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Qualitative representation of positional information

Eliseo Clementini [a,*], Paolino Di Felice [a,1], Daniel Hernández [b,2]

[a] *Dip. di Ing. Elettrica, Università di L'Aquila, 67040 Poggio di Roio, Italy*
[b] *Fakultät für Informatik, Technische Universität München, 80290 Munich, Germany*

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Abstract

A framework for the qualitative representation of positional information in a two-dimensional space is presented. Qualitative representations use discrete quantity spaces, where a particular distinction is introduced only if it is relevant to the context being modeled. This allows us to build a flexible framework that accommodates various levels of granularity and scales of reasoning. Knowledge about position in large-scale space is commonly represented by a combination of orientation and distance relations, which we express in a particular frame of reference between a primary object and a reference object. While the representation of orientation comes out to be more straightforward, the model for distances requires that qualitative distance symbols be mapped to geometric intervals in order to be compared; this is done by defining structure relations that are able to handle, among others, order of magnitude relations; the frame of reference with its three components (distance system, scale, and type) captures the inherent context dependency of qualitative distances. The principal aim of the qualitative representation is to perform spatial reasoning: as a basic inference technique, algorithms for the composition of positional relations are developed with respect to same and different frames of reference. The model presented in this paper has potential applications in areas as diverse as Geographical Information Systems (GIS), Computer Aided Design (CAD), and Document Recognition. © 1997 Elsevier Science B.V.

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1. Introduction

Qualitative information is often mistaken to be vague or inexact, but it is not. On the contrary, it can be more efficient and provide more meaning than pure quantitative

* Corresponding author. Email: eliseo@ing.univaq.it.
[1] Email: pdifelice@aquila.infn.it.
[2] Email: danher@informatik.tu-muenchen.de.

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information. For example, saying that Alaska is 1.518.800 km^2 is sufficiently exact quantitative information about size and distances in Alaska, but very likely it is not meaningful in relation to the spatial knowledge of the average listener. On the other hand, saying that Alaska alone is bigger than all the states of the East coast from Maine to Florida is cognitively more immediate. Comparative information in a familiar frame of reference is the key in this example. Therefore, qualitative answers are often cognitively more eloquent than quantitative ones and in most cases it is not obvious how to infer qualitative information from quantitative data, since the quality of things is context-dependent.

A qualitative answer is on purpose limited to the kind of distinctions that are of interest in a certain context and eliminates unnecessary details. Therefore, a qualitative representation uses a discrete quantity space which has normally a natural ordering associated with it and for which qualitative arithmetic algebras have been devised [66, 67]. Qualitative reasoning has been applied mainly to scalar quantities and only recently to space, which is multidimensional in nature, giving rise to the subfield of qualitative spatial reasoning [10, 11, 31]. The delay in the development of qualitative models and reasoning techniques for spatial domains is partly due to the convincement that "it seems unlikely that such inference schemes will be useful for tasks that require full higher-dimensional manipulations" [20, p. 427].

Recent work in qualitative spatial representations has made more evident that the application of quantity spaces in more than one dimension can lead to promising results. In fact, several aspects of spatial information are currently being investigated [11]. In general, the description of a scene of objects in space involves spatial aspects that have an expression both in terms of inherent characteristics of each object and in the context of other objects. The inherent characteristics of an object are its topology (holes and separations) and its extension (size and shape), while, with respect to other objects, topological, orientation, and distance relations have to be considered.

The aim of this study is the qualitative representation of *positional information*, which is one of the basic cognitive spatial concepts and thus important in all application domains of spatial knowledge. For objects that can be modeled as points (because their extension can be disregarded with respect to the distances that are involved), positional information is determined by the orientation and distance relations [15]. This has an obvious correspondence in quantitative terms, where positional information can be expressed using polar coordinates: the distance from the origin and the angle between the radius and a reference axis.

Knowledge about locations in large-scale space are learned not only through senso-motoric experience of the domain, but also from symbolic representations such as maps, and from facts inferred from experience in other spatial domains [6]. Thus, positions in space are likely to be represented in the mind in a mixture of imaginal and propositional formats, which can be suitably modeled in terms of qualitative concepts. It has been suggested that spatial knowledge is hierarchically organized in the human mind [8, 30]. People have too little short-term memory to support a whole map "in the head". Therefore, spatial knowledge is inferred from the representation of global features and relations. There are mainly two kinds of spatial knowledge: route and survey knowledge. Route knowledge is more elementary and it is made up of order information between known landmarks. Survey knowledge is more elaborate and has the characteristics of a "view from above" on a spatial situation. That is, it is independent of a particular order of visited landmarks and, thus, keeps people from getting lost when they leave a known route. Another common categorization of spatial knowledge distinguishes between egocentric (centered on observer), allocentric (relative to distinguished reference structures), and geocentric (relative to coordinated system of reference frames) views [30]. Positional information, as we intend it in this paper, is survey knowledge made up of orientation and distance relations, which depending on scale can be egocentric, allocentric or geocentric. Both orientation and distance might depend on many other factors, like the objects' sizes, point of view, etc., so that it is not enough to express them in terms of isolated relations. We shall rather need to introduce frames of reference to take into account those internal and external factors. Furthermore, cognitive studies show the existence of multiple frames of reference [46]. For example, when someone emerges from a subway system or a driver gets off a highway to enter a local street network, there is a sudden change in the frame of reference. Distance is also crucially dependent on *scale*. For example, the meaning of *close* in a statement "A is *close* to B" depends not only on the actual relative position of both objects but also their relative sizes and other scale-dependent factors.

In previous work, the qualitative description of space has been mostly restricted to topological and orientation relations. Topological relations are able to describe all aspects of the scene which are invariant with respect to common linear transformations (translation, rotation, rubber sheeting) and therefore provide a description of important characteristics of the objects involved in the scene [9,18,57]. However, topological relations alone, being independent of the position and extension of objects, are not sufficient to provide a full description of a scene. Orientation relations describe where objects are placed relative to one another, and can be defined in terms of three basic concepts: the primary object, the reference object, and the frame of reference. The orientation of the primary object is then expressed with respect to the reference object as it is determined by the frame of reference. The combination of topological and orientation relations provides a restricted form of positional information that is mainly useful in small-scale environments such as "the objects in a room" [31,60]. For large-scale environments such as geographic space, however, we must also consider distance relations for describing positions.

In this paper, we build a unified framework for orientation and distance relations. We represent the position of a primary object by a pair of distance and orientation relations with respect to a reference object. As mentioned above, we introduce frames of reference to determine what we mean by, e.g., *front* in the case of orientation, and the range of what we consider to be *close* or *far*, in the case of distance.[3] The frames of reference are in general different for distance and orientation, since they can be influenced by different factors. The framework features reasoning capabilities, i.e., means to find new information from what is already known. The basic step in reasoning with spatial relations is, given "A, r_1, B" and "B, r_2, C", to find the relation "A, r_3, C".

[3] In other words, the concept of distance varying with the scale of reasoning is embedded in the frame of reference.

This is called *composition* of spatial relations and is usually defined by an exhaustive table of all possible relation pairs [24, 32]. Since relations cannot always be expected to be given with respect to the same frame of reference, spatial reasoning cannot just be plain composition of spatial relations, but must include articulation rules between frames of reference. Composition is needed, among others, for constraint propagation and relaxation in networks of spatial relations [47]. Furthermore, we need to conceptualize the world at various levels of granularity in order to reason in terms of simpler concepts. Here, Hobbs [35] theory of granularity fits in neatly: An indistinguishability relation between elements allows us to build equivalence classes that are elements of a coarser theory. Coarser theories can be hierarchically structured, but in general their structure is a lattice. The framework developed in this paper forms part of a wealth of qualitative spatial reasoning techniques with applications in, for example, Geographical Information Systems (GISs), conceptual design in CAD systems, and document recognition. In general, whenever a human being needs to interact with a system dealing with spatial knowledge, qualitative approaches are important. In a GIS devoted to vehicle navigation, for example, driving instructions need to be qualitative in nature and need to be extracted from the quantitative data available or derived from other qualitative information.

Since the representation of orientation was previously described elsewhere [31], in what follows we will concentrate mainly on distances. We do, however, summarize the previous work on orientation in Section 2. In Section 3, we develop a framework for the qualitative representation of distance, by defining the structure of the distance domain at different levels of granularity and by analyzing the mapping between qualitative distances and geometric intervals; furthermore, we introduce a general notion of frame of reference for distances, which captures contextual information. Qualitative spatial reasoning is exploited in Section 4. Basic inference mechanisms are proposed with particular attention to the composition of two distance relations, that we then integrate with varying relative orientation to yield qualitative positional information. Also, articulation rules to mediate between different frames of reference and levels of granularity are discussed. A comparison with other approaches to qualitative position is presented in Section 5. Conclusions and further work are discussed in Section 6.

2. A qualitative approach to orientation

Orientation relations describe where the objects are placed relative to one another.

From a cognitive point of view, body centered orientations can be seen as the result of body construction and the environment we live in. The vertical axis is determined by earth gravitation, which defines up and down. The front/back distinction arises from the corresponding asymmetry of the human body. The left/right distinction, finally, corresponds to the body symmetry along the sagittal axis.

Orientation distinctions are also clearly reflected in language [37]. A large number of prepositions such as *on top of*, *in front of*, *in back of*, *behind*, etc., make reference to an orientation with respect to an axis.

Orientation relations can be derived formally from the fundamental observation on how three points in the plane relate to each other. Let us call them *point of view*,

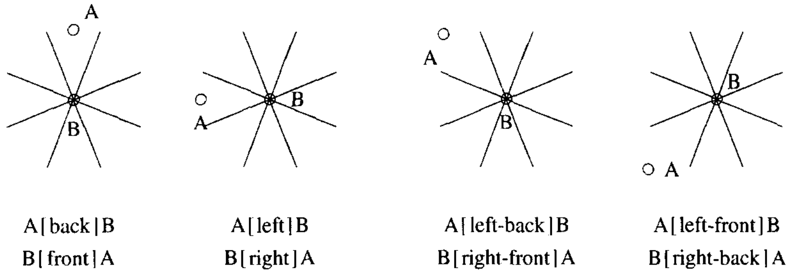

Fig. 1. Orientation relations at a level with eight distinctions.

primary object, and *reference object*, respectively: the point of view and the reference object are connected by a straight line such that the primary object can be to the left, to the right or on that line. This is what we call the *basic* (or first) level of granularity for orientation relations. In particular, if the three objects lie on the same line, the case is called *collinearity*. There are various levels of orientation relations of different granularities, partitioning the plane in several cone-shaped regions. From the basic level, which partitions the plane in two half-planes, we can consider the level with four partitions (second level), eight partitions (third level), and so on.[4] Depending on the context, different sets of relation names are used. In the context of local, small-scale environments (but also in procedural descriptions in large-scale environments) relations such as the following (at a level with eight distinctions) are used: *front* (f), *back* (b), *left* (l), *right* (r), *left-back* (lb), *right-back* (rb), *left-front* (lf), and *right-front* (rf)—see Fig. 1.[5] In the context of geographic space, where a fixed reference point such as the North Pole exists, the usual geographic labels are used to name the relations: *north* (N), *south* (S), *east* (E), *west* (W), *north-east* (NE), *north-west* (NW), *south-east* (SE), *south-west* (SW). It is, of course, possible to have finer orientation distinctions, but neither the formal properties of the model nor its usability depend on the number of distinctions being made.

The *frame of reference* for orientation determines the "front" side of the reference object, and thus the labels given to the orientations based on it. One can distinguish three basic types of reference frames: *intrinsic* (when the orientation is given by some inherent property of the reference object), *extrinsic* (when external factors, e.g., motion, impose a particular orientation on the reference object), and *deictic* (when the orientation is given by the point of view from which the reference object is seen) [58].

Orientations have a uniform circular neighboring structure on each level, e.g., at the level with eight distinctions l and lb are neighbors but l and b are not. In general, the

[4] An alternative approach to orientation relations proposed by Freksa and Zimmermann [27] distinguishes 15 positions and orientations in a grid determined by a vector and orthogonal lines at its endpoints (see also Section 5).

[5] Thus, in this case the orientation labels results from the transfer of distinguished body reference axes from an observer to the reference object.

set $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ denotes the $n+1$ orientation relations at a level of granularity k, with $n = 2^k - 1$. Given a reference object A and a primary object B, the orientation relation of B with respect to A is given by the function $\theta_{AB} = \theta(A, B)$, which can assume any of the values α_i. We define a function *successor* such that $\mathrm{succ}(\alpha_0) = \alpha_1$, $\mathrm{succ}(\alpha_1) = \alpha_2$, ..., $\mathrm{succ}(\alpha_n) = \alpha_0$. Analogously, we define a function *predecessor* as $\mathrm{pred}(\alpha_0) = \alpha_n$, $\mathrm{pred}(\alpha_1) = \alpha_0$, ..., $\mathrm{pred}(\alpha_n) = \alpha_{n-1}$. Each orientation α_i has an *opposite* orientation, which is obtained applying $(n+1)/2$ times the function succ to α_i, that is: $\mathrm{opp}(\alpha_i) = \mathrm{succ}^{(n+1)/2}(\alpha_i)$. For example, opp(N) = S.

The *range* between two orientations is defined as the number of steps necessary to go from one orientation to the other along the circular neighboring structure following the shortest possible path. This means that the range between two orientations α_i and α_j is the number of times we need to apply either the function succ or the function pred (depending on which requires less steps) to α_i in order to obtain α_j. Interestingly, this range corresponds to the number of possible results of composing the two orientations involved. For example, for θ_{AB} = lb and θ_{BC} = r, the resulting composition θ_{AC} belongs to the set {lb, b, rb, r}. The range assumes the maximum value for $\alpha_i = \mathrm{opp}(\alpha_j)$ and zero for $\alpha_i = \alpha_j$. On the level shown, for example, the range between an orientation and its opposite orientation is 4. Two orientations (except for the basic level) are *orthogonal* if the range between them is $(n+1)/4$. Each orientation α_i has two orthogonal orientations: $\mathrm{orth}(\alpha_i) = \{\mathrm{succ}^{(n+1)/4}(\alpha_i), \mathrm{pred}^{(n+1)/4}(\alpha_i)\}$. For example, orth(NE) = {SE, NW}.

Reasoning with qualitative orientations involves being able to relate orientations at different levels of granularity, to transform between different frames of reference, to compute the composition of two orientation relations, and to propagate and maintain the resulting constraints in a network. All of these can take advantage of the circular neighboring structure of the orientation domain. The relation between orientations at different levels of granularity is not a straightforward one. For example, Hoegg and Schwarzer [36] have proposed a solution that introduces an internal subdivision in 16 sectors, which is fine enough to express any of the relations of the coarser levels. The transformation between frames of reference corresponds to a rotation of labels, and—as it was mentioned above—the composition of two orientation relations corresponds to their range. Finally, an efficient form of constraint relaxation can be implemented by coarsening contradictory relations according to the neighboring structure instead of retracting them altogether.

When dealing with extended objects, the area in which a given orientation is accepted as correctly describing the relative position of two objects (called "acceptance area"), becomes dependent on the size and shape of the objects. Various mechanisms have been proposed to determine overlapping and non-overlapping acceptance areas that also take the relative distance between the objects into account [31].

3. A qualitative approach to distance

In this section, we establish a qualitative framework for distances by looking at their main properties and investigating the structure of the distance domain. Analogously to orientation, three elements are needed to establish a distance relation: the *primary object* (PO), the *reference object* (RO), and the *frame of reference* (FofR). We start by reviewing common distance concepts in mathematics and cognition (Section 3.1). In Section 3.2, we define sets of distance relations that are organized along various levels of granularity as suggested by cognitive considerations. Distance systems are introduced in Section 3.3 with the purpose of comparing distance relations besides naming them: an acceptance function maps distance relations to geometric intervals and an algebraic structure allows us to add up and compare those intervals. Distance systems are then specialized to particular distance domains that follow a recurrent pattern (Section 3.4). Section 3.5 deals with frames of reference, made up by three components (distance system, scale, and type) which together express contextual information. In Section 3.6, the metric definition of distance is interpreted within our qualitative framework: the concept of zero needs to be substituted by the concept of the smallest qualitative distinction, while symmetry and triangle inequality hold in the case of same frame of reference only.

3.1. Common distance concepts

In a metric space, three axioms define the concept of distance between points:

$$dist(P_1, P_1) = 0 \quad (reflexivity) \tag{1}$$

$$dist(P_1, P_2) = dist(P_2, P_1) \quad (symmetry) \tag{2}$$

$$dist(P_1, P_2) + dist(P_2, P_3) \geqslant dist(P_1, P_3) \quad (triangle\ inequality) \tag{3}$$

The distance between points $P_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ of an n-dimensional vector space can be expressed in terms of the Minkowsky L_p-metric [53]:

$$d_p(P_1, P_2) = \left(\sum_{j=1}^{n} |x_{1j} - x_{2j}|^p\right)^{1/p} \tag{4}$$

Conventional Euclidean distance, for example, is defined by the L_2-metric. Similarly, the city block (or Manhattan) distance is defined by the L_1-metric.

Our intuitive concept of distance, however, does not rely on coordinates and in fact, in a qualitative framework, we do not have any way of establishing them. People's concepts of space (and, therefore, of distance) are rather dependent upon many cultural and experiential factors [45]. What it means for A to be near B depends not only on their absolute positions (and the metric distance between them), but also on their relative sizes and shapes, the position of other objects, the frame of reference, and "what it takes to go from A to B". The distance between two points can be measured in different ways according to different perspectives, each of them appropriate under certain circumstances. The measures most often adopted by humans are: spatial, temporal, economic, and perceptual. They produce a measure of the effort needed to go from one point to another in terms of, respectively, metric distance, travel time, costs to be invested, spatial perception.

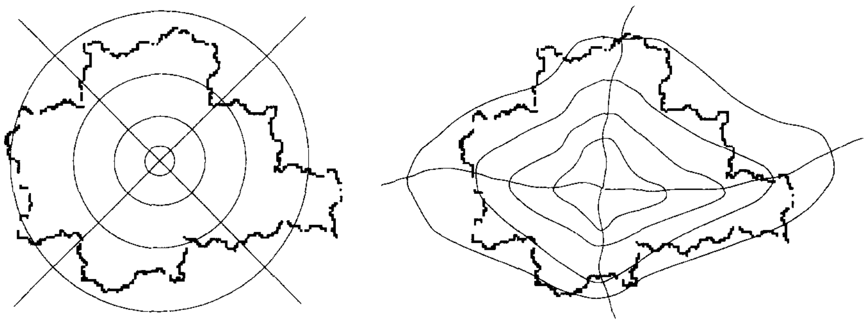

Fig. 2. Isotropic and anisotropic surfaces.

In the presence of obstacles or in structured environments (city blocks), the shortest path between two points might not be a straight line and thus both our metric and cognitive distance concepts be different from the Euclidean distance. Furthermore, we distinguish between distance concepts that can be depicted by isotropic surfaces and those that result in anisotropic surfaces (Fig. 2). On the former the movement effort is the same in all directions from every point, while on the latter this is not true. Isolines are curves connecting all the points at the same distance. In an isotropic space isolines are concentric circles while, in the other case, isolines are variously shaped closed curves. In the remainder of the paper, we consider isotropic surfaces only, since we assume that it is always possible to apply a transformation from the physical domain to other domains (e.g., the costs involved) in order to obtain an isotropic surface.

3.2. Naming distances

The types of objects involved and the context in which they are embedded are decisive factors for establishing the set of relations to be used for naming distances. The first level of granularity that comes to mind distinguishes between *close* and *far*. Those two relations subdivide the plane in two regions centered around the reference object, where the outer region goes to infinity. For isotropic space, qualitative distance relations partition the physical space in circular regions of different sizes (where the difference can be even in the order of magnitude).

In a similar way, we can introduce further levels of granularity. For example: a level with three distinctions *close*, *medium*, and *far*, a level with four distinctions *very close*, *close*, *far*, and *very far*, a level with five distinctions *very close*, *close*, *commensurate*, *far*, and *very far*, and so on (see Fig. 3). Notice that the names given to relations are arbitrary, since we do not discuss linguistic reasons to associate a meaning to a given term (see Section 5 for some pertinent references). Obviously, more than five distinctions could be introduced if necessary, since the formal properties of the distance domain and the applicability of the approach do not depend on the number of distance relations.

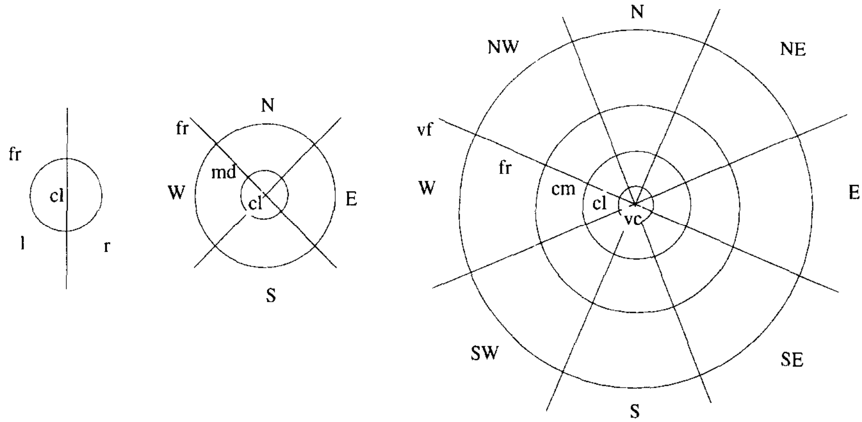

Fig. 3. Various levels of distance and orientation distinctions. These figures show typical distance/orientation granularity configurations, but, of course, both granularities are essentially independent from each other.

These concepts can be formalized as follows. Let us consider a level of granularity with $n+1$ distance distinctions that partition the space surrounding a reference object RO, and let us name them with a finite set of distance symbols $Q = \{q_0, q_1, q_2, \ldots, q_n\}$, where q_0 is the distance closest to RO and q_n is the one farthest away. Given a set of objects O, the qualitative distance between a PO and a RO, both belonging to O, is a function $d : O \times O \rightarrow Q$, which associates to the PO the distance symbol identifying the qualitative distance from the RO. If an object A acts as the RO and an object B acts as the PO, the distance between A and B is expressed by $d_{AB} = d(A, B)$. Since there is a total order on the distance symbols $(q_0 < q_1 < q_2 < \cdots < q_n)$, we can define a function *successor* that gives the next symbol in the list, that is: $\mathrm{succ}(q_i) = q_{i+1}$ for each $i < n$ and $\mathrm{succ}(q_n) = q_n$. Analogously, the function *predecessor* gives the previous symbol in the list, that is: $\mathrm{pred}(q_i) = q_{i-1}$ for each $i > 0$ and $\mathrm{pred}(q_0) = q_0$. Also, a function *ordinal* can be defined as $\mathrm{ord} : Q \rightarrow \{1, \ldots, n+1\}$, such that $\mathrm{ord}(q_i) = i+1$, and $\mathrm{ord}^{-1}(i) = q_{i-1}$, except $\mathrm{ord}^{-1}(i) = q_n$ for $i > n$, $\mathrm{ord}^{-1}(i) = q_0$ for $i \leqslant 1$. Note that whereas a subscripted d_{AB} denotes a distance *variable* (i.e., the distance of primary object B from a given reference object A), q_i denotes a qualitative distance *value*.

3.3. Distance systems

Besides naming distances, we also need to specify how they relate to each other, i.e., compare their magnitudes. To this end, we consider a mapping from the distance symbols to 1-dimensional geometric intervals representing distance ranges. Then, an algebraic structure over intervals with order relations is introduced with the purpose of comparing distance ranges. These comparative relations, which should express among others also order-of-magnitude relations, will be called structure relations in the following.

The three notions mentioned above are organized in *distance systems*. Formally, a distance system D is defined as

$$D = (\mathcal{Q}, \mathcal{A}, \mathcal{I}),$$

where

- $\mathcal{Q}$ is the totally ordered set of distance relations,
- $\mathcal{A}$ is an *acceptance function* defined as $\mathcal{A} : \mathcal{Q} \times O \to I$, such that, given a reference object RO, $\mathcal{A}(q_i, RO)$ returns the geometric interval $\delta_i \in I$ corresponding to the distance relation q_i,
- $\mathcal{I}$ is an algebraic structure with operations and order relations defined over a set of intervals I. $\mathcal{I}$ defines the *structure relations* between intervals.

Each distance relation can be associated to an *acceptance area* surrounding a reference object. In the case of isotropic space, acceptance areas are circular regions which can be uniquely identified with a series of consecutive intervals $\delta_0, \delta_1, \ldots, \delta_n$ (distance ranges). The acceptance function $\mathcal{A}$ performs such a mapping from the symbolic domain of distance relations to geometric intervals. This mapping is necessary to calculate the composition of distances in the domain of intervals, rather than in the domain of distance relations. Then, the inverse function $\mathcal{A}' : I \times O \to \mathcal{Q}$ is used to find the corresponding result back to the domain of distance relations; overall:

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "\oplus"] \arrow[d, "\mathcal{A}"] & \mathcal{Q} \times \mathcal{Q} \to \mathcal{Q} \\ I \arrow[r, "+"] & I \times I \to I \arrow[u, "\mathcal{A}'"'] \end{tikzcd}$$

The composition of distances in the domain of distance relations is indicated with $\oplus$ and will be discussed thoroughly in Section 4.

A strict interpretation of the intervals $\delta_0, \delta_1, \ldots, \delta_n$ implies that they are separated by sharp boundaries and therefore there are points $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ making up the intervals $[0, a_1], [a_1, a_2], \ldots, [a_n, +\infty]$. This, however, is not always the case in reality, since cognitive considerations suggest that intervals may have indeterminate boundaries. Two opposite interpretations are those of *non-exhaustive* and *overlapping* intervals [48]. Non-exhaustive intervals apply if there exists a void space between the acceptance areas of two consecutive distance relations; in this case the intervals are $[a_0, a_1], [a_2, a_3], \ldots, [a_{2n}, +\infty]$ with $a_0, a_1, \ldots, a_{2n} \in \mathbb{R}^+$ and $a_{2i} > a_{2i-1}$ for $i \geqslant 1$. Overlapping intervals apply if the acceptance areas of two consecutive distance relations share a common region; in this case the intervals can be represented as $[0, a_1], [a_2, a_3], \ldots, [a_{2n}, +\infty]$ with $a_{2i} \leqslant a_{2i-1}$ and $a_{2i} > a_{2i-2}$. Our approach is independent of the interpretations above, which all could be used. In fact, our model uses only the notion of consecutive intervals, which is inherited from the total order of $\mathcal{Q}$.

In the following, we discuss the algebraic structure $\mathcal{I} = (I, +, \leqslant, \ll)$, which defines a sum operation and two order relations. Let us first consider $(I, +)$, where I is the set of closed intervals over $\mathbb{R}^+$ and $+$ is a binary operation that given two intervals returns the minimal interval that contains both of them. Formally, the operation $+ : I \times I \to I$ is defined such that, given two intervals $[a,b]$ and $[c,d]$, we have

$$[a,b] + [c,d] = [\min(a,c), \max(b,d)]. \tag{5}$$

Such a sum is a totally-defined internal operation satisfying the associative and commutative properties and, therefore, $(I,+)$ is a commutative semigroup.

Let us now define the order relations for comparing intervals. To this end, we first introduce the length of intervals, which is a function $\| \; \| : I \to \mathbb{R}^+$, such that $\|[a,b]\| = b-a$. The function length has the following properties:

(1) $\|i\| \geqslant 0$, $\forall i \in I$,

(2) $\forall a \in \mathbb{R}^+, i = [a,a] \Leftrightarrow \|i\| = 0$,

(3) $\|i_1\| + \|i_2\| \geqslant \max(\|i_1\|, \|i_2\|)$, $\forall i_1, i_2 \in I$.

The order relation $\leqslant$ between intervals is defined as

$$i_1 \leqslant i_2 \Leftrightarrow \|i_1\| \leqslant \|i_2\|, \quad \forall i_1, i_2 \in I. \tag{6}$$

In the case their length is equal, we say that the two intervals are *congruent* ($\cong$), that is,

$$i_1 \cong i_2 \Leftrightarrow \|i_1\| = \|i_2\|, \quad \forall i_1, i_2 \in I. \tag{7}$$

The following properties hold for the relation $\leqslant$:

$$i \leqslant i, \quad \forall i \in I \quad \text{(reflexive)}, \tag{8}$$

$$i_1 \leqslant i_2 \wedge i_2 \leqslant i_1 \Rightarrow i_1 \cong i_2, \quad \forall i_1, i_2 \in I \quad \text{(antisymmetric)}, \tag{9}$$

$$i_1 \leqslant i_2, i_2 \leqslant i_3 \Rightarrow i_1 \leqslant i_3, \quad \forall i_1, i_2, i_3 \in I \quad \text{(transitive)}. \tag{10}$$

Therefore, the relation $\leqslant$ between intervals is a quasi-order.[6] It is a total order in the sense that, $\forall i_1, i_2 \in I$, we can say whether $i_1 \leqslant i_2$ or $i_2 \leqslant i_1$.

Besides $\leqslant$, from a qualitative point of view it is important to characterize different orders of magnitude. A basic qualitative process is to disregard the effects of a i_1 with respect to a much bigger i_2. Let us consider an "indistinguishability" relation between two quantities such that no relevant predicate distinguishes between them [35]. Such relation can be defined between lengths of intervals and we indicate it with $\|i_1\| \approx \|i_2\|$.[7] This relation is an equivalence relation and can be used to substitute values in expressions. Also, the indistinguishability relation allows us to define the order relation "much less than" ($\ll$) as follows:

$$\|i_1\| + \|i_2\| \approx \|i_2\| \Leftarrow \|i_1\| \ll \|i_2\|, \tag{11}$$

[6] It is not a partial order, since Eq. (9) does not imply that i_1 is equal to i_2. Also the relation $\leqslant$ is not compatible with the sum operation since, given $i_1 \leqslant i_2$, it does not follow that $i_1 + i_3 \leqslant i_2 + i_3$, $\forall i_1, i_2, i_3 \in I$.

[7] For example, supposing that the only relevant predicate is order of magnitude, if order of magnitude $(\|i_1\|)$ = order of magnitude $(\|i_2\|)$, then $\|i_1\| \approx \|i_2\|$.

that is, the relation $\ll$ between two lengths holds if their sum is indistinguishable from the bigger one.[8] The indistinguishability relation can be defined directly for intervals in this way:

$$i_1 \approx i_2 \Leftrightarrow \|i_1\| \approx \|i_2\|, \quad \forall i_1, i_2 \in I. \tag{12}$$

Analogously, we define the order relation $\ll$ between intervals as

$$i_1 \ll i_2 \Leftrightarrow \|i_1\| \ll \|i_2\|, \quad \forall i_1, i_2 \in I, \tag{13}$$

and we also say that the bigger interval "absorbs" the smaller one (*absorption rule*). In particular, such relation is a strict order relation since the following properties hold:

$$i \not\ll i, \quad \forall i \in I \quad \text{(anti-reflexive)}, \tag{14}$$

$$i_1 \ll i_2, i_2 \ll i_3 \Rightarrow i_1 \ll i_3, \quad \forall i_1, i_2, i_3 \in I \quad \text{(transitive)}. \tag{15}$$

Notice, however, that $\ll$ is not a total order since this relation or its inverse ($\gg$) are not applicable to each pair of elements in I. Mixed properties relate $\ll$ and $\leqslant$ as follows:

$$i_1 \cong i_2 \Rightarrow i_1 \approx i_2, \tag{16}$$

$$i_1 \ll i_2 \Rightarrow i_1 < i_2, \tag{17}$$

$$i_1 \leqslant i_2, i_2 \ll i_3 \Rightarrow i_1 \ll i_3, \tag{18}$$

$$i_1 \ll i_2, i_2 \leqslant i_3 \Rightarrow i_1 \ll i_3. \tag{19}$$

We use the algebraic structure $\mathcal{I} = (I, +, \leqslant, \ll)$ to define structure relations over the domain of intervals. Considering the intervals $\delta_0, \delta_1, \ldots, \delta_n$, originated by the acceptance function $\mathcal{A}$, and applying the sum operation to all possible combinations of them, we obtain a set of intervals Δ, which is a subset of I. The cardinality of Δ is $(n+1)(n+2)/2$. Specifically, we indicate a generic sum of consecutive intervals with $\Delta_{h..i} = \sum_{k=h}^{i} \delta_k$ for some $h, i \in [0..n]$, with $h \leqslant i$. If the sum starts from the origin ($h = 0$), we abbreviate it to $\Delta_i = \sum_{k=0}^{i} \delta_k$. Therefore, a Δ_i is the distance range from the origin up to and including δ_i. The structure relations of the distance system (which we indicate with r_Δ) are the order relations holding between all intervals in Δ. Structure relations are needed in the composition of qualitative distances to evaluate the result of comparisons between pairs of intervals (see algorithms in Section 4.1). In the general case, the structure relations r_Δ are arbitrary, that is, given two intervals of Δ, the relation between them can be any in the set $\{\ll, <, \leqslant, \cong, \approx, >, \geqslant, \gg\}$. In the next subsection, we discuss structure relations with uniform properties.

[8] Other definitions for the relation $\approx$ would imply that the transitivity property does not hold, restricting the kind of substitutions that can be made in expressions to fixed computations for a particular problem [35,48,55]. Fo example, the relation $\ll$ could be used to define "slightly less than" in this way: $\|i_1\| \ll \|i_2\| \Rightarrow \|i_2\| \lesssim \|i_1\| + \|i_2\|$. The relation $\lesssim$ is not an order relation since the transitivity property does not hold in general. Combining $\lesssim$, its inverse $\gtrsim$, and $=$, we can define a relation "roughly equal" ($\simeq$), which is similar to "indistinguishable" ($\approx$), but is not an equivalence relation. By negation of the relations above, we can define an intermediate relation: what is between "slightly less than" and "much less than" can be named "moderately less than", obtaining the same set of primitive relations given in [48].

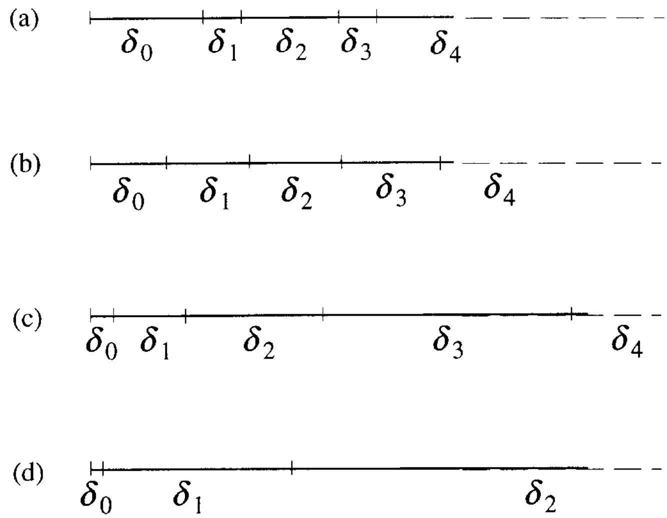

Fig. 4. Illustration of various distance ranges and homogeneous properties.

3.4. Distance systems with homogeneous properties

By *homogeneous* we mean that the structure relations of the distance domain follow a recurrent pattern (for example, each range is bigger than the previous one). The general type of distance system where this is not the case is called accordingly *heterogeneous*. In the following, we identify an increasingly restrictive set of properties that can apply to the structure of intervals. The properties have an underlying cognitive plausibility and help in building composition rules (see Section 4.2). By adding more constraints in the composition of relations, it is possible to restrict the indeterminacy of the result.

A first consideration is that people are more inclined to make finer distance distinctions in the neighborhood of the reference object, while they are less and less motivated to do so as the distances involved get bigger. Therefore, in most contexts it is unlikely that a farther distance relation maps to a smaller distance range (see Fig. 4(a)).

The monotonicity restriction constrains the distance domain to increasingly bigger ranges (see Fig. 4(b)).

Monotonicity. Any given interval is bigger or equal than the previous one:

$$\delta_0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \delta_n. \tag{20}$$

In certain contexts, people may want to exclude equally spaced intervals, and impose that more distant relations correspond to considerably bigger ranges (see Fig. 4(c)). As said above, finer distinctions near to the point of origin and coarser distinctions in the periphery seems more akin to the cognitive concept of distance. This can be conveniently formalized as follows.

Range restriction. Any given interval is bigger than the entire range from the origin to the previous interval:

$$\delta_i \geqslant \Delta_{i-1}, \quad \forall i > 0. \tag{21}$$

The tendency of distance ranges to become bigger and bigger can be further emphasized to obtain differences even in the order of magnitude. These differences allow us to disregard small intervals δ_i with respect to much bigger δ_j, with $i < j$, (see Fig. 4(d)).

Orders of magnitude. For a given difference p between the orders of two distance relations q_i and q_j, with $1 \leqslant p \leqslant (j - i)$, the following holds:

$$(\mathrm{ord}(q_j) - \mathrm{ord}(q_i)) \geqslant p \quad \Rightarrow \quad \delta_j \gg \delta_i, \ \forall i, j \geqslant 0, \ i < j. \tag{22}$$

As the difference p decreases, we have a more restrictive rule. Thus, if $p = 1$, we impose that a given interval absorbs all the smaller intervals including the immediate predecessor.

3.5. Frames of reference

In a quantitative representation of space, the Cartesian reference system (made up of orthogonal graduated axes) acts as the frame of reference. In previous work on qualitative descriptions of orientation (see Section 2) the frame of reference was meant to fix the "front" side of the reference object. For qualitative distances, however, there seems to be no simple equivalent of the concept. Thus, in this paper, we want to introduce a more general "frame of reference" concept, which is able not only to specify the granularity and scale of reasoning but also to capture relevant contextual information. We propose that a distance frame of reference be made up of three components:

$$\mathrm{FofR} = (D, S, T), \tag{23}$$

where D is a distance system, S is a scale, and T is a type (see below).

Distance systems were discussed in detail in Section 3.3. However, it is important to stress the fact that they carry substantial contextual information, since it is the context which determines the relevant distinctions and how they are structured.

As pointed out in the Introduction, distance is crucially dependent on scale. Scale is commonly understood as "the proportion used in determining the relationship of a representation to that which it represents" [2]. In a strict sense this is only true for extensional or analogical representations such as maps. For other types of representations this applies only figuratively. But scale also makes sense without reference to a representation: The proportion between different spatial extensions such as, for example, those defined by the living environment of different species (birds, mammals, insects, microorganisms) [9] is also commonly called scale. It is in this sense that another dictionary definition can be understood: "A progressive classification, as of size, amount,

[9] More abstractly these different spatial extensions can be defined by reachability constraints. If an entity, given its size and life time, cannot possibly reach a limiting distance, it is bound to a given scale.

importance, or rank" [2]. In each case, however, either the possibility of going from one scale to another or an observer able to see or reason about two or more different scales at a time are required for the concept to make sense at all!

Scale is usually thought of in a quantitative manner as in a map's legend. This is so, because numbers are "featureless" and thus apt to compare what would otherwise be incomparable. In other words, "we turn to using quantities when we can't compare the qualities of things" [50]. However, scale can also be expressed qualitatively when we do have qualitative means of comparison. This can be the case both for scale as proportion between represented and representing worlds (e.g., when using a city map, people are seldom aware of a numeric scale and rather get a feeling of what the corresponding distance on the map means by walking around) and for scale as proportion between different spatial extensions (e.g., getting around the central area of a small town requires a different effort than doing so in a large city). Within our framework, scale plays a flexible role: It can put a "unit" to the length of the intervals obtained by the acceptance function $\mathcal{A}$, if that kind of information is available. But it can also just be a member of a "progressive classification" (total order) that helps us to relate different frames of reference.

The third and last component of a FofR is the type T, which can be either intrinsic, extrinsic or deictic in analogy to the treatment of orientation:

- *Intrinsic.* The distance is determined by some inherent characteristics of the reference object, like its topology, size or shape. For example, a house taken as a reference object can implicitly influence the distance relations with respect to itself, without the need of any external factors. Hence, the size of the house determines the acceptance area for *close* in a statement like "the bicycle is parked close to the house".
- *Extrinsic.* The distance is determined by some external factor, like the arrangement of objects, the traveling time, or the costs involved. For example, in a subway system, it might be appropriate to say that two stops are *close* if the time required to go from one to the other is short; or we may consider the two stops *close* if they lie in the same cost zone. The "canonical" frames of reference usually superimposed on geographic space, like the scale of a cartographic projection, are also part of this category.
- *Deictic.* The distance is determined here by an external point of view. The most immediate case is the one of objects that are visually perceived by an observer. Of course, the observer's position influences the distance he or she perceives. Deictic frames of reference include also cases in which the point of view is used figuratively, i.e., not in the sense of sight. Often the point of view is related to how an individual builds a mental map of the space. For example, a traveler may have a personal conviction of a distance being "far away", independently of all other intrinsic and extrinsic factors.

Having described the three components of a distance frame of reference more or less independently of each other, we now turn to the question of how they relate to each other. Although there is a characteristic mutual causality involved that makes some configurations more likely than others, in principle each particular component can be combined with any of the others. For example, a given type can have an expression at different scales and in different distance systems; a given scale might host many different distance systems arising from various type perspectives; or the same structure defined by a given distance system might occur in changing scale/type contexts.

Often the type of FofR influences the scale chosen which in turn modifies the distance system as explained above. In the case of an intrinsic FofR, for example, the scale is likely to be a function of inherent characteristics of the reference object such as its size (e.g., if the RO is a desk, the distance distinctions made are likely to be within the realm of typical office spaces). In the case of an extrinsic FofR, the scale might well be a reference unit such as "1 km" given by external factors. Note that fixing such a unit does not require the distance information to be "exact", but rather establishes the setting in which approximate quantities or qualitative order-of-magnitude relations can be understood. Finally, in the case of a deictic FofR, the scale could be a function of the distance between the point of view and the reference object.

3.6. Qualitative distance axioms

Besides having a clear quantitative meaning, the three distance axioms mentioned in Section 3.1 suggest the following qualitative interpretation within our framework:

The distance of a point to itself is zero (Reflexivity): In a qualitative framework, the concept of zero is not a sharp concept like the quantitative "zero" distance, but it is defined by the kind of distinctions that we want to make. If two objects are both "close" to a reference object and both in "front" of it, they do not necessarily share exactly the same position. Actually, nothing can be said without considering a more detailed level of granularity. A qualitative theory substitutes the equality relation with an indistinguishability relation. We can assess if two objects share the same position up to the smallest qualitative distance distinction we are allowed to make in a certain granularity level. Therefore, the smallest distance relation (like "very close") acts as the zero element. Also, when considering objects with extension, the case in which two objects meet at their boundary may be considered a case of distance "zero" and can be described by means of a topological relation as well.

The distance between two points is symmetric (Symmetry): In Euclidean geometry space is assumed to be symmetric, which means that the distance between points A and B is the same as between B and A. In a qualitative framework, however, asymmetry arises, in particular, when it is not possible to exchange the role of the RO and the PO without changing the qualitative distance relation. If the type of the frame of reference is intrinsic or deictic, then the choice of the RO influences the distance system D and, hence, the distance between the RO and the PO. For example, the distance between two points might be perceived as different, depending on the direction of travel in a deictic frame of reference. In the intrinsic case, objects A and B taken as ROs have in general different properties that may determine different frames of reference. The adoption of a common frame of reference of the extrinsic type, instead, implies that symmetry holds. Summarizing, when the same frame of reference can be maintained for evaluating both d_{AB} and d_{BA}, we can assume $d_{AB} = d_{BA}$, otherwise those two distances are different.

The direct distance between two points is either shorter or equal to the sum of the distances through a third point (Triangle inequality): The triangle inequality applies at the qualitative level as long as the distance symbols are conceptualized as the qualitative equivalents of straight line distances within a common frame of reference (these are the assumptions made in Sections 4.1 and 4.2, which deal with composition and thus verify the triangle inequality). However, when modeling distances in anisotropic spaces or when qualitative distances are given with respect to different frames of reference, the triangle inequality does not necessarily hold. For example, when the direct route from A to C involves arduous hill-climbing (or is traffic jammed), whereas the route through a point B is effortless (or an expressway).

4. Reasoning about positional information

In this section, we concentrate on the composition of distance and orientation relations, as the basic step of qualitative reasoning. Given the position of an object B with respect to an object A in terms of qualitative distance and orientation, and the position of a third object C with respect to B, what we want to infer is the position of C with respect to A. Section 4.1 discusses composition assuming that the frames of reference for orientation and distance are the same in A and B, when they act as reference objects. In general, the result is not a single value but a range of possible resulting distances and orientations within lower and upper bounds. We develop three different algorithms to compute the composition of distances in the three basic cases of same, opposite, and orthogonal relative orientation. The composition of distances in the general case of relative orientation is treated as an interpolation of those three cases. In Section 4.2, we treat the special case of distance systems with homogeneous structure as an application of the algorithms. Finally, in Section 4.3, we remove the restriction of same frame of reference in the composition; this leads to a revised algorithm in the case of opposite orientation, while the remaining two algorithms are still valid.

4.1. Composition of distance and orientation relations

The *position* of an object B with respect to an object A is represented by the pair (d_{AB}, θ_{AB}). Given three objects A, B, and C, if we know the two pairs (d_{AB}, θ_{AB}) and (d_{BC}, θ_{BC}), their *composition* is the pair (d_{AC}, θ_{AC}). Such a composition is the qualitative counterpart of the sum of two vectors.

The following considerations help to clarify the interplay of distances and orientations in the composition process. Given two objects A and B, and moving the orientation of the third object C while keeping fixed its distance from B, let us indicate with C_0, C_1, ..., C_4 the third object at different positions (see Fig. 5). The following set of inequalities holds:

$$d(A, C_4) < d(A, C_3) < d(A, C_2) < d(A, C_1) < d(A, C_0). \tag{24}$$

That is, the resulting distance d_{AC} varies as an inverse function of the range between the orientations θ_{AB} and θ_{BC}.

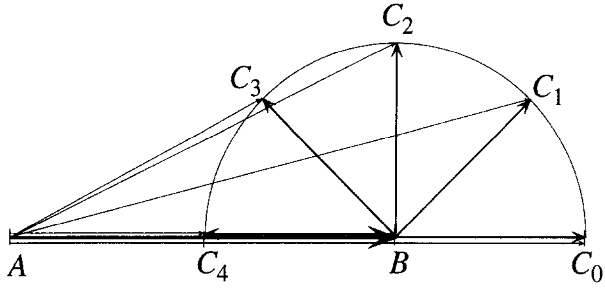

Fig. 5. Interplay of distance and orientation relations for fixed d_{BC}.

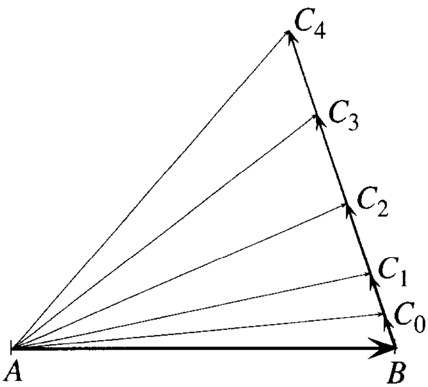

Fig. 6. Interplay of distance and orientation relations for fixed θ_{BC}.

Referring to Fig. 6, we have the third object C at increasing distances from B. Again, let us denote with $C_0, C_1, \ldots, C_4$ the third object at different positions. The following set of inequalities holds:

$$\theta(A,C_0) < \theta(A,C_1) < \theta(A,C_2) < \theta(A,C_3) < \theta(A,C_4). \tag{25}$$

That is, the resulting orientation varies as a direct function of the distance d_{AC}. Notice that if one distance is much shorter than the other, the orientation of the longer distance will prevail; if the distances d_{AB} and d_{BC} are similar, an orientation in the middle of the range between θ_{AB} and θ_{BC} will be the most likely resulting orientation.

From the considerations above, the role of comparison versus naming in composition becomes apparent: given that typical descriptions are a mixture of qualitative names or classes (e.g., "far") and comparisons (e.g., "but closer than") we must maintain and reason internally with both positional relations and comparative information and try to use naming only as a final step in a particular context.

Unlike the quantitative sum of vectors, the composition of positional relations cannot be expressed as a formula to compute the resulting position, since angles are only available as orientations and lengths are only available as distance symbols.

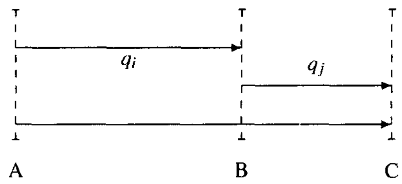

Fig. 7. Adding distances.

In the following, we investigate how to compute the composition (d_{AC}, θ_{AC}). To that end, we refer to the three basic cases of same $(\theta_{BC} = \theta_{AB})$, opposite $(\theta_{BC} = \mathrm{opp}(\theta_{AB}))$, and orthogonal $(\theta_{BC} \in \mathrm{orth}(\theta_{AB}))$ orientation. Same and opposite orientation correspond to the biggest and smallest resulting distance, respectively (see Fig. 5); orthogonal orientation is an intermediate case. Criteria to reduce the indeterminacy of the result, as well as algorithms to compute the composition in the three basic cases mentioned above are the main contributions of this section.

Same orientation

Let us first suppose that the orientation of B with respect to A is the same as the orientation of C with respect to B: $\theta_{BC} = \theta_{AB}$ (see Fig. 7). As said before, we cannot always find a unique result for the composition of qualitative distances, but rather a logical disjunction of possible results. However, since the disjunctive result must be made up of consecutive distances, we proceed by finding a lower and an upper bound for the resulting distances.

In the case of same orientation, since we are adding two "positive quantities," the composition of them cannot be less than the bigger distance. Therefore, for the lower bound we have that:

$$LB(d_{AC}) \geqslant \max(d_{AB}, d_{BC}), \tag{26}$$

while for the upper bound we can say that $UB(d_{AC}) \leqslant q_n$.

In order to find more restrictive upper and lower bounds for the result of composition, we introduce Algorithm 1 (see also Fig. 8), which takes the two distances $d_{AB} = q_i$ and $d_{BC} = q_j$, as well as the structure relations among intervals r_Δ as input.[10] The algorithm first applies the absorption rule to check whether the interval Δ_j can be disregarded with respect to δ_i; if so, $LB = UB = q_i$. Otherwise, to find the upper bound, Δ_j is compared to δ_{i+1} in order to see if Δ_j's outer limit falls within δ_{i+1}. Via recursive calls, the test is repeated for the sum of all successors of δ_i until Δ_j becomes smaller than this sum. The algorithm terminates since, at most, such a sum will eventually include the infinitely big δ_n. The lower bound is computed by comparing Δ_{j-1} with δ_i. If Δ_{j-1} is bigger, then the lower bound must necessarily overcome q_i. The check is repeated recursively with the sum of δ_i and all its successors until Δ_{j-1} becomes smaller.

[10] As mentioned in Section 3.3, the knowledge given by r_Δ supports the evaluation of predicates. In the case of Algorithm 1, these are $\Delta_j \ll \delta_i$, $\Delta_j < \Delta_{\mathrm{inc}}$, and $\Delta_{j-1} < \Delta_{\mathrm{inc}}$.

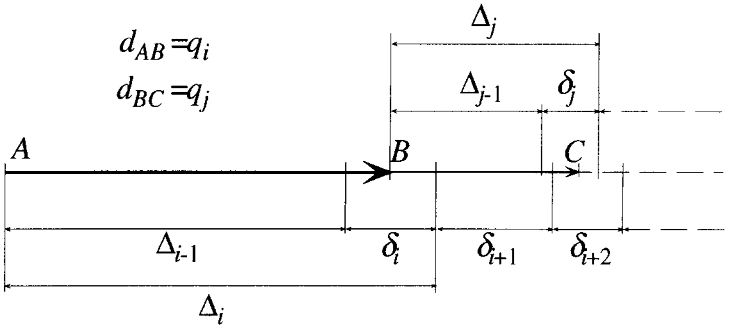

Fig. 8. Composition of distances for same orientation.

Algorithm 1. Algorithm for computing the composition of distance relations (same orientation).

begin
  $Input(q_i, q_j, r_\Delta)$;
  if $\Delta_j \ll \delta_i$ **then** $UB \leftarrow q_i$; $LB \leftarrow q_i$
    else $FindUB(\delta_{i+1}, i+1, UB)$;
      $FindLB(\delta_i, i, LB)$ **fi**;
  $\theta_{AC} \leftarrow \theta_{AB}$;
  $Output(LB, |UB|, \theta_{AC})$

where

proc $FindUB(\Delta_{\mathrm{inc}}, k, \mathbf{var}\, UB) \equiv$
  if $\Delta_j < \Delta_{\mathrm{inc}}$ **then** $|UB| \leftarrow q_k$
    else $FindUB(\Delta_{\mathrm{inc}} + \delta_{k+1}, k+1, UB)$ **fi.**

proc $FindLB(\Delta_{\mathrm{inc}}, k, \mathbf{var}\, LB) \equiv$
  if $\Delta_{j-1} < \Delta_{\mathrm{inc}}$ **then** $LB \leftarrow q_k$
    else $FindLB(\Delta_{\mathrm{inc}} + \delta_{k+1}, k+1, LB)$ **fi.**

end

Opposite orientation

Now, let us consider the composition of distances in the case of opposite orientation, that is, $\theta_{BC} = \mathrm{opp}(\theta_{AB})$ (see Fig. 9). A coarse upper bound is given by the maximum of the two distances since the case of opposite orientation corresponds to the difference between two "positive quantities". Therefore:

$$UB(d_{AC}) \leqslant \max(d_{AB}, d_{BC}). \tag{27}$$

Similarly, for the lower bound we have that $LB(d_{AC}) \geqslant q_0$.

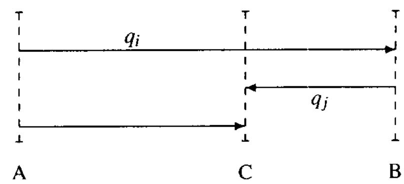

Fig. 9. Subtracting distances.

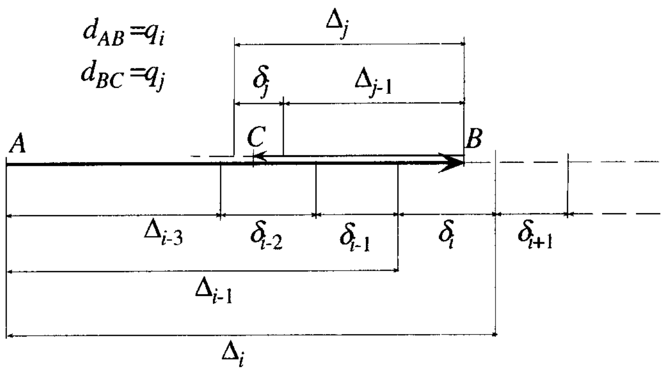

Fig. 10. Composition of distances for opposite orientation ($q_i > q_j$).

Algorithm 2 finds more restrictive upper and lower bounds for the result of composition. It handles separately the three cases $q_i > q_j$, $q_i < q_j$, and $q_i = q_j$, which correspond to three different resulting orientations: equal to θ_{AB}, θ_{BC}, or the logical disjunction of them, respectively. Below, we illustrate how the algorithm manages those three cases.

Case $q_i > q_j$. Algorithm 2 first applies the absorption rule to check whether the interval Δ_j can be disregarded with respect to δ_i; if so, $LB = UB = q_i$. Otherwise, to find the lower bound (procedure *FindLB*), Δ_j is initially compared to δ_{i-1} to see whether $LB = q_{i-1}$; if not, the test is repeated, via recursive calls, for the sum of all predecessors of δ_{i-1} until Δ_j becomes smaller. The procedure *FindUB* finds the upper bound exactly in the same way with the only difference that the initial test compares Δ_{j-1} to δ_i (see Fig. 10).

Case $q_i < q_j$. Algorithm 2 first applies the absorption rule to check whether the interval $\Delta_{i..j}$ can be disregarded with respect to δ_0; if so, $LB = UB = q_0$. Otherwise, to find the lower bound (procedure *FindLBopp*), $\Delta_{i+1..j-1}$ is initially compared to δ_0 to see if $LB = q_0$; if not, the test is repeated, via recursive calls, for the sum of all successors of δ_0 until $\Delta_{i+1..j-1}$ becomes smaller. The strategy implemented by the procedure *FindLBopp* can be informally explained as follows. Since the first distance is smaller than the second one, the first piece of the distance d_{BC} reaches the object A and the remaining part goes beyond A: to calculate how much C overcomes A, the

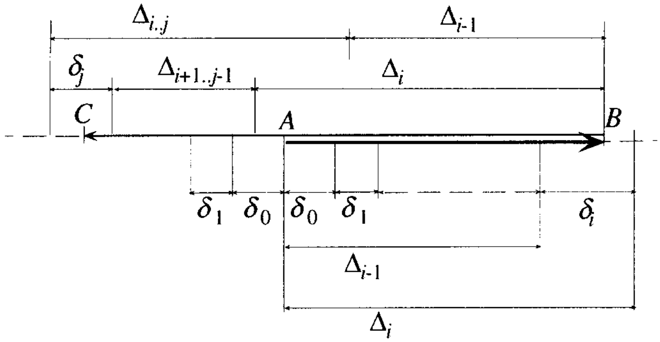

Fig. 11. Composition of distances for opposite orientation ($q_i < q_j$).

sum of δ_0 and its successors is compared to $\Delta_{i+1..j-1}$. The procedure *FindUBopp* works exactly as *FindLBopp* with the only difference that the initial test compares $\Delta_{i..j}$ to δ_1 (see Fig. 11).

Case $q_i = q_j$. Algorithm 2 first sets $LB = q_0$, hence it proceeds to compute the upper bound (procedure *FindUBeq*). δ_j is initially compared to δ_0 to see if $UB = q_0$; if not, the test is repeated, via recursive calls, for the sum of δ_0 to its successors.

Algorithm 2. Algorithm for computing the composition of distance relations (opposite orientation).

begin
  $Input(q_i, q_j, r_\Delta)$;
  case
    $\Delta_i > \Delta_j$: $\theta_{AC} = \theta_{AB}$;
      if $\Delta_j \ll \delta_i$ **then** $LB \leftarrow q_i$; $UB \leftarrow q_i$
        else $FindLB(\delta_{i-1}, i-1, LB)$;
          $FindUB(\delta_i, i, UB)$ **fi**;
    $\Delta_i < \Delta_j$: $\theta_{AC} = \theta_{BC}$;
      if $\Delta_{i..j} \ll \delta_0$ **then** $LB \leftarrow q_0$; $UB \leftarrow q_0$
        else $FindLBopp(\delta_0, 0, LB)$;
          $FindUBopp(\delta_1, 1, UB)$ **fi**;
    $\Delta_i \cong \Delta_j$: $\theta_{AC} = \theta_{AB} \vee \theta_{BC}$;
      $LB \leftarrow q_0$;
      $FindUBeq(\delta_0, 0, UB)$;
  endcase;
  $Output(LB, UB, \theta_{AC})$

where

proc $FindLB(\Delta_{\text{inc}}, k, \textbf{var}\, LB) \equiv$
  if $\Delta_j \leqslant \Delta_{\text{inc}}$ **then** $LB \leftarrow q_k$
    else $FindLB(\Delta_{\text{inc}} + \delta_{k-1}, k-1, LB)$ **fi**.

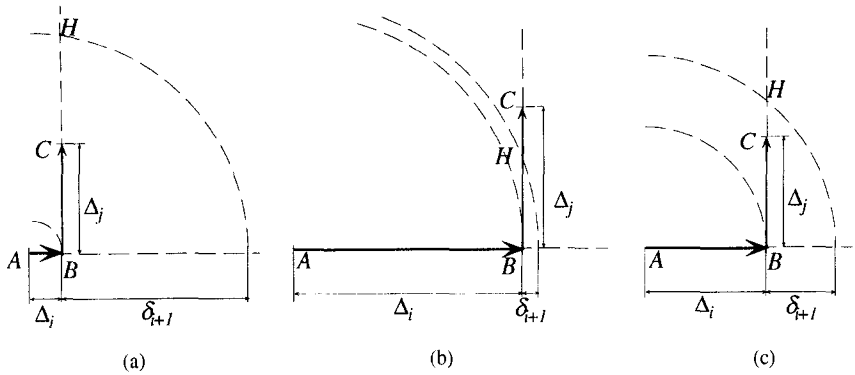

Fig. 12. Composition of distances for orthogonal orientation.

proc *FindUB*$(\Delta_{\text{inc}}, k, \textbf{var}\, UB) \equiv$
 if $\Delta_{j-1} \leqslant \Delta_{\text{inc}}$ **then** $UB \leftarrow q_k$
  else *FindUB*$(\Delta_{\text{inc}} + \delta_{k-1}, k-1, UB)$ **fi.**

proc *FindLBopp*$(\Delta_{\text{inc}}, k, \textbf{var}\, LB) \equiv$
 if $\Delta_{i+1..j-1} < \Delta_{\text{inc}}$ **then** $LB \leftarrow q_k$
  else *FindLBopp*$(\Delta_{\text{inc}} + \delta_{k+1}, k+1, LB)$ **fi.**

proc *FindUBopp*$(\Delta_{\text{inc}}, k, \textbf{var}\, UB) \equiv$
 if $\Delta_{i..j} < \Delta_{\text{inc}}$ **then** $UB \leftarrow q_k$
  else *FindUBopp*$(\Delta_{\text{inc}} + \delta_{k+1}, k+1, UB)$ **fi.**

proc *FindUBeq*$(\Delta_{\text{inc}}, k, \textbf{var}\, UB) \equiv$
 if $\delta_j < \Delta_{\text{inc}}$ **then** $UB \leftarrow q_k$
  else *FindUBeq*$(\Delta_{\text{inc}} + \delta_{k+1}, k+1, UB)$ **fi.**

end

Orthogonal orientation

Let us consider the composition of distances in the case of orthogonal orientation, that is, $\theta_{BC} \in \text{orth}(\theta_{AB})$. In this case, the upper bound tends to be smaller than in the case of same orientation and the lower bound is always the biggest of the distances q_i and q_j. Algorithm 3, which is a modified version of Algorithm 1, computes the upper and lower bounds of distance composition for orthogonal orientation.

To find the upper bound, the procedure *FindUB* takes into account the following three cases: $\Delta_i \ll \Delta_{\text{inc}}$, $\Delta_i \gg \Delta_{\text{inc}}$, and Δ_i "is comparable to" Δ_{inc}. These three cases are depicted in Fig. 12, where we also introduce an auxiliary point H corresponding to the intersection of the segment BC (or its imaginary prolongation) with the radial distance Δ_{i+1} from A. The segment BH can thus be used to decide under which circumstances the upper bound might exceed the current range (in which case a further recursion step will be necessary). To see how this works consider the first step of recursion where $\Delta_{\text{inc}} = \delta_{i+1}$.

If $\Delta_i \ll \delta_{i+1}$ (Fig. 12(a)), then the segment BH is "slightly" bigger than δ_{i+1}, due to elementary geometric considerations. Therefore, only a value for Δ_j greater than δ_{i+1} can (but not necessarily) increment the resulting distance (next recursive call). Otherwise the upper bound is q_{i+1}.

If $\Delta_i \gg \delta_{i+1}$ (Fig. 12(b)), then the segment BH is much bigger than δ_{i+1}; hence, it is unlikely that Δ_j is able to increment the resulting distance unless Δ_j is much greater than δ_{i+1}.

In the intermediate cases, that is, when Δ_i is comparable to δ_{i+1} (Fig. 12(c)), the segment BH is bigger than δ_{i+1}. Hence, Δ_j needs to be considerably bigger than δ_{i+1} in order to overcome the boundary of the next distance range, otherwise the upper bound is q_{i+1}.

Algorithm 3. Algorithm for computing the composition of distance relations (orthogonal orientation).

```
begin
  Input(q_i, q_j, r_Δ);
  if Δ_j ≪ δ_i then UB ← q_i
             else FindUB(δ_{i+1}, i + 1, UB) fi;
  if Δ_i ⩾ Δ_j then LB ← q_i
             else LB ← q_j fi;
  case
    Δ_j ≫ Δ_i: θ_AC ← θ_BC;
    Δ_j ≪ Δ_i: θ_AC ← θ_AB;
    else θ_AB < θ_AC < θ_BC
  endcase;
  Output(LB, UB, θ_AC)

where

proc FindUB(Δ_inc, k, var UB) ≡
  case
    Δ_i ≪ Δ_inc: if Δ_j ⩽ Δ_inc then UB ← q_k
                               else FindUB(Δ_inc + δ_{k+1}, k + 1, UB) fi;
    Δ_i ≫ Δ_inc: if ¬(Δ_j ≫ Δ_inc) then UB ← q_k
                                   else FindUB(Δ_inc + δ_{k+1}, k + 1, UB) fi;
    else if (Δ_j < Δ_inc) ∨ (Δ_j ≈ Δ_inc)
         then UB ← q_k
         else FindUB(Δ_inc + δ_{k+1}, k + 1, UB) fi
  endcase.

end
```

Composition for generic orientation

As long as the granularity level of orientation relations makes only four distinctions, the application of the algorithms just described is straightforward. If more distinctions are

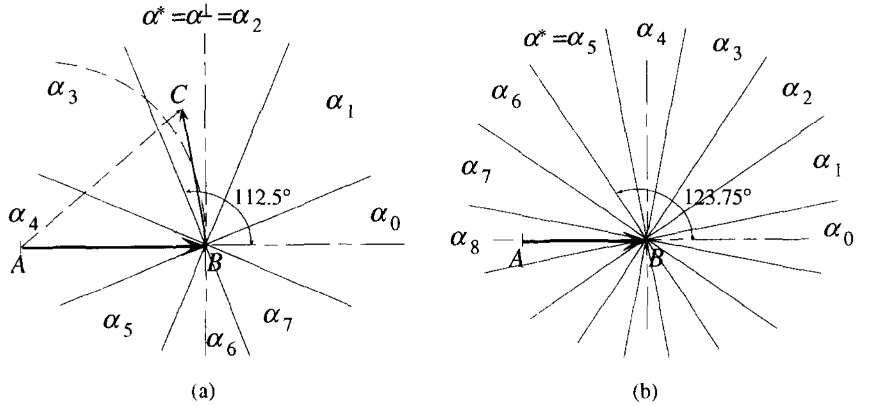

Fig. 13. Composition for intermediate orientation. For eight orientation distinctions (a), a combination of Algorithms 1 and 3 is used for $\theta_{BC} = \alpha_1$ and Algorithm 2 is used for $\theta_{BC} = \alpha_3$. For sixteen orientation distinctions (b), the combination of Algorithms 1 and 3 is used for orientations α_1 through α_3, Algorithm 3 is used for α_5, and Algorithm 2 is used for α_6 and α_7.

made, however, there are intermediate orientation relations between *same*, *orthogonal*, and *opposite* orientation, for which we have to decide which algorithm gives the best approximation.

For that purpose, it is useful to consider three subranges for intermediate values of θ_{BC}, referring only to the upper half-plane, since all considerations are symmetrical on the axis defined by AB:

(i) $\theta_{AB} < \theta_{BC} < \alpha^{\perp}$,

(ii) $\alpha^{\perp} < \theta_{BC} \leqslant \alpha^*$,

(iii) $\alpha^* < \theta_{BC} < \mathrm{opp}(\theta_{AB})$,

where $\alpha^{\perp} \in \mathrm{orth}(\theta_{AB})$ and α^* is a particular orientation relation such that the angle between the two lines AB and BC is approximately 120°.[11] Depending on the number of orientation distinctions, α^* is chosen such that most of its cone lies before 120°. For example, for eight distinctions α^* is α_2, and for sixteen distinctions α^* is α_5.

For the first subrange, both Algorithms 1 and 3 could be used. Lower and upper bounds inferred by Algorithm 1 are greater than those given by Algorithm 3. The most constrained answer to the composition can be obtained by taking *LB* from Algorithm 1 and *UB*from Algorithm 3. For $\theta_{BC} \leqslant \alpha^*$, the resulting distance d_{AC} is always greater than the biggest distance between d_{AB} and d_{BC} (Eq. (26)), while, for $\theta_{BC} > \alpha^*$, the resulting distance d_{AC} is always less than the biggest distance between d_{AB} and d_{BC} (Eq. (27)). Therefore, Algorithm 3 is the most appropriate for the second subrange and Algorithm 2 for the third subrange. The choice of the algorithms for the intermediate orientation relations is illustrated in Fig. 13 for the case of eight and sixteen orientation relations.

[11] The value 120° corresponds to the angle for which the three distances d_{AB}, d_{BC}, and d_{AC} are equal.

Table 1
Resulting distances for the monotonicity restriction (same orientation, five distance symbols)

$\oplus$	q_0	q_1	q_2	q_3	q_4
q_0	q_0, q_1	q_1, q_2	q_2, q_3	q_3, q_4	q_4
q_1	q_1, q_2	q_1, q_2, q_3	q_2, q_3, q_4	q_3, q_4	q_4
q_2	q_2, q_3	q_2, q_3, q_4	q_2, q_3, q_4	q_3, q_4	q_4
q_3	q_3, q_4	q_3, q_4	q_3, q_4	q_3, q_4	q_4
q_4	q_4	q_4	q_4	q_4	q_4

Table 2
Resulting distances for the range restriction (same orientation, five distance symbols)

$\oplus$	q_0	q_1	q_2	q_3	q_4
q_0	q_0, q_1	q_1, q_2	q_2, q_3	q_3, q_4	q_4
q_1	q_1, q_2	q_1, q_2	q_2, q_3	q_3, q_4	q_4
q_2	q_2, q_3	q_2, q_3	q_2, q_3	q_3, q_4	q_4
q_3	q_3, q_4	q_3, q_4	q_3, q_4	q_3, q_4	q_4
q_4	q_4	q_4	q_4	q_4	q_4

Table 3
Resulting distances for the orders of magnitude restriction ($p = 2$, same orientation, five distance symbols)

$\oplus$	q_0	q_1	q_2	q_3	q_4
q_0	q_0, q_1	q_1, q_2	q_2	q_3	q_4
q_1	q_1, q_2	q_1, q_2	q_2, q_3	q_3	q_4
q_2	q_2	q_2, q_3	q_2, q_3	q_3, q_4	q_4
q_3	q_3	q_3	q_3, q_4	q_3, q_4	q_4
q_4	q_4	q_4	q_4	q_4	q_4

4.2. Composition for homogeneous distance systems

This section constitutes an application of the general algorithms developed above to the special case of homogeneous structure relations (see Section 3.4). Due to space limits, only the "same orientation" case is treated. By considering the restrictions given by Eqs. (20), (21), and (22) (monotonicity, range restriction, and orders of magnitude with $p = 2$), Algorithm 1 gives the resulting distances shown in Tables 1, 2, and 3, respectively, for a system with five distance symbols.

Notice that the upper bound becomes progressively more restrictive. By considering the monotonicity restriction (Eq. (20)), since an interval is at least as big as its predecessor, the upper bound may be formulated as

$$UB(d_{AC}) = \mathrm{ord}^{-1}(\mathrm{ord}(d_{AB}) + \mathrm{ord}(d_{BC})). \tag{28}$$

By considering the range restriction (Eq. (21)), the composition of two distances can at most be one step bigger than the maximum of the two distances. The upper bound becomes:

$$UB(d_{AC}) = \text{succ}(\max(d_{AB}, d_{BC})). \tag{29}$$

The orders of magnitude restriction (Eq. (22)) allows us to disregard the effect of the smaller relation. The upper bound can be expressed as

$$|\text{ord}(d_{AB}) - \text{ord}(d_{BC})| \geqslant p \Rightarrow UB(d_{AC}) = \max(d_{AB}, d_{BC}). \tag{30}$$

4.3. Composition between different frames of reference

The qualitative description of distances among a set of objects is reasonably assumed to be given according to different frames of reference. A "basic" type of qualitative reasoning is therefore to relate the distances to each other and be able to infer new information. Ideally, we would like to transform all distance descriptions to the same ("canonical") frame of reference. However, different distance frames of reference refer to different granularities or scales, thus making a transformation into an implicit frame difficult. We rather must restrict ourselves to using articulation rules [35] that state how two particular frames of reference compare. This comparative information, which consists mainly of order information between reference magnitudes, can further constrain the relations maintained in the constraint network, and suggests deferring naming to those cases where it can be done in a disambiguating context. From the definition of frames of reference given in Section 3.5 it follows that articulation rules must relate mainly the distance systems and scales of the frames involved. The frame type does not need to be explicitly related, since it already determines the scale factor and is thus contained in it. The distance systems must be compared as to the sets of relations involved and their structure (e.g., the order-of-magnitude relations between the distances). Note, however, that it might not always be feasible to find all comparative relations between distance ranges in different frames of reference. In general, only similar distance systems might be successfully related to each other, and some might be incomparable to each other.

Two particular cases are of interest: different scales and same distance system, and same scale and different distance systems. As an example of the first special case suppose you get the following answer to the question "How far is it to the Partnachklamm?": "It's about *an hour by car* with a *30 min walk* up hill." Both the distance to be traveled by car and the distance to be covered on foot afterwards are given w.r.t. a set of distinctions that we could call "qualitative time" (the labels being *5 min* for a very short distance, *30 min* for a close distance, *1 h* for a middle far distance, and anything *>1 h* for places far away; notice that these are "qualitative" in the sense that *5 min* could correspond to 2 min 23 sec actual time, and *30 min* could turn out to be 41 min actual time). The scale, however, is a different one in each case, *1 h by car* being something in the order of 100 km, whereas *1 h on foot* amounts to more or less 5 km. In order to reason about distances in this case it is not necessary to give all the structure relations, that is, if we know that scale S' is smaller than scale S'', then each δ_i' is smaller than the corresponding δ_i''. Note that as far as geographic positional information is concerned, the half hour walk can be neglected, and we can conclude that this place is about 100 km away from our present location. As far as the time and physical effort to get there is concerned, of course, the half hour walk is a significant component.

The second special case is illustrated by the statements: "It is only *5 min* to the bakery. All other stores are also *close* by." Here the scale is the same in the two sentences, as implied by an assumed means of transportation (e.g., on foot), but different distance systems are used in each. In the first sentence, the "qualitative time" system described above is used, whereas in the second sentence, a simpler and less specific close/far distinction is made.

Let us consider the composition of two distance relations expressed in two different frames of reference F' and F''. Given three objects A, B, C, with $d_{AB} = q_i'$ in the frame of reference F' and $d_{BC} = q_j''$ in the frame of reference F'', the result of the composition is d_{AC}, expressed in the frame of reference F'. In the most general case, both scales and distance systems are different in F' and F''. In order to perform the composition, we need to know the structure relations between the different distance ranges in F' and F''. If Δ' is the set of intervals of the first distance system and Δ'' is the set of intervals of the second distance system, the structure relations we need to consider are the order relations between each interval in Δ' and each interval in Δ'', indicated with $r_{\Delta',\Delta''}$. Then, Algorithms 1 and 3 can be applied providing $r_{\Delta',\Delta''}$ in place of r_Δ as input, while Algorithm 2 needs an adaptation described below and given in Algorithm 4. Such an algorithm distinguishes three cases: $\Delta_i' \approx \Delta_j''$, $\Delta_i' > \Delta_j''$, and $\Delta_i' < \Delta_j''$. The first case ($\Delta_i' \approx \Delta_j''$) replaces the case $\Delta_i \cong \Delta_j$ of Algorithm 2, because when we have different frames of reference it is unlikely to have the equality of distance ranges. The second case ($\Delta_i' > \Delta_j''$) is the same as in Algorithm 2, while the last case ($\Delta_i' < \Delta_j''$) needs two additional procedures *FindMin* and *FindMax* to estimate the part of distance d_{BC} that is equal to d_{AB} in the frame of reference centered in B. This is not a problem in the case of same frame of reference where the distance of A with respect to B is equal to q_i (i.e., object A falls into the interval comprised between Δ_{i-1} and Δ_i (see Fig. 11)). With different frames of reference, such a distance cannot be determined with uncertainty smaller than δ_i. The procedures *FindMin* and *FindMax* find two indices x and y such that the interval $\Delta''_{x+1..y}$ contains the object A with $\Delta''_{x+1..y} \geqslant \delta_i'$. Hence, the procedures *FindLBopp* and *FindUBopp* take the intervals $\Delta''_{y+1..j-1}$ and $\Delta''_{x+1..j}$, respectively, as the part of distance d_{BC} that goes beyond A.

Algorithm 4. Algorithm for computing the composition with different frames of reference (opposite orientation)

begin
  Input$(q_i', q_j'', r_{\Delta',\Delta''})$;
  case
    $\Delta_i' \approx \Delta_j''$: $\theta_{AC} = \theta_{AB} \vee \theta_{BC}$;
        $LB \leftarrow q_0'$;
        FindUBeq$(\delta_0', 0, UB)$;

$\Delta'_i > \Delta''_j$: $\theta_{AC} = \theta_{AB}$;
 if $\Delta''_j \ll \delta'_i$ **then** $LB \leftarrow q'_i$; $UB \leftarrow q'_i$
 else $FindLB(\delta'_{i-1}, i-1, LB)$;
 $FindUB(\delta'_i, i, UB)$ **fi**;
$\Delta'_i < \Delta''_j$: $\theta_{AC} = \theta_{BC}$;
 $FindMin(\delta''_0, 0, x)$;
 $FindMax(\Delta''_{x+1}, x+1, y)$;
 if $\Delta''_{x+1..j} \ll \delta'_0$ **then** $LB \leftarrow q'_0$; $UB \leftarrow q'_0$
 else $FindLBopp(\delta'_0, 0, LB)$;
 $FindUBopp(\delta'_1, 1, UB)$ **fi**;

endcase;
$Output(LB, UB, \theta_{AC})$

where

proc $FindMin(\Delta''_{\text{inc}}, k, \textbf{var}\, x) \equiv$
 if $\Delta''_{\text{inc}} \approx \Delta'_{i-1}$ **then** $x \leftarrow k$
 else if $\Delta''_{\text{inc}} > \Delta'_{i-1}$
 then $x \leftarrow k-1$
 else $FindMin(\Delta''_{\text{inc}} + \delta''_{k+1}, k+1, x)$ **fi.**

proc $FindMax(\Delta''_{\text{inc}}, k, \textbf{var}\, y) \equiv$
 if $(\Delta''_{\text{inc}} \approx \Delta'_i) \vee (\Delta''_{\text{inc}} > \Delta'_i)$
 then $y \leftarrow k$
 else $FindMax(\Delta''_{\text{inc}} + \delta''_{k+1}, k+1, y)$ **fi.**

proc $FindLB(\Delta'_{\text{inc}}, k, \textbf{var}\, LB) \equiv$
 if $\Delta''_j \leqslant \Delta'_{\text{inc}}$ **then** $LB \leftarrow q'_k$
 else $FindLB(\Delta'_{\text{inc}} + \delta'_{k-1}, k-1, LB)$ **fi.**

proc $FindUB(\Delta'_{\text{inc}}, k, \textbf{var}\, UB) \equiv$
 if $\Delta''_{j-1} \leqslant \Delta'_{\text{inc}}$ **then** $UB \leftarrow q'_k$
 else $FindUB(\Delta'_{\text{inc}} + \delta'_{k-1}, k-1, UB)$ **fi.**

proc $FindLBopp(\Delta'_{\text{inc}}, k, \textbf{var}\, LB) \equiv$
 if $\Delta''_{y+1..j-1} < \Delta'_{\text{inc}}$ **then** $LB \leftarrow q'_k$
 else $FindLBopp(\Delta'_{\text{inc}} + \delta'_{k+1}, k+1, LB)$ **fi.**

proc $FindUBopp(\Delta'_{\text{inc}}, k, \textbf{var}\, UB) \equiv$
 if $\Delta''_{x+1..j} < \Delta'_{\text{inc}}$ **then** $UB \leftarrow q'_k$
 else $FindUBopp(\Delta'_{\text{inc}} + \delta'_{k+1}, k+1, UB)$ **fi.**

proc $FindUBeq(\Delta'_{\text{inc}}, k, \textbf{var}\, UB) \equiv$
 if $\delta''_j < \Delta'_{\text{inc}}$
 then $UB \leftarrow q'_k$
 else $FindUBeq(\Delta'_{\text{inc}} + \delta'_{k+1}, k+1, UB)$ **fi.**

end

5. Related work

This section reviews some of the literature on the subject of the representation of distance and orientation, and in particular on their combination to handle positional information. We look into studies in the fields of cognitive science and linguistics as well as into previous work in AI, in particular in the areas of qualitative reasoning and fuzzy logic.

5.1. Cognitive science

There have been many psychophysical studies of the perception of distance and related cognitive issues. Downs and Stea [15] identify several characteristics of cognitive maps such as their incompleteness, the tendency to schematization and associated distortions and augmentations as well as significant intergroup and individual differences. Thus, the resulting representation resembles rather a collage of bits and pieces of spatial information rather than a map [64].

Cognitive distances in urban environments have been studied by Lee [43], Briggs [6], and Canter and Tagg [7] among others. Briggs [6] identifies four levels of hierarchically organized knowledge: Knowledge of points, knowledge about the relative proximity of pairs of nodes (one-dimensional), knowledge of relative location (two-dimensional), knowledge of sets of nodes and their interlinking paths (region clustering).

Studies of intercity distances [59, 62] have suggested the influence of factors such as the attractiveness of and familiarity of the subjects with origin, goal, and interconnecting paths, as well as the kind and number or barriers along the way and the magnitude of the geographic distance.

In his extensive review of the psychological literature on the perception and cognition of environmental distances (i.e., those that cannot be perceived in entirety from a single point of view and thus require moving around for their apprehension) Montello [51] identifies several processes as well as sources for distance knowledge. In general, knowledge of environmental distance depends on perception or awareness of body movement or change of position (based on vision, vestibular sensation, motor efference or auditory processes). Processes by which distance knowledge is acquired and used include working-memory, non-mediated, hybrid, and simple-retrieval processes. Knowledge about distances might be derived from many sources, including environmental features (turns, landmarks, barriers), travel time and travel effort. Not a single process or source can account for human distance knowledge. Rather, "there are alternative processes that account for distance knowledge in different situations, and multiple, partially redundant knowledge sources that differentially influence it as a function of scale and availability" [51, p. 13].

One of the best studied factors is the influence of the number of environmental features in the perception of distance for which four main hypotheses are put forward: *feature-accumulation hypothesis*, *segmentation hypothesis*, *scaling hypothesis*, *analog-time hypothesis*. Each of these provides a seemingly different explanation for the influence of this factor, but at the same time unveils characteristics of cognitive distance, which we attempt to model in our framework. The feature-accumulation hypothesis, for example, states that the number of features makes a difference because "longer distances contain more features". The qualitative model is compatible with this hypothesis as a larger number of environmental features leads to more distinctions being made. Similarly, the scaling hypothesis aims at explaining why the relative magnitude of estimated distances increases as physical distance decreases by observing that subjective distances near reference points tend to be exaggerated, and that features break the path to be estimated into smaller distances. This is related to the logarithmic distance phenomenon [29], which is modeled in the qualitative framework through appropriate structure relations.

The "asymmetry of cognitive distances" (i.e., the distance between two points being perceived as different depending on travel direction) often mentioned in the literature is not as straightforward as it might seem to be. Everyday experience often presents us with actual distance asymmetries (due to road conditions, rush hour schedules, etc.), which should not be confused with psychological asymmetry illusions due to perceptual, organismic, or affectional factors. Furthermore, common sense spatial knowledge allows us to distinguish between "longer distances" and "when it takes longer in the other direction", while being aware that actual distance is the same in both ways. Our model is perfectly capable of dealing with distance asymmetries by explicitly storing different distances (with respect to different frames of reference) for each way when required.

Travel time is frequently proposed as source of distance knowledge, in particular when travel occurs over larger scales: "Cognized distances increase with, and are directly related to time required to traverse path." [28]. However, no straightforward relation between time and distance can be established: neither their simple equation nor 'distance equal to time times speed' explain the data available from psychological experiments satisfactorily.

5.2. Linguistics

The basic dichotomy of "proximity" and "distality" has been studied from a linguistic point of view by Bierwisch [4], Herweg [34], and Wunderlich and Herweg [68] among many others.

Pribbenow [54] distinguishes five linguistic distance concepts: "inclusion" (acceptance area restricted to projection of reference object), "contact/adjacency" (immediate neighborhood of RO), "proximity" and "geo-distance" (surroundings of RO), and "remoteness" (defined as complement of the proximal region around the RO). A similar system of distinctions in English described by Jackendoff and Landau [37] consists of 3 degrees of distance: interior of reference object (*in*, *inside*), exterior but in contact (*on*, *against*), proximate (*near*), plus the corresponding "negatives" ("farther away than"): interior (*out of*, *outside*), contact (*off of*), proximate (*far from*). Furthermore, prepositions such as *among* and *between* also convey some distance information.

The qualitative approach to the representation of spatial knowledge is heavily inspired by the way spatial information is expressed verbally. In particular, the relational nature of spatial expressions has a direct correspondance in the qualitative relations. Also the semantics of such a relation is not seen in a direct relation between the objects involved, but rather, as in language, the reference object is used to define a region in which the primary object is located. However, the relation between the inventory of spatial constructs of a language and the possible combinations of basic spatial concepts is not a one-to-one mapping [54,63]. For example, some prepositions might correspond to several different spatial relations depending on the context in which they are used [33].

Many other aspects discussed in this paper, such as the asymmetry of distances between primary object and reference object, have been observed in linguistic studies ("The standard expression of spatial location is strikingly asymmetrical." [37, p. 107]). The fact that distance and orientation interact to provide further positional constraints, for example, can also be seen in spatial prepositions like *in back of* and *behind*, which share the same orientation but indicate different distances (*in back of* is restricted to proximal distances, whereas *behind* is unrestricted).

5.3. Early AI-approaches

Previous work in AI on the representation of positional information has used a variety of approaches, some of them combining both qualitative and quantitative information. [12]

The SPAM program [49], for example, combines topological information represented propositionally with metric information represented by "fuzzy maps", where information is stored in terms of "fuzz ranges", e.g., [5.0, 7.0][2.0, 3.0] for an x/y coordinate pair. [13] In this system, frames of reference are associated with each object, establishing the origin of the coordinate system, its scale and orientation. In fact, the distinction between frames of reference and objects is replaced by an hybrid construct called "frob", which can be dealt with uniformly. Positional information is derived from relations between frames of reference, i.e., the origin of one frame of reference in terms of the other, the difference between their orientations, the ratio of their scales. Even though separate frob trees index position, orientation and scale finding minimum and maximum bounds for those quantities is computationally inefficient.

Another system which is primarily concerned with information about the relations between the boundaries of objects but that nevertheless allows the computation of relative position is MERCATOR [12,13]. A grid of vertices connected by arcs of fuzzy length is used to represent the boundaries of objects. Here pairs of distances and directed edges (i.e., orientations) expressed each as fuzz ranges (as in SPAM) are used to relate the shapes of objects to the grid of edges. The system is able to determine the relative position of extended objects but again the interval bounds representation of lengths and orientations is awkward and limited. For example, it is not possible to represent a square of unknown dimensions.

It should be mentioned, that positional information might not be needed at all. In structured environments (i.e., where roads define the possible paths between loca-

[12] For descriptions of the mathematical coordinate-based representation of position and the corresponding transformation mechanism (e.g., for translation, rotation, and scaling) see Appendix 1 in [3] or Section 6.3 in [14].

[13] This way of dealing with uncertainty is different from both the qualitative approach presented in this paper and the fuzzy approaches discussed in the next section. It also leads to more complicated algorithms.

tions) incidence relations between roads and places are all that is needed for navigation. The TOUR system [41] uses topological route maps, consisting of places, paths and regions to assimilate spatial information and reason about the movement of a robot in such an environment. The world state is represented using predicates such as on_path($PP, X_1, X_2, \ldots, X_3$) to indicate that places X_1, X_2 and so on occur on path PP in that order; border(RR, PP, S) to state that path PP is on the border of region RR, where S indicates if the paths's forward direction goes clockwise or counter-clockwise around RR; star($X, \langle PP_1, S_1\rangle, \ldots \langle PP_k, S_k\rangle$) to indicate that paths PP_1 through PP_k meet at place X, where S_i denotes forward or backward direction on a path.[14] Together with actions such as turn and go-to this representation can be used to assimilate a cognitive map while wandering around and to plan a route for moving from one place to another.

In open large-scale spaces a simple type of positional information is sufficient to navigate, provided there are visible landmarks available. The QUALNAV system [44] exploits the fact that, given any two visible landmarks, the current position can be either on the "left of", on the "right of" or "on" the directed virtual connecting line between the landmarks. Thus, seeing the landmarks in a particular *order* already provides some positional information. Furthermore, combining such observations it is possible to determine positional regions and to plan a route from a start to a goal position as a series of "crossings" of virtual connecting lines.

5.4. *Qualitative physics*

The motivation for qualitative reasoning arose predominantly from research on engineering problem solving (see, for example, the two special issues on the topic published in the journal *Artificial Intelligence* in 1984 and in 1991, respectively [5,67]), which sought techniques for automating engineering practice for a variety of important tasks (such as, for example, circuit analysis, qualitative explanation of physical systems, and complex mechanical systems [20]). Most work in the area has concentrated on modeling processes as opposed to the descriptive approach needed to handle space. However, two examples of contributions relevant to our work deal with qualitative vectors: Kim [40] represents angles (and, hence, orientation) in terms of *quadrants* and *inclination*. This approach allows to define a qualitative vector arithmetic. A qualitative description of distances, however, is outside the scope of Kim's paper. Weinberg et al. [65] propose a *qualitative vector algebra* suitable for reasoning with qualitative estimates of the *magnitudes* and the *direction* of vectors in the plane, the latter expressed by the angle w.r.t. the x-axis (*polar notation*). They apply this algebra to a comparative analysis of how angles between objects change as objects move, and describe a problem solver that deals successfully with translational 2D mechanical systems. A drawback of their model is that it does not support different levels of granularity both for magnitudes and directions. More recent approaches in qualitative physics emphasize on representing the spatial extent of objects. For example, Rajagopalan [56] proposes a qualitative descrip-

[14] This simplified description of TOUR follows Davis [14] rather than the original paper, which includes among other things local reference frames.

tion of the spatial extent of an object in terms of its extremal points, in order to simulate the effects of translational and rotational motion of objects in a magnetic field [42]. Within this model, the relative position/orientation of objects is represented through inequality relations between their extremal points, and the discontinuities modeled by piecewise continuous variables and circular quantity spaces.

5.5. Qualitative spatial reasoning

Building on insights from cognitive science, linguistics, early AI approaches to the representation of spatial knowledge, and qualitative physics, the field of qualitative spatial reasoning, in which our own work can be classified, has flourished in recent years. We comment here only on the closest related approaches, and refer for a more general review of the literature to [11,26,31].

Freksa [25] and Freksa and Zimmermann [27] present an approach to qualitative spatial reasoning based on directional orientation information. They distinguish 15 possible positions and orientations of a point based on the left/straight/right distinction w.r.t. a vector $\vec{ab}$ as well as the front/neutral/back distinction w.r.t. the lines orthogonal to $\vec{ab}$ on the end points of a and b. Note that this model reduces to an orthogonal set of orientations when used for cardinal directions, since the "b" or "north" reference point is considered to be either infinitely far away or cyclicly changing to south at the pole. Furthermore, establishing that objects are on the parallel line defined by "b" does not seem feasible from an egocentric perspective. Reasoning is done by applying primitive operators such as identity, inversion, homing, shortcut, and the inverses of the last two. Zimmermann [71] extends this model to combine orientation, position, and distance. While this particular combined model makes fewer distance distinctions than the model presented in this paper, Zimmermann's general Δ-calculus [72] is able to deal with multiples of symbolic quantities and their differences and thus able to model homogeneous distance systems. As a matter of fact, the Δ-calculus could be used as an alternate algebraic structure $\mathcal{I}$ to define a different type of structure relations between intervals (see Section 3.3).

Mukerjee and Joe [52] propose a representation for the relative position of objects at arbitrary angles based on four directions and four quadrants centered on the reference object. The representation of relative direction is grounded on intrinsic fronts, based on the assumption that most objects have a distinguished front, which might or might not be the case depending on the application type. The representation of relative positions is based on projecting the boundaries of the object in the direction of the intrinsic front to obtain its "lines of travel". The lines of travel of two objects form a "collision parallelogram", w.r.t. which the relations *behind*, *after*, *inside* (corresponding to qualitative areas), and *back*, *front* (qualitative points) are defined. The relative position of two objects is specified by the quadrant information (dir(A/B)), and the positional relations pos(A/B) and pos(B/A). As compared to our representation this approach leads to larger composition tables, which—even using various redundancies and symmetries—can only be marginally reduced.

Frank [21] develops an algebra for qualitative spatial reasoning about distances and cardinal directions. This algebra consists of

- a set of distance and direction symbols (such as: *close* and *far*, and *North*, *South*, *East*, and *West*),
- a set of operations (i.e., *inverse* and *compose*); and
- a set of axioms.

The representation of spatial relations is based on the concept of a *directed path* from one endpoint to another. The qualitative distance system consists of a number of qualitative distance values, that describe distances from the nearest to the farthest, and a distance function that maps a path to a qualitative distance value. Note that, as opposed to our approach, only equally spaced intervals are used here. Two cardinal direction systems based on the concepts of cone and half-plane are examined. Both systems are defined similarly to the qualitative distance one. The paper presents alternatives for the combination of distances and directions. The result of the qualitative reasoning can be either *Euclidean exact* (which denotes a homomorphism between the qualitative reasoning result obtained using the axioms of the algebra and the quantitative reasoning result obtained by translating the qualitative values to analytical geometry and applying the equivalent functions to them) or *Euclidean approximate*. To be able to compare qualitative and quantitative results, the axioms are developed to only provide a *single answer*. Frank's experiments showed that the algebra can achieve satisfactory results under some restricted conditions.

Jong [38] introduces a model for qualitative distances and one for qualitative orientations, whose combination defines the *model for locational relations*. Both these models are flexible with respect to the number of distance/direction symbols. As in our model, the number of distinctions about distances (*granularity level for distances*) does not influence the number of distinctions about directions (*granularity level for directions*), and vice versa. At a given level of granularity, the intervals of the Jong's model satisfy the *monotonicity assumption* (Section 3.3). Jong introduces three different models for making qualitative reasoning about distances and directions: the *all-answer* model, the *likely-answer* model, and the *single-answer* model. The qualitative reasoning discussed in Jong's thesis is based on inference rules and it is similar to those qualitative-based models that use composition tables to represent qualitative reasoning results, see for example [21,24,31]. A large part of the thesis is devoted to the definition of a link between the representations of qualitative and quantitative distances/directions.

5.6. *Fuzzy logic*

Many authors, among them Dutta [17], Altman [1], and Jorge and Vaida [39] have suggested the use of linguistic variables [69,70] either to model space directly or extend available qualitative frameworks. Linguistic variables are "variables whose values are not numbers but words or sentences in a natural or artificial language". The motivation being that "linguistic characterizations are, in general, less specific than numerical ones" [69, p. 3]. This motivation is in the same spirit as the "making only as many distinctions as necessary" characterization of qualitative representations advocated in this paper. There are, however, important differences, which we will discuss after a more detailed review of some of the approaches just mentioned.

The work of Dutta [17] aims at representing spatial constraints between a set of objects given imprecise, incomplete and possibly conflicting information about them. To do so, Dutta distinguishes between position and motion spatial constraints. Position constraints in turn can be "propositional" (i.e., qualitative), "metrical" (i.e., quantitative), a quantitatively specified range or "visual" (i.e., only graphically depictable as a fuzzy area). The main idea is then to transform those constraints into an equivalent possibility distribution, in the two-dimensional case into distributions on the X and Y axes. These distributions can be generally approximated by triangular shapes and thus represented by fuzzy numbers of the form (mean value, left spread, right spread). Spatial reasoning is done in this framework by computing joint possibility distributions according to the compositional rule of inference [70].

Altman [1] presents a treatment more specific to GIS using fuzzy regions where the membership function is the concentration of some feature attribute at some point. He argues against converting imprecise data (e.g., *near*) to "hard" data (e.g., *less than 5 km*) at too early a stage in the analysis process, very much in the spirit of qualitative reasoning. Altman defines special distance and directional metrics on fuzzy regions as well as methods for their analysis and synthesis. The "distance" between two fuzzy regions is not a scalar as in other approaches but a new fuzzy set made out of the distances between elements in the Cartesian product of the two sets, and whose membership is the minimum membership of each element pair in the product. The type of uncertainty this deals with results from imprecision in the spatial delineation of the regions rather than from fuzziness in the position of point-like objects as in Dutta's work.

Jorge and Vaida [39] develop a fuzzy relational path algebra to reason about spatial relations involving distances and directions based on semirings. They aim at complementing qualitative distance and direction functions like those developed by Frank [21] with degree of likelihood functions corresponding to membership functions of fuzzy predicates. Semirings are used as algebraic structures subsuming the formal properties of basic fuzzy set theory and the relational and path algebras of Smith and Park [61] and Frank [21] respectively.

As compared to qualitative representations, fuzzy approaches do not take the structure of the represented domain into consideration and, consequently, do not exploit the resulting neighborhood constraints. One of the sources of the uncertainty that both qualitative and fuzzy approaches intend to deal with is the need to express a certain relationship independently of the context in which it occurs. In fuzzy approaches contextual information is encoded in a single number, the value of the membership function, which is not as expressive as the elaborated and explicit frame of reference concept developed in this paper. The quantitative characterization of the elasticity of constraints in fuzzy approaches contrasts with the more flexible concept of acceptance areas or distance ranges and corresponding overlapping or non-exhaustive semantic interpretations in our qualitative model. Finally, the composition of fuzzy relations (done, e.g., by min-max operations) yields a single membership value rather than a disjunction of possible relations as in the qualitative case. In order to verbalize the result of such reasoning a defuzzification process is required in the fuzzy model as opposed to a constraint selection in the qualitative model.

6. Discussion and future research

Positional information is a significant and important type of human knowledge. The aim of this paper was to develop a qualitative model for representing the position of objects in two-dimensional space and for performing basic spatial reasoning as a qualitative replacement of quantitative vector algebra. Qualitative models for more than one-dimensional (scalar) values have historically been considered not expressive enough for general use [20]. The results of this paper, together with other results obtained in the emerging field of qualitative spatial reasoning [11], lead to a new attitude towards qualitative models of space, which should be seen as a viable alternative to quantitative models whenever quantitative information is not fully available or is not desired.

The qualitative framework proposed in this paper is the basis for representing and reasoning with positional information, although it should not be considered conclusive by itself. A number of additional research directions can be envisioned. In this paper, we have not explicitly treated the case of extended objects: in fact, the extension of objects influences the concepts of distance and orientation, depending on the scale adopted for reasoning. If the distances involved at a given scale are such that the extension of the objects can be disregarded, we can use the point abstraction, which was adopted in this paper as a model situation. If the extension of the objects is of the same order of magnitude of the distances among them, then the morphology of the distance and orientation domains should be modified according to the size and shape of the reference object.

Beyond the basic steps in reasoning (the composition algorithms introduced in Section 4), constraint propagation mechanisms should be investigated as the main source of spatial inference. Mechanisms similar to those proposed by [48] are needed to deal with assignments, constraints, and rules, in order to be able to bridge the gap between qualitative relations and quantitative information, whenever the latter is available.

The framework can be customized to different application areas, reflecting both cognitive uses and technical needs of the specific field. For example, the framework could be specialized to the domain of vehicle navigation systems. The issue of navigation has been largely investigated in robotics, see for example [44,49]. In vehicle navigation systems, positional information at different scales and granularity has a crucial role, since different issues arise to help guiding vehicles at the road level (small-scale environment), at the city level (urban scale), and at the large region level (geographic scale).

A good starting point for WWW-explorations related to the subject is the list of "Spatial Reasoning Resources": http://www.cs.albany.edu/~amit/spatsites.html.

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