Optimal recovery of isotropic classes of twice-differentiable multivariate functions

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ABSTRACT

We consider the class of functions defined on a convex body in \( \mathbb{R}^d \), \( d \in \mathbb{N} \), whose second derivatives in any direction are uniformly bounded and the class of \( d \)-variate functions periodic with respect to a given full-rank lattice \( \mathcal{L} \) and having uniformly bounded second derivative in any direction. The problem of the optimal algorithm which recovers functions from these classes using their values and values of their gradients at \( n \) points (nodes) is considered. We first obtain an estimate for the error of the optimal algorithms with fixed nodes. In the periodic case, for every \( n \) sufficiently large, we describe the optimal set of \( n \) nodes. When \( d = 2 \), for certain periodic cases, optimality of the hexagonal arrangement of nodes is shown. For both the periodic case and the non-periodic case we present asymptotic results as \( n \) gets large.

1. Introduction

Let \( d \in \mathbb{N} \) and \( G \subset \mathbb{R}^d \) be a convex body (i.e. a compact convex set with non-empty interior). Define

\[
\|f\|_G = \text{ess sup}_{x \in G} |f(x)|,
\]

for a measurable function \( f : G \to \mathbb{R} \). Instead of considering the problem of finding the weakest possible conditions defining the class, we will restrict ourselves to the following definition. Denote by \( W_G \) the class of continuously differentiable functions \( f : G \to \mathbb{R} \) such that for every unit vector \( r \in \mathbb{R}^d \),

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the directional derivative \( \frac{\partial^2 f}{\partial r^2} \) exists inside \( G \) at least in a generalized sense and

\[
\left\| \frac{\partial^2 f}{\partial r^2} \right\|_G \leq 1.
\]

Let \( \mathcal{L} \) be a full-rank lattice in \( \mathbb{R}^d \), that is a lattice generated by \( d \) linearly independent vectors. For every full-rank lattice \( \mathcal{L} \), choose a system of vectors \( \{v_1, \ldots, v_d\} \) which generates it, and define

\[
\Pi(\mathcal{L}) := \{ \alpha_1 v_1 + \cdots + \alpha_d v_d : \alpha_1, \ldots, \alpha_d \in [0, 1) \},
\]

the fundamental parallelepiped of lattice \( \mathcal{L} \). We say that a function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( \mathcal{L} \)-periodic if for every point \( x \in \mathbb{R}^d \) and for every vector \( v \in \mathcal{L} \), it holds that \( f(x + v) = f(x) \).

Denote by \( \mathcal{W}_\mathcal{L} \) the class of \( \mathcal{L} \)-periodic continuously differentiable functions \( f : \mathbb{R}^d \to \mathbb{R} \) such that for every unit vector \( r \in \mathbb{R}^d \), the directional derivative \( \frac{\partial^2 f}{\partial r^2} \) exists at least in a generalized sense in \( \mathbb{R}^d \) and

\[
\left\| \frac{\partial^2 f}{\partial r^2} \right\|_{\Pi(\mathcal{L})} \leq 1.
\]

The class \( \mathcal{W}_G \) can be considered as a multidimensional analogue of the standard class \( W^2_\infty[0, a] \) of functions \( g : [0, a] \to \mathbb{R} \) such that \( g' \) is absolutely continuous on \( [0, a] \) and \( |g'(t)| \leq 1 \) for almost every \( t \in [0, a] \). The class \( \mathcal{W}_\mathcal{L} \) can be viewed as a multidimensional analogue of the periodic class \( W^2_\mathcal{L}[0, a] \).

We remark here that the existence of the generalized derivative \( \frac{\partial^2 f}{\partial r^2} \) is understood in the sense that for a.e. line \( l \) parallel to vector \( r \) and passing through the interior of the domain of \( f \), the restriction of \( \frac{\partial^2 f}{\partial r^2} \) to the intersection of \( l \) with the domain of \( f \) is locally absolutely continuous and \( \frac{\partial^2 f}{\partial r^2} \) is measurable.

For other equivalent ways to define the generalized derivative \( \frac{\partial^2 f}{\partial r^2} \) see e.g. [21,23].

We consider the problem of finding an optimal method of global recovery of functions from classes \( \mathcal{W}_G \) and \( \mathcal{W}_\mathcal{L} \) which uses as information values of functions and their gradients at \( n \) points in \( G \) and \( \Pi(\mathcal{L}) \) respectively. We study this problem both in the case where the set of nodes is given (finding the error of the optimal recovering algorithm) and in the case where the nodes are free and only their number \( n \) is fixed (finding optimal nodes and the recovering operator). In the latter case we obtain the asymptotic behavior as \( n \to \infty \) of the error of the optimal algorithm with \( n \) nodes and present a sequence of asymptotically optimal configurations of nodes. In certain periodic cases on the plane we find the optimal configuration of nodes exactly.

Let \( \mathcal{V} = \mathcal{W}_G \), \( D = G \), and \( \mathcal{H} \) be the space of all continuous functions \( f : G \to \mathbb{R} \), or \( \mathcal{V} = \tilde{\mathcal{W}}_\mathcal{L}, D = \Pi(\mathcal{L}) \) and \( \mathcal{H} = C_\mathcal{L} \), where \( C_\mathcal{L} \) is the space of all continuous \( \mathcal{L} \)-periodic functions \( f : \mathbb{R}^d \to \mathbb{R} \). Let \( n \in \mathbb{N}, X = \{x_1, \ldots, x_n\} \subset D \) be an arbitrary set of points, and \( I_X \) be the information operator with values in \( \mathbb{R}^{(d+1)n} \) of the form

\[
I_X(f) = (f(x_1), \ldots, f(x_n), \nabla f(x_1), \ldots, \nabla f(x_n))
\]

defined on the set of all functions whose domain contains \( X \) with each point of \( X \) being a limiting point of the domain and the gradient being well defined at every point of \( X \).

Every mapping \( \Phi : \mathbb{R}^{(d+1)n} \to \mathcal{H} \) generates the algorithm of recovery of functions from the class \( \mathcal{V} \) of the form

\[
S(f) = \Phi(I_X(f)).
\]

For every function \( f \in \mathcal{V} \) and every algorithm of the form (1), define

\[
R(f; X, \Phi) = \|f - \Phi(I_X(f))\|_D
\]

and let

\[
R(\mathcal{V}; X, \Phi) = \sup_{f \in \mathcal{V}} R(f; X, \Phi).
\]

The quantity \( R(\mathcal{V}; X, \Phi) \) is known as the global or worst-case error of algorithm (1) over the class \( \mathcal{V} \).
Problem 1. Given a set $X \subset D$ of $n$ distinct nodes, find the value
\[
R(F; X) := \inf_{\Phi: \mathbb{R}^{(d+1)n} \to \mathcal{H}} R(F; X, \Phi)
\]  \hfill (2)
and optimal algorithms, i.e. mappings $\Phi^*: \mathbb{R}^{(d+1)n} \to \mathcal{H}$, which attain the infimum on the right-hand side of (2), if they exist.

Problem 2. Given $n \in \mathbb{N}$, it is required to find the value
\[
R_n(F) := \inf_{X \subset D, \#X = n} R(F; X)
\]  \hfill (3)
and optimal algorithms, i.e. sets of nodes $X^*_n$, attaining the infimum on the right-hand side of (3), and mappings $\Phi^*$, attaining the infimum on the right-hand side of (2) when $X = X^*_n$, if they exist (here $\#X$ stands for the cardinality of the set $X$).

We remark that the approximants assigned by the algorithms introduced above do not necessarily belong to the above mentioned classes of functions (and may even be non-differentiable) and do not necessarily interpolate the function being recovered at the nodes.

The problem of optimal recovery of the classes of univariate functions having a given majorant for their modulus of continuity or the modulus of continuity of their derivatives was considered by Korneichuk in [16,17]. The problem of optimal recovery of functions from their values and values of their derivatives of order up to $r - 1$ at $n$ nodes was considered by Bojanov [4] on the class of functions defined on an interval whose $(r - 1)$-th derivative is absolutely continuous and whose $r$-th derivative is bounded in $L_p$-norm. He proved optimality of equally spaced nodes. The problem of optimal recovery of the class of functions defined on a $d$-dimensional convex polyhedron and having a given majorant for the modulus of continuity from the values of functions at a fixed set of $n$ points, which includes all the vertices, was solved by Babenko and Ligun [1] and Babenko [2]. The asymptotic behavior of the optimal recovery error on this class as $n$ gets large was obtained in [1,3], and the problem of optimal lattice configurations of nodes was considered by Babenko in [2]. For more information and a further review of known results on optimal recovery see for example the books [24,26,18,19,25,20,27].

2. Auxiliary definitions and related problems

2.1. The optimal covering problem

Problem 2 for the classes introduced above turns out to be related to the following discrete geometric problems. For a set $Y \subset \mathbb{R}^d$ and a point $y \in \mathbb{R}^d$, define
\[
\text{dist}(y, Y) := \inf_{u \in Y} |y - u|,
\]
where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. For a set $V \subset \mathbb{R}^d$ ($V$ does not necessarily contain $Y$), let
\[
r(V, Y) := \sup_{x \in V} \text{dist}(x, Y).
\]  \hfill (4)
It is not difficult to see that value (4) equals the infimum of the set of all radii $r > 0$ such that
\[
\bigcup_{x \in Y} B[x, r] \supset V,
\]
where $B[x, r]$ is the closed ball in $\mathbb{R}^d$ centered at point $x$ of radius $r > 0$. Value (4) is also known as the best approximation of the set $V$ by the set $Y$ in the Euclidean metric in $\mathbb{R}^d$.

Further, in our paper, we reduce the asymptotic solution to Problem 2 on the class $W_G$, where $G \subset \mathbb{R}^d$ is a convex body, to the problem of the asymptotic behavior of the value
\[
E_n(G) := \inf_{X \subset G, \#X = n} r(G, X) = \inf_{X \subset G, \#X = n} r(G, X).
\]  \hfill (5)
Problem 2 on the class $W_L$ reduces to finding the quantity

$$
\epsilon_n(L) := \inf_{X \subseteq \Pi(L), \#X = n} r(\mathbb{R}^d, X + L),
$$

where

$$
A + B = \{x + y : x \in A, y \in B\}
$$
is the sum of the sets $A, B \subseteq \mathbb{R}^d$.

The classical formulation of the problem of the most economical covering of the Euclidean space $\mathbb{R}^d$ is as follows. A countable collection of equal balls in $\mathbb{R}^d$ whose union contains $\mathbb{R}^d$ is called a covering of $\mathbb{R}^d$. Denote by $\mu_d$ the Lebesgue measure in $\mathbb{R}^d$ and let

$$
\beta_d = \mu_d(B[0, 1]).
$$

For a given covering $\mathcal{M}$ of $\mathbb{R}^d$, its density is defined as

$$
\Gamma(\mathcal{M}) = \lim_{R \to \infty} \frac{\sum_{B \in \mathcal{M}} \mu_d(B \cap [-R, R]^d)}{(2R)^d},
$$

provided that this limit exists. (Quantity $\Gamma(\mathcal{M})$ will not change if $\mathcal{M}$ is scaled by any positive factor.) The problem is to find the value

$$
\Gamma_d := \inf \Gamma(\mathcal{M}),
$$

where the infimum is taken over all coverings $\mathcal{M}$ of $\mathbb{R}^d$ for which the density $\Gamma(\mathcal{M})$ exists, and the most economical coverings of $\mathbb{R}^d$, i.e. coverings $\mathcal{M}^*$, which attain the infimum on the right-hand side of (7).

Let $L^*$ be the lattice in $\mathbb{R}^2$ generated by vectors $v_1 = (1, 0)$ and $v_2 = (1/2, \sqrt{3}/2)$. It is known that

$$
\Gamma_2 = \Gamma(\mathcal{M}_*) = \frac{2\pi}{\sqrt{27}},
$$

where $\mathcal{M}_*$ is the collection of circles in $\mathbb{R}^2$ of radius $1/\sqrt{3}$ with centers at the points of the lattice $L^*$ (see [13] or [8,9]). The solution to this problem for $d \geq 3$ is unknown. This problem is dual to the problem of the densest packing of non-overlapping balls of equal radii in the space $\mathbb{R}^d$. The solution to the best-packing problem was found by Fejes Tóth for $d = 2$ (cf. [7]) and by Hales for $d = 3$ (cf. [12]). For $d \geq 4$ the solution to this problem remains unknown. More information on known results related to the optimal covering and best-packing problems can be found for example in [9,10,22,6,5].

Define

$$
\theta_d = \left(\frac{\Gamma_d}{\beta_d}\right)^{1/d}
$$

and let $\partial A$ be the boundary of a set $A \subseteq \mathbb{R}^d$. The result given by Kolmogorov and Tikhomirov [15, Theorem IX] (see also [14, Section 7.4]) implies that for every compact set $A \subseteq \mathbb{R}^d$ with $\mu_d(A) > 0$ and $\mu_d(\partial A) = 0$, it holds that

$$
E_n(A) = \inf_{X \subseteq \mathbb{R}^d, \#X = n} r(A, X) = \frac{\theta_d \mu_d(A)^{1/d}}{n^{1/d}}(1 + o(1)), \quad n \to \infty.
$$

This relation was later re-proved by a number of authors, in particular in [11, Theorem 10.7]. We remark that (9) always holds when $A$ is a convex body.
2.2. The optimal coding problem

When we try to recover functions from a given class in Problems 1 and 2 using as approximating functions only elements from the same class (i.e. when we take \( \mathcal{H} \) to be \( W_G \) or \( W_L \)), the optimal recovery error is determined by the error of coding the function \( f \) by vector \( I_X(f) \), and we obtain the problem of optimal coding (see [19, Section 8.3.2] and [25, Section 4.3]).

Recall that \( \mathcal{F} = W_G \) and \( D = G \) or \( \mathcal{F} = W_L \) and \( D = \Pi(L) \), and \( X = \{x_1, \ldots, x_n\} \subset D \). Since the vector \( I_X(f) \) is the only information that we have about the function \( f \), any element \( g \in \mathcal{F} \) such that \( I_X(g) = I_X(f) \) has “the same right” to be chosen as the approximation for \( f \). Denote by \( I_X(\mathcal{F}) \) the image of the class \( \mathcal{F} \) with respect to the information operator \( I_X \), and for every vector \( w \in \mathbb{R}^{(d+1)n} \), let \( I_X^{-1}(w) \) be the set of all functions in the domain of \( I_X \) for which \( I_X(f) = w \). Let

\[
K(\mathcal{F}; X) = \sup_{w \in I_X(\mathcal{F})} \operatorname{diam} I_X^{-1}(w) \cap \mathcal{F}
\]

\[
= \sup_{w} \|f_1 - f_2\| : f_1, f_2 \in \mathcal{F}, \; I_X(f_1) = I_X(f_2)
\]

be the error of the coding method using a given set \( X \) as the set of nodes. It measures how “informative” a given information operator \( I_X \) is for recovering functions from the class \( \mathcal{F} \). The problem of optimal coding is stated in the following way.

**Problem 3.** Find the value

\[
\lambda_n(\mathcal{F}) = \inf_{\mathcal{F} \subset \mathcal{H}, \; X = \mathbb{R}^n} K(\mathcal{F}; X)
\]  

(10)

and optimal information operators \( I_{X*} \) (i.e. sets of nodes \( X_n^* \) attaining the infimum on the right-hand side of (10)).

2.3. The central algorithm

We next describe a recovering mapping \( \Phi \), which will give the solution to Problem 1. This construction is known as the central algorithm and can be found for example in [25, Section 4.3]. Define \( D_1 = \mathcal{G} \) when \( \mathcal{F} = W_G \) and \( D_1 = \mathbb{R}^d \) when \( \mathcal{F} = W_L \).

Define the mappings \( \overline{\psi}_X, \underline{\psi}_X : \mathbb{R}^{(d+1)n} \rightarrow \mathcal{H} \) in the following way: for every \( w \in I_X(\mathcal{F}) \), let

\[
\overline{\psi}_X(w)(x) := \sup_{f \in \mathcal{F}, \; I_X(f) = w} f(x), \; x \in D_1,
\]

and

\[
\underline{\psi}_X(w)(x) := \inf_{f \in \mathcal{F}, \; I_X(f) = w} f(x), \; x \in D_1,
\]

and let functions \( \overline{\psi}_X(w) \) and \( \underline{\psi}_X(w) \) be identically zero if \( w \not\in I_X(\mathcal{F}) \). Then the algorithm \( \Phi_{X,\mathcal{F}} : \mathbb{R}^{(d+1)n} \rightarrow \mathcal{H} \) is defined in the following way:

\[
\Phi_{X,\mathcal{F}}(w) := \frac{1}{2} \left( \overline{\psi}_X(w) + \underline{\psi}_X(w) \right).
\]

(11)

The general result in [25, Theorem 3.2.1] implies that

\[
R(\mathcal{F}, X) = \sup_{w \in I_X(\mathcal{F})} \operatorname{rad}(I_X^{-1}(w) \cap \mathcal{F}),
\]

(12)

where \( \operatorname{rad}(A) \) is the radius of the set \( A \subset \mathcal{H} \) with respect to the norm \( \| \cdot \|_D \). This general result also implies that algorithm (11) is a central algorithm, i.e. for every vector \( w \in I_X(\mathcal{F}) \),

\[
\sup_{f \in I_X^{-1}(w) \cap \mathcal{F}} \|f - \Phi_{X,\mathcal{F}}(w)\|_D = \operatorname{rad}(I_X^{-1}(w) \cap \mathcal{F}) = \frac{1}{2} \|\overline{\psi}_X(w) - \underline{\psi}_X(w)\|_D.
\]

(13)

and that it is optimal on the class \( \mathcal{F} \) among all algorithms with nodes at the set \( X \). We summarize properties of algorithm (11) in the following statement.
Lemma 1. For every finite set $X \subset D$, it holds that

$$R(\mathcal{F}; X, \Phi_{X, \mathcal{F}}) = R(\mathcal{F}; X) = \frac{1}{2} K(\mathcal{F}; X) = \sup_{f \in \mathcal{F}} \|f\|_D.$$  \hspace{1cm} (14)

The first two equalities in Lemma 1 follow from (12) and (13). The proof of the third equality repeats the proof of Proposition 8.3.2 in [19]. Taking the infimum over $X$ in (14), we immediately obtain the following proposition.

Lemma 2. For every $n \in \mathbb{N}$, it holds that

$$R_n(\mathcal{F}) = \frac{1}{2} \lambda_n(\mathcal{F}).$$

3. Main results

We obtain the following results for Problem 1.

Theorem 1. Let $G \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a convex body, and $X \subset G$ be a finite set of nodes. Then the algorithm

$$S^*(f) = \Phi_{X, W_G}(l_X(f)),$$

where $\Phi_{X, W_G}$ is defined in (11), is optimal on the class $W_G$ in the sense of Problem 1. In addition, in the case $r(G, X) \geq \sqrt{2}r(\partial G, X)$, it holds that

$$R(W_G; X) = \frac{1}{4} r^2 (G, X).$$ \hspace{1cm} (15)

For a given lattice $\mathcal{L} \subset \mathbb{R}^d$, define

$$f(\mathcal{L}) = \min_{v \in \mathcal{L}, \ v \neq 0} |v|.$$

Theorem 2. Let $\mathcal{L}$ be a full-rank lattice in $\mathbb{R}^d$, $d \in \mathbb{N}$, and $X \subset \Pi(\mathcal{L})$ be a finite set of nodes. Then the algorithm

$$S^*(f) = \Phi_{X, \tilde{W}_{\mathcal{L}}}(l_X(f)),$$

where mapping $\Phi_{X, \tilde{W}_{\mathcal{L}}}$ is defined in (11), is optimal on the class $\tilde{W}_{\mathcal{L}}$ in the sense of Problem 1. In addition, if

$$r(\mathbb{R}^d, X + \mathcal{L}) < \frac{1}{2} f(\mathcal{L}),$$ \hspace{1cm} (16)

it holds that

$$R(\tilde{W}_{\mathcal{L}}; X) = \frac{1}{4} r^2 (\mathbb{R}^d, X + \mathcal{L}).$$ \hspace{1cm} (17)

Remark 1. In Theorem 1, in the general case it holds that

$$R(W_G; X) \leq \max \left\{ \frac{1}{4} r^2 (G, X), \frac{1}{2} r^2 (\partial G, X) \right\}. \hspace{1cm} (18)$$

The second quantity under the maximum sign in (18) is attained in certain cases when $r(G, X) < \sqrt{2}r(\partial G, X)$. For example, if $y_0$ is the farthest point on $\partial G$ from $X$ and $z$ is the closest point in $X$ to $y_0$, consider the ball whose radius is twice the distance between $z$ and $y_0$ and the center is at point $t$, which lies on the line passing through $z$ and $y_0$, and is symmetric with point $z$ about $y_0$. If the interior of this ball does not contain points from $X$, we have $R(W_G; X) = r^2 (\partial G, X)/2$. 
Denote by \( \mathcal{U}_d \) the collection of centers of the most economical covering of \( \mathbb{R}^d \) with unit balls. In the case \( d = 2 \), in view of (8), we can take \( \mathcal{U}_2 = \mathcal{L}^\ast \). Let \( a \) be an interior point in \( G \) and \( Y_h = Y_h[G] = (a + h\mathcal{U}_d) \cap G, h > 0 \). Denote by \( D_h = D_h[G] \) the \( h \)-neighborhood of \( \partial G \). A subset \( Y \subset \mathbb{R}^d \) is called \( h \)-separated if for any two distinct points \( x, y \in Y \), we have \( |x - y| \geq h \). Let \( Z_h = Z_h[G] \) be a point set on \( D_h \cap G \) such that \( r(D_h \cap G, Z_h) \leq \frac{h}{\sqrt{2}} \) and \( \#Z_h = o(#Y_h) \), \( h \to 0 \). (In view of Lemmas 8 and 9 in the proof, one can take as \( Z_h \) an \( h/\sqrt{2} \)-separated subset of \( D_h \cap G \) having the largest possible number of points.) Define

\[
V_h = V_h[G] = Y_h[G] \cup Z_h[G], \quad h > 0. \tag{19}
\]

**Theorem 3.** Let \( G \subset \mathbb{R}^d, d \in \mathbb{N}, \) be a convex body. Then

\[
R_n(W_G) = \frac{1}{2} \lambda_n(W_G) = \frac{E_n^2(G)}{4} (1 + o(1)) = \frac{\theta_d^2 \mu_d(G)^2/d}{4n^{2/d}} (1 + o(1)), \quad n \to \infty. \tag{20}
\]

The sequence of sets of nodes \( V_h = V_h[G], h > 0 \), defined by (19) is asymptotically optimal on the class \( W_G \), i.e.,

\[
R(W_G; V_h) = R_{n_h}(W_G)(1 + o(1)), \quad h \to 0, \tag{21}
\]

where \( n_h = \#V_h, h > 0 \).

In the periodic case we have the following result.

**Theorem 4.** Let \( \mathcal{L} \) be a full-rank lattice in \( \mathbb{R}^d, d \in \mathbb{N}, \) and number \( n \in \mathbb{N} \) be such that \( \epsilon_n(\mathcal{L}) < J(\mathcal{L})/2 \). An \( n \)-point set \( X_n^\ast \subset \Pi(\mathcal{L}) \) such that

\[
r(\mathbb{R}^d, X_n^\ast + \mathcal{L}) = \epsilon_n(\mathcal{L}) \tag{22}
\]

is optimal for the class \( \widetilde{W}_\mathcal{L} \) in the sense of **Problem 3** and the algorithm

\[
S^\ast(f) = \Phi_{X_n^\ast, \widetilde{W}_\mathcal{L}}(I_{\Phi}(f))
\]

is optimal on this class in the sense of **Problem 2**. In addition,

\[
R_n(\widetilde{W}_\mathcal{L}) = \frac{1}{2} \lambda_n(\widetilde{W}_\mathcal{L}) = \frac{1}{4} \epsilon_n^2(\mathcal{L}) = \frac{\theta_d^2 \mu_d(\Pi(\mathcal{L}))^{2/d}}{4n^{2/d}} (1 + o(1)), \quad n \to \infty. \tag{23}
\]

A sequence of sets of nodes \( T_h = V_h[\Pi(\mathcal{L})], h > 0 \), where the configuration \( V_h[\Pi(\mathcal{L})] \) is defined by (19) with \( G = \Pi(\mathcal{L}) \), is asymptotically optimal on the class \( \widetilde{W}_\mathcal{L} \), i.e.,

\[
R(\widetilde{W}_\mathcal{L}; T_h) = R_{n_h}(\widetilde{W}_\mathcal{L})(1 + o(1)), \quad h \to 0, \tag{24}
\]

where \( m_h = \#T_h, h > 0 \).

In some cases, under the assumptions of **Theorem 4** the optimal set of nodes can be found explicitly. Recall that \( \mathcal{L}^\ast \) is the lattice in \( \mathbb{R}^2 \) generated by vectors \( \mathbf{v}_1 = (1, 0) \) and \( \mathbf{v}_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \).

**Theorem 5.** Let \( \mathcal{P} \) be a full-rank sublattice of \( \mathcal{L}^\ast \), and let \( k = \#(\mathcal{L}^\ast \cap \Pi(\mathcal{P})) \). The set of nodes \( X_N = (\frac{1}{k} \mathcal{L}^\ast) \cap \Pi(\mathcal{P}), N \geq 2 \), is optimal in the sense of **Problem 3** on the class \( W_\mathcal{P} \) among all configurations of \( kN^2 \) nodes, and the algorithm

\[
S^\ast(f) = \Phi_{X_N, \widetilde{W}_\mathcal{P}}(I_{\Phi}(f))
\]

is optimal on this class in the sense of **Problem 2** among all algorithms with \( kN^2 \) nodes. In addition, for \( n = kN^2 \),

\[
R_n(\widetilde{W}_\mathcal{P}) = \frac{1}{2} \lambda_n(\widetilde{W}_\mathcal{P}) = \frac{1}{4} \epsilon_n^2(\mathcal{P}) = \frac{k}{12n}.
\]
Remark 2. If \( \mathcal{L}_0 \) is the lattice in \( \mathbb{R}^2 \) generated by vectors \( \mathbf{v}_1 = (1, 0) \) and \( \mathbf{v}_2 = (0, \sqrt{3}) \), then every function from the class \( \tilde{W}_{\mathcal{L}_0} \) is 1-periodic along the first variable and \( \sqrt{3} \)-periodic along the second variable. Theorem 5 in particular implies that the system of nodes \( X_N = (\frac{1}{N}\mathcal{L}^n) \cap (0, 1) \times \{0, \sqrt{3}\} \) will be optimal on the class \( \tilde{W}_{\mathcal{L}_0} \) among all configurations of \( 2N^2 \) nodes, and that the system of nodes \( Y_N = (\frac{1}{N}\mathcal{L}^n) \cap \Pi(\mathcal{L}^n) \) will be optimal on the class \( \tilde{W}_{\mathcal{L}_0} \) among all configurations of \( N^2 \) nodes.

4. Auxiliary statements

4.1. One-dimensional extremal problems

For \( a > 0 \), define

\[
\varphi_a(t) := \begin{cases} \frac{a^2 - t^2}{4} & t \in [0, \frac{a}{2}) \\ \frac{(a - t)^2}{2} & t \in \left[ \frac{a}{2}, a \right] \\ 0 & t \in (a, \infty) \end{cases}
\]

(25)

It is not difficult to see that \( \varphi_a \in W^2_{\infty}[0, a] \).

Lemma 3. Let \( a > 0 \). Then

\[
\sup_{g \in W^2_{\infty}[0, a]} |g(0)| = \frac{a^2}{4}.
\]

The supremum is attained by the function \( \varphi_a \).

Proof. For every function \( g \in W^2_{\infty}[0, a] \) such that \( g'(0) = g(a) = g'(a) = 0 \), and every \( t \in [0, \frac{a}{2}] \), we have

\[
|g'(t)| = |g'(t) - g'(0)| = \left| \int_0^t g''(u)du \right| \leq \int_0^t |g''(u)|du \leq \int_0^t du = t.
\]

For \( t \in [\frac{a}{2}, a] \), we also get

\[
|g'(t)| = |g'(a) - g'(t)| = \left| \int_t^a g''(u)du \right| \leq \int_t^a |g''(u)|du \leq \int_t^a du = a - t.
\]

Then

\[
|g(0)| = |g(a) - g(0)| = \left| \int_0^a g'(t)dt \right| \leq \int_0^a |g'(t)|dt \leq \int_0^{a/2} tdt + \int_{a/2}^a (a - t)dt = \frac{a^2}{4}.
\]

Since \( \varphi_a \in W^2_{\infty}[0, a] \), \( \varphi'_a(0) = \varphi'_a(a) = 0 \), and \( \varphi(a) = \frac{a^2}{4} \), we obtain the assertion of Lemma 3. \( \square \)

For \( a > 0 \), let

\[
\psi_a(t) := \begin{cases} \frac{(a - t)^2}{2} & t \in [0, a], \\ 0 & t \in (a, \infty) \end{cases}
\]

It is not difficult to see that \( \psi_a \in W^2_{\infty}[0, a] \).
Lemma 4. Let $a > 0$. Then
\[
\sup_{g \in W^2_\infty[0,a]} \sup_{g(a) = g'(a) = 0} |g(0)| = \frac{a^2}{2}.
\] (26)

The supremum in (26) is attained by the function $\psi_a$.

Proof. Let $g \in W^2_\infty[0,a]$ be an arbitrary function such that $g(a) = g'(a) = 0$. For every $t \in [0,a]$, we obtain
\[
|g(0)| = \left| \int_0^a g''(u) \cdot u \, du \right| \leq \int_0^a |g''(u)| \cdot u \, du \leq \frac{a^2}{2}.
\]

Since $\psi_a \in W^2_\infty[0,a]$, $\psi_a(a) = \psi_a'(a) = 0$, and $\psi_a(0) = \frac{a^2}{2}$, we obtain the assertion of Lemma 4. \(\square\)

4.2. Auxiliary statements in the multivariate case

Below, we find a bound for the growth of functions from the multivariate classes considered.

Lemma 5. Let $U \subset \mathbb{R}^d$ be a convex body and $f \in W_U$. Then for every two distinct points $x, y \in U$, the function
\[
g(t) = f \left( x + \frac{t}{\alpha} (y - x) \right),
\]
where $\alpha = |y - x|$, belongs to the class $W^2_\infty[0,\alpha]$.

Proof. Define $u = \frac{y-x}{\|y-x\|}$. Then
\[
g'(t) = \text{grad} f \left( x + tu \right) \cdot u = \frac{\partial f}{\partial u} (x + tu), \quad t \in [0,\alpha].
\]

Let $t_1 < t_2$ be two arbitrary points in $[0,\alpha]$. Choose any $\epsilon > 0$ and let $\delta \in (0,\epsilon)$ be that from the definition of continuity of $\frac{\partial f}{\partial u}$ (in the totality of variables) at points $x + t_1 u$ and $x + t_2 u$ for the chosen $\epsilon$.

Let $x_1^1, x_2^1 \in \text{int} U$ be points satisfying the following conditions:

1. $|x + t_1 u - x_1^1| < \delta$, $i = 1, 2$,
2. there is a number $t^*$ such that $x_2^1 = x_1^1 + t^* u$, and
3. the function $h(t) = \frac{\partial f}{\partial u}(x_1^1 + tu)$ is locally absolutely continuous inside its domain with $|h'(t)| \leq 1$ for almost all $t$.

Such $x_1^1$ and $x_2^1$ exist in view of the definition of the generalized derivative and the fact that $\|\frac{\partial f}{\partial u}\|_U \leq 1$. Then
\[
|g'(t_1) - g'(t_2)| = \left| \frac{\partial f}{\partial u} (x + t_1 u) - \frac{\partial f}{\partial u} (x + t_2 u) \right|
\leq \left| \frac{\partial f}{\partial u} (x_1^1) - \frac{\partial f}{\partial u} (x_2^1) \right| + \left| \frac{\partial f}{\partial u} (x + t_1 u) - \frac{\partial f}{\partial u} (x_1^1) \right|
\leq \left| h(0) - h(t^*) \right| + 2\epsilon \leq |t^*| + 2\epsilon
\leq |t_1 - t_2| + 2\delta + 2\epsilon \leq |t_1 - t_2| + 4\epsilon.
\]

Letting $\epsilon \to 0$, we have
\[
|g'(t_1) - g'(t_2)| \leq |t_1 - t_2|, \quad t_1, t_2 \in [0,\alpha].
\]
Hence, \( g' \) is absolutely continuous on \([0, \alpha]\) and \( |g''(t)| \leq 1\) for almost every \( t \in [0, \alpha]\). Lemma 5 is proved. \( \square \)

**Lemma 6.** Let \( U \subset \mathbb{R}^d \) be a convex body and \( f \in W_U \). If \( y \in U \) is such that \( f(y) = 0 \) and \( \nabla f(y) = 0 \), then for every \( x \in U \), it holds that
\[
|f(x)| \leq \frac{|x - y|^2}{2}.
\] (27)

**Proof.** If \( y = x \) relation (27) holds trivially. Assume that \( x \neq y \) and define \( \gamma = |y - x| \). By Lemma 5, the function
\[
g(t) = f \left( x + \frac{t}{\gamma} (y - x) \right)
\]
belongs to the class \( W^2_{\infty}[0, \gamma] \) and \( g(\gamma) = g'(\gamma) = 0 \). Then in view of Lemma 4,
\[
|f(x)| = |g(0)| \leq \frac{\gamma^2}{2} = \frac{|x - y|^2}{2}.
\]
Lemma 6 is proved. \( \square \)

5. **Proof of Theorem 1 and Remark 1**

5.1. **Proof of Theorem 1**

Optimality of the algorithm \( \Phi_{X, W_C}(I_X(f)) \) follows from Lemma 1. We now show relation (15). Let \( f \in W_C \) be an arbitrary function such that \( I_X(f) = 0 \) and \( x_0 \in G \) be a point of maximum of \( |f(x)| \) on \( G \). Let also \( z \) be the closest point in \( X \) to \( x_0 \) and define \( \beta = |z - x_0| \).

First, consider the case when \( x_0 \) lies inside \( G \). In this case, \( \nabla f(x_0) = 0 \). By Lemma 5, the function
\[
g(t) = f \left( x_0 + \frac{t}{\beta} (z - x_0) \right), \quad t \in [0, \beta],
\]
belongs to the class \( W^2_{\infty}[0, \beta] \). Since \( \nabla f(x_0) = 0 \), we have \( g'(0) = 0 \). Since \( z \) is a point where both \( f \) and its gradient vanish, we obtain that \( g(\beta) = g'(\beta) = 0 \). Then by Lemma 3, we have
\[
\|f\|_G = |f(x_0)| = |g(0)| \leq \frac{\beta^2}{4} = \frac{|z - x_0|^2}{4} = \frac{1}{4} (\text{dist}(x_0, X))^2 \leq \frac{1}{4} r^2(G, X).
\] (28)

Next we consider the case \( x_0 \in \partial G \). Since \( f(z) = 0 \) and \( \nabla f(z) = 0 \), in view of Lemma 6 we obtain
\[
\|f\|_G = |f(x_0)| \leq \frac{1}{2} |z - x_0|^2 = \frac{1}{2} (\text{dist}(x_0, X))^2 \leq \frac{1}{2} r^2(\partial G, X).
\] (29)

From relations (28) and (29) taking into account Lemma 1, we have
\[
R(W_G; X) = \sup_{f \in W_G \atop f(y) = 0} \|f\|_G \leq \max \left\{ \frac{r^2(G, X)}{4}, \frac{r^2(\partial G, X)}{2} \right\}, \quad \text{(30)}
\]

which proves relation (18) in Remark 1. Let \( y \) be an arbitrary point in \( G \setminus X \). Define \( a := \text{dist}(y, X) \) and let \( f_y(x) := \varphi_a(|x - y|) \). It is not difficult to see that \( f_y \in W_G \) and \( I_X(f_y) = 0 \). Then
\[
\|f_y\|_G = |\varphi_a(0)| = \frac{a^2}{4} = \frac{1}{4} (\text{dist}(y, X))^2,
\]
and hence, by Lemma 1,

\[
R(W_C; X) = \sup_{f \in W_C \atop I_X(f) = 0} \|f\|_C \geq \sup_{y \in C \setminus X} \|f_y\|_C = \frac{1}{4} r^2(G, X). \tag{31}
\]

If \( r(G, X) \geq \sqrt{2}r(\partial G, X) \), then from relations (30) and (31) we have

\[
R(W_C; X) = \frac{1}{4} r^2(G, X).
\]

Theorem 1 is proved.

5.2. Proof of Remark 1

Relation (18) has been proved above. Assume that \( r(G, X) < \sqrt{2}r(\partial G, X) \). Let \( y_0 \in \partial G \) be such that \( \text{dist}(y_0, X) = r(\partial G, X) \), \( z \) be the point in \( X \) closest to \( y_0 \), and \( t = 2y_0 - z \). Assume that \( \text{dist}(t, X) \geq 2|z - y_0| \). Since \( \text{dist}(t, z) = 2|z - y_0| \), we actually have \( \text{dist}(t, X) = 2|z - y_0| \). Let \( b = 2|z - y_0| \) and \( f_s(x) = \phi_b(x - t) \). Since \( f_s \in W_C \) and \( I_X(f_s) = 0 \), we have

\[
R(W_C; X) = \sup_{f \in W_C \atop I_X(f) = 0} \|f\|_C = \|f_s\|_C = \|f_s(y_0)\| = \phi_b(|z - y_0|) = \phi_b \left( \frac{b}{2} \right)
\]

\[
= \frac{b^2}{2} = \frac{1}{2} |z - y_0|^2 = \frac{1}{2} (\text{dist}(y_0, X))^2 = \frac{1}{2} r^2(\partial G, X).
\]

Taking into account relation (18) we will have

\[
R(W_C; X) = \frac{1}{2} r^2(\partial G; X).
\]

Remark 1 is proved.

6. Proof of Theorem 2

The optimality of the algorithm \( \Phi_{X, \tilde{W}_L} (I_X(f)) \) follows from Lemma 1. We now show relation (17). Let \( f \in \tilde{W}_L \) be any function such that \( I_X(f) = 0 \), and let \( y_0 \in \Pi_L \) be such that \( \|f\|_{\Pi_L} = |f(y_0)| \). Denote by \( z \) a point from \( X + L \), such that \( |y_0 - z| = \text{dist}(y_0, X + L) \), and let \( \beta = |y_0 - z| \) and \( u = \frac{z - y_0}{\beta} \). By Lemma 5, the function

\[
g(t) = f \left( y_0 + tu \right), \quad t \in [0, \beta],
\]

belongs to the class \( W^2_{\infty} \{0, \beta\} \). By the choice of \( y_0 \), we have \( g'(0) = \frac{\partial f}{\partial u} (y_0) = 0 \). It is not difficult to see that \( g(\beta) = f(y_0 + \beta u) = f(z) = 0 \), \( g'(\beta) = \frac{\partial f}{\partial u} (y_0 + \beta u) = \frac{\partial f}{\partial u}(z) = 0 \). Hence, by Lemma 3, we obtain

\[
\|f\|_{\Pi_L} = \|f(y_0)\| = |g(0)| \leq \frac{\beta^2}{4} = \frac{|y_0 - z|^2}{4} \leq \frac{\text{dist}^2(y_0, X + L)}{4} \leq \frac{r^2(\mathbb{R}^d, X + L)}{4}.
\]

Hence, in view of Lemma 1, it holds that

\[
R(\tilde{W}_L; X) = \sup_{f \in \tilde{W}_L \atop I_X(f) = 0} \|f\|_{\Pi_L} \leq \frac{r^2(\mathbb{R}^d, X + L)}{4}. \tag{32}
\]

To show that equality holds in (32) we need the following proposition.
Lemma 7. Let \( \mathcal{L} \) be a full-rank lattice in \( \mathbb{R}^d \) and \( 0 < a < J(\mathcal{L})/2 \). Then the function
\[
 f_a(x) = \sum_{v \in \mathcal{L}} \varphi_a(|x - v|),
\]
where \( \varphi_a \) is defined by (25), belongs to the class \( \widetilde{W}_\mathcal{L} \), vanishes together with its gradient for every \( x \) such that \( \text{dist}(x, \mathcal{L}) \geq a \), and satisfies equality \( \|f_a\|_{\mathcal{L}} = a^2/4 \).

Proof. Define
\[
P = \bigcup_{v \in \mathcal{L}} B[v, a].
\]
The balls in the union defining \( P \) are pairwise disjoint, and \( P \) is closed. Hence, for every \( x \in P \), the sum in the definition of \( f_a \) contains at most one non-zero element and \( f_a \) will vanish together with its gradient for every \( x \) such that \( \text{dist}(x, \mathcal{L}) \geq a \). It is not difficult to see that \( f_a \) is \( \mathcal{L} \)-periodic. In view of the properties of the function \( \varphi_a(|x|), x \in \mathbb{R}^d \), we have \( f_a \in W_\mathcal{L} \). Then
\[
\|f_a\|_{\mathcal{L}} = \varphi_a(0) = \frac{a^2}{4}.
\]
Lemma 7 is proved. \( \square \)

Let \( z_0 \in \Pi(\mathcal{L}) \) be a point such that
\[
b := \text{dist}(z_0, X + \mathcal{L}) = r(\mathbb{R}^d, X + \mathcal{L}).
\]
Since, by assumption, \( b < J(\mathcal{L})/2 \) and the information operator \( \text{I}_X \) vanishes on the function \( f_b(x - z_0) \), in view of Lemmas 1 and 7, we have
\[
R(\widetilde{W}_\mathcal{L}; X) = \sup_{f \in \widetilde{W}_\mathcal{L}} \|f\|_{\mathcal{L}} \geq \|f_b(\cdot - z_0)\|_{\mathcal{L}} = \frac{b^2}{4} = \frac{1}{4} r^2(\mathbb{R}^d, X + \mathcal{L}),
\]
which together with (32) gives (17). Theorem 2 is proved.

7. Proof of Theorem 3

The first equality in (20) follows from Lemma 2 and the third equality follows from (9). Recall that \( D_h(G), h > 0 \), is the \( h \)-neighborhood of \( \partial G \). To show the second equality in (20) we will need the following lemmas.

Lemma 8. Let \( U \) be a convex bounded set with non-empty interior, and \( Z_h, h > 0 \), be an \( h/\sqrt{2} \)-separated subset of \( D_h(U) \cap \overline{U} \) having the largest possible number of points. Then
\[
#Z_h = o \left( \frac{1}{h^d} \right), \quad h \to 0.
\]

Proof. Since \( D_h \cap \overline{U} = D_h(U) \cap \overline{U} \) is bounded, we have \( #Z_h < \infty \). It is not difficult to see that
\[
r(\partial U, Z_h) \leq r(D_h \cap \overline{U}, Z_h) \leq \frac{h}{\sqrt{2}}.
\]
Indeed, the first inequality holds by monotonicity. If there was a point \( y \in D_h \cap \overline{U} \) such that \( \text{dist}(y, Z_h) > h/\sqrt{2} \), then \( Z_h \cup \{y\} \) would also be an \( h/\sqrt{2} \)-separated subset of \( D_h \cap \overline{U} \). This would contradict the choice of \( Z_h \).

Since for every \( x \neq y \in Z_h \), it holds that
\[
B \left( x, \frac{h}{2\sqrt{2}} \right) \cap B \left( y, \frac{h}{2\sqrt{2}} \right) = \emptyset,
\]
we have
\[
#Z_h = o \left( \frac{1}{h^d} \right), \quad h \to 0.
\]
In the definition of the density of a covering of \( \mathbb{R}^d \), instead of the cubes \([-R, R]^d = R \cdot [-1, 1]^d\), one can take sets of the form \( R \cdot T, R > 0 \), where \( T \subset \mathbb{R}^d \) is a convex body containing the origin as its interior point. For the covering of \( \mathbb{R}^d \) by unit balls whose centers are in \( U \), the density equals \( \Gamma_d \), and hence,

\[
\Gamma_d = \lim_{h \to 0} \frac{\#(U - a) \cap \mathcal{U}_d \cdot \beta_d}{\mu_d(U)} = \lim_{h \to 0} \frac{\#(U \cap h\mathcal{U}_d + a) \cdot \beta_d h^d}{\mu_d(U)} = \lim_{h \to 0} \frac{\#Y_h \cdot \beta_d h^d}{\mu_d(U)}.
\]

In view of relation (33), we have \( \#Z_h = o(\#Y_h), h \to 0 \). Then \( V_h = Y_h \cup Z_h \) and

\[
n_h = \#V_h[U] = \#Y_h(1 + o(1)) = \frac{\Gamma_d \mu_d(U)}{\beta_d h^d}(1 + o(1)) = \left( \frac{\theta_d}{h} \right)^d \mu_d(U)(1 + o(1)), \quad h \to 0.
\]

Lemma 9 is proved. \( \square \)

It is also not difficult to see that for every bounded convex set \( U \subset \mathbb{R}^d \) with non-empty interior, we have \( r(U \setminus D_h(U), Y_h) \leq h \), and hence

\[
r(U, V_h[U]) \leq h.
\] (37)

To complete the proof of (20), denote by \( \{\alpha_n\}_{n \in \mathbb{N}} \) a sequence of positive numbers such that \( \alpha_n = E_n(G)(1 + o(1)) \) and let \( Z_{\alpha_n} = Z_{\alpha_n}(G) \) be as in Lemma 8 with \( h = \alpha_n \) and \( U = G \). In view of (9), we have \( \alpha_n = \theta_d \mu_d(G)^{1/d} n^{-1/d}(1 + o(1)) \), and hence, \( \#Z_{\alpha_n} = o(n), n \to \infty \). Let \( X_n, n \in \mathbb{N} \), be a sequence of point configurations on \( G \) such that \( \#X_n = n, n \in \mathbb{N} \), and

\[
r(G, X_n) = E_n(G)(1 + o(1)), \quad n \to \infty.
\]

Define \( k_n = \#Z_{\alpha_n} \) and \( P_n := X_{n-k_n} \cup Z_{\alpha_n}, n \in \mathbb{N} \). Then

\[
r(G, P_n) = E_n(G)(1 + o(1)), \quad n \to \infty.
\] (38)

Taking into account inequalities (18) and (34), and Lemma 9, we obtain

\[
R(W_G; P_n) \leq \max \left\{ \frac{1}{4} r^2(G, P_n), \frac{1}{2} r^2(\partial G, P_n) \right\}
\leq \max \left\{ \frac{1}{4} r^2(G, X_{n-k_n}), \frac{1}{2} r^2(\partial G, Z_{\alpha_n}) \right\} \leq \alpha_n^2 \left( \frac{1}{4} (1 + o(1)) \right).
\]
From relation (36), we derive that
\[ R_n(W_G) \leq R(W_G; P_n) \leq \frac{\theta_n^2}{4} (1 + o(1)) \]
\[ = \frac{1}{4} E_n^2(G)(1 + o(1)) = \frac{\theta^2_4 \mu_d(G)2^{2d}}{4n^{2/d}} (1 + o(1)), \quad n \to \infty. \]  
(39)

On the other hand, from (31) and (9), we obtain
\[ R_n(W_G) \geq \frac{1}{4} \inf_{X \in G} r^2(G, X) = \frac{1}{4} E_n^2(G) = \frac{\theta^2_4 \mu_d(G)2^{2d}}{4n^{2/d}} (1 + o(1)), \quad n \to \infty. \]

Combining this estimate with (39), we complete the proof of (20). We also obtain that
\[ R(W_G; P_n) = R_n(W_G)(1 + o(1)), \quad n \to \infty. \]

Now, let \( P_{n_h} = V_h(G) = Y_h \cup Z_h, h > 0. \)

In view of relations (18), (37), (34), Lemma 9, and relation (20), we have
\[ R_{n_h}(W_G) \leq R(W_G; V_h(G)) = R(W_G; P_{n_h}) \leq \max \left\{ \frac{1}{4} r^2(G, P_{n_h}), \frac{1}{2} r^2(\partial G, Z_h) \right\} \leq \frac{1}{4} h^2 \]
\[ = \frac{\theta^2_4 \mu_d(G)2^{2d}}{4(n_h)^{2/d}} \cdot (1 + o(1)) = R_{n_h}(W_G)(1 + o(1)), \]
which yields (21). Theorem 3 is proved.

8. Proof of Theorem 4

For every \( n \in \mathbb{N} \) such that \( \epsilon_n(\mathcal{L}) < J(\mathcal{L})/2 \), we will show the existence of the optimal set \( X^*_n \) defined by (22). Let \( V_m = \{ u^m_1, \ldots, u^m_n \}, m \in \mathbb{N} \), be a sequence of \( n \)-point sets in \( \mathcal{P}(\mathcal{L}) \) such that
\[ \lim_{m \to \infty} r(\mathbb{R}^d, V_m + \mathcal{L}) = \epsilon_n(\mathcal{L}). \]

There exists an infinite subset \( \mathcal{N} \subset \mathbb{N} \) and a configuration \( V = \{ u_1, \ldots, u_n \} \subset \overline{\mathcal{P}(\mathcal{L})} \) such that
\[ \lim_{\mathcal{N} \ni m \to \infty} u^m_k = u_k, \quad k = 1, \ldots, n. \]

Let \( p \in \mathbb{R}^d \) be an arbitrary point. For every \( m \in \mathbb{N} \), there exists a point \( \tau_m = u^m_{k_m} + v_m \in V_m + \mathcal{L} \) such that \( |p - \tau_m| \leq r(\mathbb{R}^d, V_m + \mathcal{L}) + \frac{1}{m} \). Since each sequence \( \{ u^m_{k_m} \}_{m \in \mathbb{N}}, k = 1, \ldots, n \), is uniformly bounded, the sequence \( \{ v_m \}_{m \in \mathbb{N}} \) is also uniformly bounded. This implies that sequence \( \{ v_m \}_{m \in \mathbb{N}} \) contains only a finite number of pairwise distinct elements. Since sequence \( \{ k_m \}_{m \in \mathbb{N}} \) also has finitely many distinct elements, there is an infinite subset \( \mathcal{N}_1 \subset \mathcal{N} \) such that \( k_m = k_0 \) and \( v_m = v, m \in \mathcal{N}_1 \), for some index \( 1 \leq k_0 \leq n \) and vector \( v \in \mathcal{L} \). Then
\[ \text{dist}(p, V + \mathcal{L}) \leq |p - u_{k_0} - v| = \lim_{\mathcal{N}_1 \ni m \to \infty} |p - u^m_{k_m} - v_m| \]
\[ \leq \lim_{\mathcal{N}_1 \ni m \to \infty} \left( r(\mathbb{R}^d, V_m + \mathcal{L}) + \frac{1}{m} \right) = \epsilon_n(\mathcal{L}). \]

In view of the arbitrariness of \( p \), we can write
\[ r(\mathbb{R}^d, V + \mathcal{L}) \leq \epsilon_n(\mathcal{L}). \]

Then as \( X^*_n \) one can take the set \( V \subset \overline{\mathcal{P}(\mathcal{L})} \) after shifting any point of it which does not belong to \( \overline{\mathcal{P}(\mathcal{L})} \) by the corresponding vector from the lattice \( \mathcal{L} \) and adding the corresponding number of points if the cardinality of \( X^*_n \) becomes less than \( n \).
The first equality in (23) follows from Lemma 2. Let us show the optimality of $X^*_n$ and the second equality in (23). Let $X \subset \Pi(\mathcal{L})$ be any $n$-point set. Define $a = \epsilon_n(\mathcal{L})$ and let $z_0 \in \mathbb{R}^d$ be such that 

$$\text{dist}(z_0, X + \mathcal{L}) = r(\mathbb{R}^d, X + \mathcal{L}) \geq a.$$ 

Since $a < J(\mathcal{L})/2$, by Lemma 7, function $f_a(x - z_0)$ belongs to the class $\tilde{W}_\mathcal{L}$, and operator $I_k$ vanishes on this function. Since configuration $X^*_n$ satisfies the assumptions of Theorem 2, in view of Lemma 1 and relation (17), we have

$$R(\tilde{W}_{\mathcal{L}}; X) = \sup_{f \in \tilde{W}_{\mathcal{L}}} \|f - I_k(\mathcal{L})\|_{\mathcal{H}(\mathcal{L})} \geq \|f_a(\cdot - z_0)\|_{\mathcal{H}(\mathcal{L})} = \frac{a^2}{4},$$

$$= \frac{e_n^2(\mathcal{L})}{4} = \frac{1}{4} r^2(\mathbb{R}^d, X^*_n + \mathcal{L}) = R(\tilde{W}_{\mathcal{L}}; X^*_n).$$

Hence, 

$$R_n(\tilde{W}_{\mathcal{L}}) = \frac{1}{2} \lambda_n(\tilde{W}_{\mathcal{L}}) = \frac{1}{2} K(\tilde{W}_{\mathcal{L}}; X^*_n) = R(\tilde{W}_{\mathcal{L}}; X^*_n) = \frac{e_n^2(\mathcal{L})}{4},$$

which completes the proof of optimality of $X^*_n$ in the sense of Problems 2 and 3, and the second equality in (23). Algorithm $S^*$ is optimal in view of Lemma 1.

To show the last equality in (23) we note that

$$E_n(\Pi(\mathcal{L})) = \inf_{X \subset \mathbb{R}^d} r(\Pi(\mathcal{L}), X) \geq \inf_{X \subset \mathbb{R}^d} r(\Pi(\mathcal{L}), X + \mathcal{L}) \geq \inf_{X \subset \mathbb{R}^d} r(\mathbb{R}^d, X + \mathcal{L}) = \epsilon_n(\mathcal{L}).$$

Let $h > 0$ be an arbitrary number and $D_h = D_h(\Pi(\mathcal{L}))$ be the $h$-neighborhood of the boundary $\partial \Pi(\mathcal{L})$ of the parallelepiped $\Pi(\mathcal{L})$. It is not difficult to see that for $n$ sufficiently large, $r(\mathbb{R}^d, X^*_n + \mathcal{L}) < h$ and

$$\epsilon_n(\mathcal{L}) = r(\mathbb{R}^d, X^*_n + \mathcal{L}) \geq r(\Pi(\mathcal{L}) \setminus D_h, X^*_n) \geq E_n(\Pi(\mathcal{L}) \setminus D_h).$$

In view of relation (9) we have

$$\theta_d \mu_d(\Pi(\mathcal{L}))^{1/d} = \lim_{n \to \infty} E_n(\Pi(\mathcal{L}))^{1/d} \geq \lim_{n \to \infty} \epsilon_n(\mathcal{L})^{1/d} \geq \inf_{n \to \infty} \epsilon_n(\mathcal{L})^{1/d} \geq E_n(\Pi(\mathcal{L}) \setminus D_h)^{1/d} = \theta_d \mu_d(\Pi(\mathcal{L}) \setminus D_h)^{1/d}.$$ 

Letting $h \to 0$ and taking into account the fact that $\mu_d(D_h) \to 0$, $h \to 0$, we obtain

$$\lim_{n \to \infty} \epsilon_n(\mathcal{L})^{1/d} = \theta_d \mu_d(\Pi(\mathcal{L}))^{1/d},$$

which implies the last equality in (23).

To prove (24) note that for every $x \in \mathbb{R}^d$, there is a vector $v_1 \in \mathcal{L}$ such that $x + v_1 \in \Pi(\mathcal{L})$. In view of (37), we obtain

$$\text{dist}(x, T_h + \mathcal{L}) \leq \text{dist}(x, T_h - v_1) = \text{dist}(x + v_1, T_h) \leq r(\Pi(\mathcal{L}), T_h) \leq h.$$ 

Taking into account relation (40), Lemma 9, and the notation $m_h = \#T_h$, we will have

$$\epsilon_m(\mathcal{L}) \leq r(\mathbb{R}^d, T_h + \mathcal{L}) \leq h = \frac{\theta_d \mu_d(\Pi(\mathcal{L}))^{1/d}}{(m_h)^{1/d}} (1 + o(1)) = \epsilon_m(\mathcal{L})(1 + o(1)), \quad h \to 0.$$ 

Since for every $h$ sufficiently small, we have $r(\mathbb{R}^d, T_h + \mathcal{L}) \leq h < J(\mathcal{L})/2$, taking into account Theorem 2 and equality (23), we will obtain

$$R(\tilde{W}_{\mathcal{L}}; T_h) = \frac{1}{4} r^2(\mathbb{R}^d, T_h + \mathcal{L}) = \frac{1}{4} \epsilon_m^2(\mathcal{L})(1 + o(1)) = R_m(\tilde{W}_{\mathcal{L}})(1 + o(1)), \quad h \to 0,$$

which completes the proof of (24). Theorem 4 is proved.
9. Proof of Theorem 5

It is not difficult to see that \( X_N + \mathcal{P} = \frac{1}{N} \mathcal{L}^s \), and for every \( N \geq 2 \), we have
\[
\epsilon_{kn^2}(\mathcal{P}) \leq r(\mathbb{R}^2, X_N + \mathcal{P}) = \frac{1}{N} r(\mathbb{R}^2, \mathcal{L}^s) = \frac{1}{\sqrt{3N}} \leq \frac{1}{2} = \frac{1}{2} J(\mathcal{L}^s) \leq \frac{1}{2} J(\mathcal{P}). \tag{41}
\]

Let \( X \subset \Pi(\mathcal{P}) \) be any set of \( kn^2 \) points, and let \( v = r(\mathbb{R}^2, X + \mathcal{P}) \). Denote by \( \mathcal{M}_X \) the collection of all closed circles of radius \( v \) centered at points of \( X + \mathcal{P} \). Clearly, \( \mathcal{M}_X \) covers \( \mathbb{R}^2 \). Taking into account relation (8), we will have
\[
\Gamma(\mathcal{M}_X) = \frac{kn^2 \cdot \pi v^2}{\mu_2(\Pi(\mathcal{P}))} = \frac{\pi N^2 v^2}{\mu_2(\Pi(\mathcal{L}^s))} \geq \Gamma(\mathcal{M}_X) = \frac{2\pi}{\sqrt{27}} = \frac{\pi r^2(\mathbb{R}^2, \mathcal{L}^s)}{\mu_2(\Pi(\mathcal{L}^s))}.
\]

Then
\[
r(\mathbb{R}^2, X + \mathcal{P}) = v \geq \frac{1}{N} r(\mathbb{R}^2, \mathcal{L}^s) = r(\mathbb{R}^2, X_N + \mathcal{P}).
\]

Hence,
\[
\epsilon_{kn^2}(\mathcal{P}) = r(\mathbb{R}^2, X_N + \mathcal{P}) = \frac{1}{\sqrt{3N}}.
\]

In view of (41), we can apply Theorem 4 and obtain optimality of \( X_N \) in the sense of Problem 3 and optimality of algorithm \( S^* \) in the sense of Problem 2. In view of relation (23), for \( n = kn^2 \), we obtain
\[
R_n(\widetilde{W}_{\mathcal{P}}) = \frac{1}{2} \lambda_n(\widetilde{W}_{\mathcal{P}}) = \frac{1}{4} (\epsilon_n(\mathcal{P}))^2 = \frac{1}{12N^2} = \frac{k}{12n}.
\]

Theorem 5 is proved.

References


