})$ is the Busby invariant of a covering extension E_{\min} with first and third terms \mathcal{D} and \mathcal{F} . Given any other covering extension of E with first term \mathcal{D} and last term \mathcal{G} say, there is a $*$ -homomorphism $\pi : \mathcal{G} \rightarrow \mathcal{Q}(\mathcal{D})$ such that $\pi(C) = \tilde{\mu}(\tau(C)) \subset \mathcal{F}$. It follows that $\pi : \mathcal{G} \rightarrow \mathcal{F}$. The conditions of Theorem 3.4 are met, so that E_{\min} is a quotient extension of the extension of \mathcal{G} . \square

Remark. As in [13], the above universal covering extension may be described as a pushout.

Proposition 4.7. *With notation as in Proposition 4.4, if \mathcal{D} and \mathcal{F} are C^* -envelopes (respectively maximal C^* -covers), and if a covering extension of E does exist with \mathcal{D} and \mathcal{F} as first and third terms, then the middle term \mathcal{E} is also a C^* -envelope (respectively a maximal C^* -cover).*

Proof. To see the first claim, note that if this middle term is \mathcal{E} , which dominates $C_e^*(B)$, then by the fact in Lemma 4.3 about the two maps being order preserving, we see that there exists a covering extension with middle term $C_e^*(B)$,

and other two terms dominated by, and hence equal to, $C_e^*(A)$ and $C_e^*(C)$, respectively. By the remark at the start of the paragraph, $\mathcal{E} = C_e^*(B)$. We leave the second as an exercise, using a similar idea. \square

A covering extension of the type in the last proposition with all terms C^* -envelopes, if it exists, will be called a C^* -enveloping extension, and the extension itself (of C by A) will be called a C^* -enveloped extension. If the Busby invariant τ of an extension of C by a C^* -algebra A extends to a $*$ -representation $C_e^*(C) \rightarrow \mathcal{Q}(A)$, then the extension E is C^* -enveloped. In particular, any nonselfadjoint operator algebra extension of a C^* -algebra is C^* -enveloped.

Example. There are many very interesting and topical examples of C^* -enveloped extensions, for example coming from the generalizations of Gelu Popescu’s noncommutative disk algebra A_n which have attracted much interest lately. The way in which these are usually obtained is one finds a ‘Toeplitz-like’ C^* -algebra \mathcal{E} with a quotient ‘Cuntz-like’ C^* -algebra \mathcal{F} , which in turn is generated by a nonselfadjoint operator algebra A . In Popescu’s original setting [27] the picture is

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbb{K} & \longrightarrow & C^*(S_1, \dots, S_n) & \longrightarrow & \mathcal{O}_n & \longrightarrow & 0 \\
 & & \parallel & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \mathbb{K} & \longrightarrow & \circ & \longrightarrow & A_n & \longrightarrow & 0
 \end{array}$$

(When $n = 1$, A_n is just the disk algebra, and the top row is just the Toeplitz extension by the compacts.) In any such setting, by our earlier theory (e.g. Theorem 3.7), there is a unique completion of the diagram to a subextension. Indeed, in the example above, the missing term in the diagram is the inverse image under the top right arrow $\beta: C^*(S_1, \dots, S_n) \rightarrow \mathcal{O}_n$ of the bottom right algebra A_n , which is the closure in $C^*(S_1, \dots, S_n)$ of $\mathbb{K} + A_n$. If one can show that the top right C^* -algebra (\mathcal{F} in language above) is a C^* -envelope of the bottom right algebra, and doing this is currently quite an industry (initiated by Muhly and Solel, see e.g. [19,22]), then it follows from Proposition 4.7, that the covering extension is C^* -enveloping.

Remark. We give another proof of Lemma 2.5: If A is a closed approximately unital ideal in an operator algebra B , then by the proof of Lemma 4.3 we have a covering extension

$$0 \rightarrow C_e^*(A) \rightarrow \mathcal{E} \rightarrow C_{\max}^*(C) \rightarrow 0.$$

On the other hand, by Proposition 4.1, there is another covering extension with middle term $C_e^*(B)$. Since the latter is dominated by \mathcal{E} in the ordering of C^* -covers, it follows from Lemma 4.3 that the first term in the last covering extension is dominated by, and hence equals, $C_e^*(A)$. Thus $C_e^*(A)$ is an ideal in $C_e^*(B)$.

Similar reasoning gives another proof of Lemma 2.7. That is

Lemma 4.8. *If the middle C^* -algebra in a covering extension is a maximal C^* -cover, then all the C^* -algebras in the covering extension are maximal C^* -covers.*

If A is an approximately unital operator algebra, then since $\mathcal{Q}(A) \subset \mathcal{Q}(C_{\max}^*(A))$ by Corollary 2.3, any morphism $C \rightarrow \mathcal{Q}(A)$ extends uniquely to a $*$ -homomorphism $C_{\max}^*(C) \rightarrow \mathcal{Q}(C_{\max}^*(A))$. By the Busby correspondence, we see that this defines a one-to-one map from $\mathbf{Ext}(C, A)$ into $\mathbf{Ext}(C_{\max}^*(C), C_{\max}^*(A))$. This map is not in general surjective. However it will be if A is a C^* -algebra:

Corollary 4.9. *Let \mathcal{D} be a C^* -algebra and let C be an operator algebra. There is a canonical bijection from $\mathbf{Ext}(C, \mathcal{D})$ onto $\mathbf{Ext}(C_{\max}^*(C), \mathcal{D})$, taking the split extensions onto the split extensions.*

Proof. The first assertion is clear from the discussion above, since any morphism $C_{\max}^*(C) \rightarrow \mathcal{Q}(\mathcal{D})$ uniquely extends a morphism $C \rightarrow \mathcal{Q}(\mathcal{D})$, by the universal property of C_{\max}^* . We leave the last assertion to the reader (see also [28]). \square

In particular, $\mathbf{Ext}(C) = \mathbf{Ext}(C_{\max}^*(C))$, where as usual C is an operator algebra and $\mathbf{Ext}(\cdot)$ means extensions by \mathbb{K} . We study the associated semigroup/group in the sequel paper, which turns out to have very many of the important

properties that one has in the C^* -algebra case. It is probably rarely a group; if one wants a group one can look at the invertible elements of $\mathbf{Ext}(C)$, or at the variant of Ext corresponding to C^* -enveloping extensions.

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