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Explicit formulas for the multivariate resultant

Carlos D'Andrea, Alicia Dickenstein *,[1,2]

Departamento de Matemática, F.C.E y N., UBA, (1428) Buenos Aires, Argentina

Abstract

We present formulas for the multivariate resultant as a quotient of two determinants. They extend the classical Macaulay formulas, and involve matrices of considerably smaller size, whose non-zero entries include coefficients of the given polynomials and coefficients of their Bezoutian. These formulas can also be viewed as an explicit computation of the morphisms and the determinant of a resultant complex. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Given n homogeneous polynomials $f_1, \ldots, f_n$ in n variables over an algebraically closed field k with respective degrees $d_1, \ldots, d_n$, the resultant $\mathrm{Res}_{d_1,\ldots,d_n}(f_1, \ldots, f_n)$ is an irreducible polynomial in the coefficients of $f_1, \ldots, f_n$, which vanishes whenever $f_1, \ldots, f_n$ have a common root in projective space. The study of resultants goes back to classical work of Sylvester, Bézout, Cayley, Macaulay and Dixon. The use of resultants as a computational tool for elimination of variables as well as a tool for the study of complexity aspects of polynomial system solving in the last decade, has renewed the interest in finding explicit formulas for their computation (cf. [1,3–5,14,18, 20,22,23]).

By a determinantal formula it is meant a matrix whose entries are polynomials in the coefficients of $f_1, \ldots, f_n$ and whose determinant equals the resultant $\mathrm{Res}_{d_1,\ldots,d_n}(f_1, \ldots, f_n)$. The interest on such a formula is the computation of the resultant, and so it is implicit

* Corresponding author.

E-mail addresses: cdandrea@dm.uba.ar (C. D'Andrea), alidick@dm.uba.ar (A. Dickenstein).

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that the entries should be algorithmically computed from the inputs. It is also meant that all non-zero entries have degree strictly less than the degree of the resultant.

In case all d_i have a common value d, all currently known determinantal formulas are listed by Weyman and Zelevinsky [27]. This list is short: if $d \geq 2$, there exist determinantal formulas for all d just for binary forms (given by the well known Sylvester matrix), ternary forms and quaternary forms; when $n = 5$, the only possible values for d are 2 and 3; finally, for $n = 6$, there exists a determinantal formula only for $d = 2$. We find similar strict restrictions on general $n, d_1, \dots, d_n$ (cf. Lemma 5.3).

Given $d_1, \dots, d_n$, denote $t_n := \sum_{i=1}^{n} (d_i - 1)$ the critical degree. Classical Macaulay formulas [21] describe the resultant $\mathrm{Res}_{d_1,\dots,d_n}(f_1, \dots, f_n)$ as an explicit quotient of two determinants. These formulas involve a matrix of size at least the number of monomials in n variables of degree $t_n + 1$, and a submatrix of it.

Macaulay's work has been revisited and sharpened by Jouanolou in [17], where he proposes for each $t \geq 0$, a square matrix M_t of size

$$\rho(t) := \binom{t + n - 1}{n - 1} + i(t_n - t) \tag{1}$$

whose determinant is a nontrivial multiple of $\mathrm{Res}_{d_1,\dots,d_n}(f_1, \dots, f_n)$ (cf. [17, 3.11.19.7]). Here, $i(t_n - t)$ denotes the dimension of the k-vector space of elements of degree $t_n - t$ in the ideal generated by a regular sequence of n polynomials with degrees $d_1, \dots, d_n$. Moreover, Jouanolou shows that the resultant may be computed as the ratio between the determinant of M_{t_n} and the determinant of one of its square submatrices. (cf. [17, Corollaire 3.9.7.7]).

In this paper, we explicitly find the extraneous factor in Jouanolou's formulation, i.e. the polynomial $\det(M_t)/\mathrm{Res}_{d_1,\dots,d_n}(f_1, \dots, f_n)$, for all $t \geq 0$, which again happens to be the determinant of a submatrix $\mathbb{E}_t$ of M_t for every t, and this allows us to present new resultant formulas *à la Macaulay* for the resultant, i.e. as a quotient of two determinants

$$\mathrm{Res}_{d_1,\dots,d_n}(f_1, \dots, f_n) = \frac{\det(M_t)}{\det(\mathbb{E}_t)}. \tag{2}$$

For $t > t_n$, we recover Macaulay's classical formulas. For $t \leq t_n$, the size of the matrix M_t is considerably smaller.

In order to give explicit examples, we need to recall the definition of the *Bezoutian associated with* $f_1, \dots, f_n$ (cf. [2,15,19,25] and [17] under the name "Formes de Morley"). Let $(f_1, \dots, f_n)$ be a sequence of generic homogeneous polynomials with respective degrees $d_1, \dots, d_n$

$$f_i := \sum_{|\alpha_i| = d_i} a_{\alpha_i} X^{\alpha_i} \in A[X_1, \dots, X_n],$$

where A is the factorial domain $A := \mathbb{Z}[a_{\alpha_i}]_{|\alpha_i| = d_i, i = 1,\dots,n}$.

Introduce two sets of n variables X, Y and for each pair (i,j) with $1 \le i,j \le n$, write $\Delta_{ij}(X,Y)$ for the incremental quotient

$$\frac{f_i(Y_1,\dots,Y_{j-1},X_j,\dots,X_n) - f_i(Y_1,\dots,Y_j,X_{j+1},\dots,X_n)}{X_j - Y_j}. \tag{3}$$

Note that $f_i(X) - f_i(Y) = \sum_{j=1}^n \Delta_{ij}(X,Y)(X_j - Y_j)$.

The determinant

$$\Delta(X,Y) := \det(\Delta_{ij}(X,Y))_{1 \le i,j \le n} = \sum_{|\gamma| \le t_n} \Delta_\gamma(X) \cdot Y^\gamma \tag{4}$$

is a representative of the *Bezoutian* associated with $(f_1,\dots,f_n)$. It is a homogeneous polynomial in $A[X,Y]$ of degree t_n.

Recall also that

$$\deg \mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n) = \sum_{i=1}^n d_1 \dots d_{i-1} \cdot d_i \dots d_n.$$

As a first example, let $n=3$, $(d_1,d_2,d_3) = (1,1,2)$, and let

$$f_1 = a_1X_1 + a_2X_2 + a_3X_3,$$

$$f_2 = b_1X_1 + b_2X_2 + b_3X_3,$$

$$f_3 = c_1X_1^2 + c_2X_2^2 + c_3X_3^2 + c_4X_1X_2 + c_5X_1X_3 + c_6X_2X_3$$

be generic polynomials of respective degrees $1,1,2$. Here, $t_3 = 1$. Macaulay's classical matrix M_2 looks as follows:

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & c_1 \\ 0 & a_2 & 0 & b_2 & 0 & c_2 \\ 0 & 0 & a_3 & 0 & b_3 & c_3 \\ a_2 & a_1 & 0 & b_1 & 0 & c_4 \\ a_3 & 0 & a_1 & 0 & b_1 & c_5 \\ 0 & a_3 & a_2 & b_3 & b_2 & c_6 \end{pmatrix}$$

and its determinant equals $-a_1 \mathrm{Res}_{1,1,2}$. The extraneous factor is the 1×1 minor formed by the element in the fourth row, second column.

On the other hand, because of Lemma 5.3, we can exhibit a determinantal formula for $\pm\mathrm{Res}_{1,1,2}$, and it is given by Proposition 5.6 for $t = [t_3/2] = 0$ by the determinant of

$$\begin{pmatrix} \Delta_{(1,0,0)} & a_1 & b_1 \\ \Delta_{(0,1,0)} & a_2 & b_2 \\ \Delta_{(0,0,1)} & a_3 & b_3 \end{pmatrix},$$

where Δ_γ are coefficients of the Bezoutian (4). Explicitly, we have

$$\Delta_{(1,0,0)} = c_1(a_2b_3 - a_3b_2) - c_4(a_1b_3 - a_3b_1) + c_5(a_1b_2 - a_2b_1),$$

$$\Delta_{(0,1,0)} = c_6(a_1b_2 - a_2b_1) - c_2(a_1b_3 - b_1a_3)$$

and

$$\Delta_{(0,0,1)} = c_3(a_1b_2 - b_1a_2).$$

This is the matrix M_0 corresponding to the linear transformation Ψ_0 which is defined in (9).

Take now $n=4$, and $(d_1,d_2,d_3,d_4)=(1,1,2,3)$. The critical degree is 3. Macaulay's classical matrix M_4 has size 35×35. Because the degree of $\mathrm{Res}_{1,1,2,3}$ is $2+3+6+6=17$, we know that its extraneous factor must be a minor of size 18×18. By Proposition 5.6, we can find the smallest possible matrix for $t=1$ or $t=2$. Set $t=2$. We get the following 12×12 matrix:

$$\begin{pmatrix}
\Delta^1_{(2,0,0,0)} & \Delta^2_{(2,0,0,0)} & \Delta^3_{(2,0,0,0)} & \Delta^4_{(2,0,0,0)} & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & c_1\\
\Delta^1_{(0,2,0,0)} & \Delta^2_{(0,2,0,0)} & \Delta^3_{(0,2,0,0)} & \Delta^4_{(0,2,0,0)} & 0 & a_2 & 0 & 0 & b_2 & 0 & 0 & c_2\\
\Delta^1_{(0,0,2,0)} & \Delta^2_{(0,0,2,0)} & \Delta^3_{(0,0,2,0)} & \Delta^4_{(0,0,2,0)} & 0 & 0 & a_3 & 0 & 0 & b_3 & 0 & c_3\\
\Delta^1_{(0,0,0,2)} & \Delta^2_{(0,0,0,2)} & \Delta^3_{(0,0,0,2)} & \Delta^4_{(0,0,0,2)} & 0 & 0 & 0 & a_4 & 0 & 0 & b_4 & c_4\\
\Delta^1_{(1,1,0,0)} & \Delta^2_{(1,1,0,0)} & \Delta^3_{(1,1,0,0)} & \Delta^4_{(1,1,0,0)} & a_2 & a_1 & 0 & 0 & b_1 & 0 & 0 & c_5\\
\Delta^1_{(1,0,1,0)} & \Delta^2_{(1,0,1,0)} & \Delta^3_{(1,0,1,0)} & \Delta^4_{(1,0,1,0)} & a_3 & 0 & a_1 & 0 & 0 & b_1 & 0 & c_6\\
\Delta^1_{(1,0,0,1)} & \Delta^2_{(1,0,0,1)} & \Delta^3_{(1,0,0,1)} & \Delta^4_{(1,0,0,1)} & a_4 & 0 & 0 & a_1 & 0 & 0 & b_1 & c_7\\
\Delta^1_{(0,1,1,0)} & \Delta^2_{(0,1,1,0)} & \Delta^3_{(0,1,1,0)} & \Delta^4_{(0,1,1,0)} & 0 & a_3 & a_2 & 0 & b_3 & b_2 & 0 & c_8\\
\Delta^1_{(0,1,0,1)} & \Delta^2_{(0,1,0,1)} & \Delta^3_{(0,1,0,1)} & \Delta^4_{(0,1,0,1)} & 0 & a_4 & 0 & a_2 & b_4 & 0 & b_2 & c_9\\
\Delta^1_{(0,0,1,1)} & \Delta^2_{(0,0,1,1)} & \Delta^3_{(0,0,1,1)} & \Delta^4_{(0,0,1,1)} & 0 & 0 & a_4 & a_3 & 0 & b_4 & b_3 & c_{10}\\
a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
b_1 & b_2 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

where

$$f_1 = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4,$$

$$f_2 = b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4,$$

$$f_3 = c_1X_1^2 + c_2X_2^2 + c_3X_3^2 + c_4X_4^2 + c_5X_1X_2 + c_6X_1X_3$$
$$+ c_7X_1X_4 + c_8X_2X_3 + c_9X_2X_4 + c_{10}X_3X_4,$$

f_4 is a homogeneous generic polynomial of degree 3 in four variables, and for each γ, $|\gamma|=2$, we write

$$\Delta_\gamma(X) = \sum_{j=1}^{4} \Delta^j_\gamma X_j,$$

Table 1

n	$(d_1,\dots,d_n)$	min size	classical
2	(10,70)	70	80
2	(150,200)	200	350
3	(1,1,2)	3	6
3	(1,2,5)	14	28
3	(2,2,6)	21	45
4	(1,1,2,3)	12	35
4	(2,2,5,5)	94	364
4	(2,3,4,5)	90	364
5	(4,4,4,4,4)	670	4845
7	(2,3,3,3,3,3,3)	2373	38760
10	$(3,3,\dots,3)$	175803	14307150
20	$(2,2,\dots,2)$	39875264	131282408400

which has degree 1 in the coefficients of each f_i, $i=1,\dots,4$. The determinant of this matrix is actually $\pm a_1 \mathrm{Res}_{1,1,2,3}$. Here, the extraneous factor is the minor 1×1 of the matrix obtained by taking the element in the fifth row, sixth column.

In Table 1, we display the minimal size of the matrices M_t and the size of classical Macaulay matrix for several values of n, $d_1,\dots,d_n$. We give in Section 4 an estimate for the ratio between these sizes. However, it should be noted that the number of coefficients of the Bezoutian that one needs to compute increases when the size of the matrix M_t decreases. We refer to [15,24] for complexity considerations on the computation of Bezoutians. In particular, this computation can be well parallelized. Also, the particular structure of the matrix and the coefficients could be used to improve the complexity estimates; this problem is studied for $n=2$ and 3 in [11].

Our approach combines Macaulay's original ideas [21], expanded by Jouanolou [17], with the expression for the resultant as the determinant of a Koszul complex inspired by the work of Cayley [7]. We also use the work [9,10] of Chardin on homogeneous subresultants, where a Macaulay style formula for subresultants is presented. In fact, we show that the proposed determinants are explicit non-zero minors of a bigger matrix which corresponds to one of the morphisms in a Koszul resultant complex which in general has many non-zero terms, and whose determinant is $\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n)$ (cf. Theorem 5.1). These are the complexes considered in [27,16] in the equal degree case, built from the spectral sequence associated with a twisted Koszul complex at the level of sheaves.

We give explicit expressions for the morphisms in these complexes in terms of the Bezoutian associated with $f_1,\dots,f_n$ for degrees under critical degree, addressing in this manner a problem raised by Weyman and Zelevinsky in [27] (cf. also [16, 13.1.C]).

In the last sections, we show that different classical formulas can be viewed as special cases of the determinantal formulas that we present here (cf. [16,27]). In particular, we also recover in this setting the "affine" Dixon formulas considered in [14] and we classify in particular all such determinantal formulas.

2. Notations and some preliminary statements

Let S_u denote the A-free module generated by the monomials in $A[X]$ with degree u. If $u<0$, then we set $S_u=0$. Define also the following free submodules $E^{t,j} \subseteq S^{t,j} \subseteq S_{t-d_j}$, for all $j=1,\dots,n$:

$$S^{t,j} := \langle X^\gamma, |\gamma| = t - d_j, \gamma_1 < d_1, \dots, \gamma_{j-1} < d_{j-1} \rangle \tag{5}$$

$$E^{t,j} := \langle X^\gamma \in S^{t,j}, \text{ there exists } i \neq j\colon \gamma_i \geq d_i \rangle. \tag{6}$$

Note that $E^{t,n}=0$, and $S^{t,1}=S_{t-d_1}$ $\forall t \in \mathbb{N}_0$.

Let $j_u : S_u \to S_u^*$ be the isomorphism associated with the monomial bases in S_u and denote by $T_\gamma := j_u(X^\gamma)$ the elements in the dual basis.

Convention. All spaces that we will consider have a monomial basis, or a dual monomial basis. We shall suppose all these bases have a fixed order. This will allow us to define matrices "in the monomial bases", with no ambiguity.

Let $\psi_{1,t}$ be the A-linear map

$$\psi_{1,t} : S^*_{t_n - t} \to S_t$$

which sends

$$T_\gamma \mapsto \Delta_\gamma(X), \tag{7}$$

where the polynomial $\Delta_\gamma(X)$ is defined in (4). Let Δ_t denote the matrix of $\psi_{1,t}$ in the monomial bases.

Lemma 2.1. *For suitable orders of the monomial bases in S_t and S_{t_n-t}, we have that*

$${}^{\mathbf{t}}\Delta_t = \Delta_{t_n-t}.$$

Proof. It holds that $\Delta(X,Y) = \Delta(Y,X)$ by the symmetry property of Bezoutians (cf. [17, 3.11.8]). This implies that

$$\sum_{|\gamma| = t_n - t} \Delta_\gamma(X) Y^\gamma = \sum_{|\lambda| = t} \Delta_\lambda(Y) X^\lambda = \sum_{|\gamma| = t_n - t,\ |\lambda| = t} c_{\gamma\lambda} X^\lambda Y^\gamma,$$

with $c_{\gamma\lambda} \in A$. It is easy to see that if $\Delta_t = (c_{\gamma\lambda})_{|\gamma| = t_n - t,\ |\lambda| = t}$ then $\Delta_{t_n-t} = (c_{\gamma\lambda})_{|\lambda| = t,\ |\gamma| = t_n - t}$. $\square$

Let us consider also the *Sylvester* linear map $\psi_{2,t}$:

$$\psi_{2,t} : S^{t,1} \oplus \cdots \oplus S^{t,n} \to S_t,$$

$$(g_1, \dots, g_n) \mapsto \sum_{i=1}^n g_i f_i, \tag{8}$$

and denote by D_t its matrix in the monomial bases. As usual, ψ^*_{2,t_n-t} denotes the dual mapping of (8) in degree $t_n - t$.

Denote

$$\Psi_t : S^*_{t_n-t} \oplus (S^{t,1} \oplus \cdots \oplus S^{t,n}) \to S_t \oplus (S^{t_n-t,1} \oplus \cdots \oplus S^{t_n-t,n})^* \tag{9}$$

the A-morphism defined by

$$(T,g) \mapsto (\psi_{1,t}(T) + \psi_{2,t}(g), \psi^*_{2,t_n-t}(T)), \tag{10}$$

and call M_t the matrix of Ψ_t in the monomial bases.

Denote also by E_t the submatrix of M_t whose columns are indexed by the monomials in $E^{t,1} \cup \cdots \cup E^{t,n-1}$, and whose rows are indexed by the monomials X^γ in S_t for which there exist two different indices i,j such that $\gamma_i \geq d_i, \gamma_j \geq d_j$. With these choices it is not difficult to see that M_t and E_t (when defined) are square matrices.

Remark 2.2. Observe that E_t is actually a submatrix of D_t. In fact, E_t is the transpose of the square submatrix named $\mathscr{E}(t)$ in [10], and whose determinant is denoted by $\Delta(n,t)$ in [21, Theorem 6].

Lemma 2.3. *M_t is a square matrix of size $\rho(t)$, where ρ is the function defined in* (1).

Proof. The assignment which sends a monomial m in $S^{t,i}$ to $x_i^{d_i} \cdot m$ injects the union of the monomial bases in each $S^{t,i}$ onto the monomials of degree t which are divisible by some $x_i^{d_i}$. It is easy to see that the cardinality of the set of complementary monomials of degree t is precisely $H_d(t)$, where $H_d(t)$ denotes the dimension of the t-graded piece of the quotient of the polynomial ring over k by the ideal generated by a regular sequence of homogeneous polynomials with degrees $d_1, \ldots, d_n$ (cf. [17, 3.9.2]). Moreover, using the assignment $(\gamma_1, \ldots, \gamma_n) \mapsto (d_1 - 1 - \gamma_1, \ldots, d_n - 1 - \gamma_n)$, it follows that

$$H_d(t) = H_d(t_n - t). \tag{11}$$

We can compute explicitly this Hilbert function by the following formula (cf. [21, Section 2]):

$$\frac{\prod_{i=1}^n (1 - Y^{d_i})}{(1 - Y)^n} = \sum_{t=0}^{\infty} H_d(t) \cdot Y^t. \tag{12}$$

Then,

$$\operatorname{rk}(S^{t,1} \oplus \cdots \oplus S^{t,n}) = \operatorname{rk} S_t - H_d(t).$$

Similarly,

$$\operatorname{rk}(S^{t_n-t,1} \oplus \cdots \oplus S^{t_n-t,n})^* = \operatorname{rk}(S_{t_n-t})^* - H_d(t_n - t).$$

Therefore, M_t is square of size $\operatorname{rk} S_t - H_d(t_n - t) + \operatorname{rk} S_{t_n-t}$. Since $i(t_n - t) = \operatorname{rk} S_{t_n-t} - H_d(t_n - t)$, the size of M_t equals $\operatorname{rk} S_t + i(t_n - t) = \rho(t)$. $\square$

Remark 2.4. Ordering properly the monomial bases, M_t is the transpose of the matrix which appears in [17, 3.11.19.7]. It has the following structure:

$$\begin{bmatrix} \Delta_t & D_t \\ {}^{\mathbf{t}}D_{t_n-t} & 0 \end{bmatrix}. \tag{13}$$

Remark 2.5. Because $\psi_{2,t}=0$ if and only if $t<\min\{d_i\}$, we have that $\Psi_t=\psi_{2,t}+\psi_{1,t}$ if $t>t_n-\min\{d_i\}$, and $\Psi_t=\psi_{2,t}$ if $t>t_n$.

Finally, denote $\mathbb{E}_t$ the square submatrix of M_t which has the following structure:

$$\mathbb{E}_t = \begin{bmatrix} * & E_t \\ {}^{\mathbf{t}}E_{t_n-t} & 0 \end{bmatrix}. \tag{14}$$

It is clear from the definition that $\det(\mathbb{E}_t)=\pm\det(E_t)\det(E_{t_n-t})$.

Remark 2.6. Dualizing (10) and using Lemma 2.1 with a careful inspection at (13) and (14), we have that ordering properly their rows and columns,

$${}^{\mathbf{t}}M_t = M_{t_n-t} \quad \text{and} \quad {}^{\mathbf{t}}\mathbb{E}_t = \mathbb{E}_{t_n-t}.$$

3. Generalized Macaulay formulas

We can extend the map $\psi_{2,t}$ in (8) to the direct sum of all homogeneous polynomials with degrees $t-d_1,\dots,t-d_n$, and the map ψ_{2,t_n-t} to the direct sum of all homogeneous polynomials with degrees $t_n-t-d_1,\dots,t_n-t-d_n$, to get a map

$$\tilde{\Psi}_t : (S_{t_n-t})^* \oplus (S_{t-d_1}\oplus\cdots\oplus S_{t-d_n}) \to S_t \oplus (S_{t_n-t-d_1}\oplus\cdots\oplus S_{t_n-t-d_n})^*.$$

We can thus see the matrix M_t of Ψ_t in (9) as a choice of a square submatrix of $\tilde{\Psi}_t$. We will show that its determinant is a non zero minor of maximal size.

Proposition 3.1. *Let M_t' be a square matrix over A of the form*

$$M_t' := \begin{bmatrix} \Delta_t & F_t \\ {}^{\mathbf{t}}F_{t_n-t} & 0 \end{bmatrix}, \tag{15}$$

where F_t has $i(t)$ columns and corresponds to a restriction of the map

$$S_{t-d_1}\oplus\cdots\oplus S_{t-d_n} \to S_t,$$
$$(g_1,\dots,g_n) \mapsto \sum_{i=1}^n g_i f_i,$$

and similarly for F_{t_n-t} in degree t_n-t. Then, $\det(M_t')$ *is a multiple of* $\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n)$ (*probably zero*).

Proof. It is enough to mimic for the matrix M_t' the proof performed by Jouanolou in [17, Proposition 3.11.19.10] to show that the determinant of the matrix M_t' is an inertia form of the ideal $\langle f_1,\dots,f_n\rangle$ (i.e. a multiple of the resultant). We include this proof for the convenience of the reader.

Let $N := \sum_{i=1}^n \#\{\alpha_i \in \mathbb{N}^n\colon |\alpha_i| = d_i\}$. Given an algebraically closed field k, and $a = (a_{\alpha_i})_{|\alpha_i| = d_i,\ i=1,\dots,n}$, a point in k^N, we denote by $f_1(a),\dots,f_n(a) \in k[X]$ the polynomials obtained from $f_1,\dots,f_n$ when the coefficients are specialized to a, and similarly for the coefficients of the Bezoutian. Because of the irreducibility of $\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n)$, it is enough to show that for all $a \in k^N$ such that $f_1(a),\dots,f_n(a)$ have a non-trivial solution in k^n, the determinant of the specialized matrix $M_t'(a)$ is equal to 0.

Suppose that this is case, and let $(p_1,\dots,p_n)$ be a non-trivial solution. Without loss of generality, we can suppose $p_1 \neq 0$. One of the rows of $M_t'(a)$ is indexed by X_1^t. Replace all the elements in that row as follows:

1. if the element belongs to a column indexed by a monomial X^γ, $|\gamma| = t_n - t$, then replace it with $\Delta_\gamma(a)$;
2. if it belongs to a column indexed by a monomial $X^\gamma \in S_{t-d_i}$, replace it with $X^\gamma f_i(a)$.

It is easy to check that, the determinant of the modified matrix is equal to $X_1^t \det(M_t'(a))$. Now, we claim that under the specialization $X_i \mapsto p_i$, the determinant of the modified matrix will be equal to zero if and only if $\det(M_t'(a)) = 0$.

In order to prove this, we will show that the following submatrix of size $(i(t_n - t) + 1) \times \binom{n+t-1}{n-1}$ has rank less or equal than $i(t_n - t)$:

$$\begin{bmatrix} \Delta_{\gamma_1}(a)(p) & \dots & \Delta_{\gamma_s}(a)(p) \\ & {}^{\mathbf{t}}F_{t_n-t}(a) & \end{bmatrix}.$$

This, combined with a Laplace expansion of the determinant of the modified matrix, gives the desired result.

If the rank of the block $[{}^{\mathbf{t}}F_{t_n-t}(a)]$ is less than $i(t_n - t)$, then the claim follows straightforwardly. Suppose this is not the case. Then the family $\{X^\gamma f_i(a),\ X^\gamma \in S_{t_n-t-d_i}\}$ is a basis of the piece of degree $t_n - t$ of the generated ideal $I(a) := \langle f_1(a),\dots,f_n(a)\rangle$. We will show that in this case the polynomial $\sum_{|\gamma|=t_n-t} \Delta_\gamma(a)(p)X^\gamma$ belongs to $I(a)$, which proves the claim.

Because of (3) and (4), the polynomial $(X_1 - Y_1)\Delta(a)(X,Y)$ lies in the ideal $\langle f_1(a)(X) - f_1(a)(Y),\dots,f_n(a)(X) - f_n(a)(Y)\rangle$. Specializing $Y_i \mapsto p_i$, we deduce that $(X_1 - p_1)\sum_{j=0}^{t_n}(\sum_{|\gamma|=j} \Delta_\gamma(a)(p)X^\gamma)$ is in the graded ideal $I(a)$. This, combined with the fact that $p_1 \neq 0$, proves that $\sum_{|\gamma|=j} \Delta_\gamma(a)(p)X^\gamma \in I(a)$ for all j. □

In particular, $\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n)$ divides $\det(M_t)$. We describe the extraneous factor explicitly in the following theorem, which is the main result in this section. Before stating it, we set the following convention: if the matrix $\mathbb{E}_t$ is indexed by an empty set, we define $\det(\mathbb{E}_t) = 1$.

Theorem 3.2. *For any* $t \geq 0$, $\det(M_t) \neq 0$ *and* $\det(\mathbb{E}_t) \neq 0$. *Moreover, we have the following formula à la Macaulay*:

$$\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n) = \pm \frac{\det(M_t)}{\det(\mathbb{E}_t)}.$$

For the proof of Theorem 3.2, we will need the following auxiliary lemma. Let D_t and E_t be the matrices defined in Section 2 before Lemma 2.3.

Lemma 3.3. *Let* $t \geq 0$, *and* Λ *a ring which contains* A. *Suppose we have a square matrix* M *with coefficients in* Λ *which has the following structure*:

$$M = \begin{bmatrix} M_1 & D_t \\ M_2 & 0 \end{bmatrix},$$

where M_1, M_2 *are rectangular matrices. Then, there exists an element* $m \in \Lambda$ *such that*

$$\det(M) = m \det(E_t)$$

Proof. D_t is square if and only if $t > t_n$ (cf. [21, Section 3]). In this case,

$$\det(M) = \pm\det(M_2)\det(D_t);$$

because of Macaulay's formula (cf. [21, Theorem 5]), we have that the right-hand side equals

$$\pm\det(M_2)\det(E_t)\mathrm{Res}_{d_1,\ldots,d_n}(f_1,\ldots,f_n),$$

and the conclusion follows easily.

Suppose now $0 \leq t \leq t_n$. As in the introduction, let $i(t)$ denote the dimension of the k-vector space of elements of degree t in the ideal generated by a regular sequence of n polynomials with degrees $d_1,\ldots,d_n$. Then D_t has $i(t)+H_d(t)$ rows and $i(t)$ columns, and there is a bijection between the family $\mathscr{F}$ of $H_d(t)$ monomials of degree t, and the maximal minors $m_{\mathscr{F}}$ of D_t. Namely, $m_{\mathscr{F}}$ is the determinant of the square submatrix made by avoiding all rows indexed by monomials in $\mathscr{F}$.

It is not hard to check that $m_{\mathscr{F}}$ is the determinant $\phi^*_{\mathscr{F}}$ which is used in [10], for computing the *subresultant* associated with the family $\{X^\gamma\}_{\gamma\in\mathscr{F}}$.

Now, using the generalized Macaulay's formula for the subresultant (cf. [10]), we have that

$$m_{\mathscr{F}} = \pm\det(E_t)\Delta^t_{\mathscr{F}},$$

where $\Delta^t_{\mathscr{F}}$ is the subresultant associated with the family $\mathscr{F}$. It is a polynomial in A which vanishes under a specialization of the coefficients $f_1(a),\ldots,f_n(a)$ if and only if the family $\{X^\gamma\}_{\gamma\in\mathscr{F}}$ fails to be a basis of the t-graded piece of the quotient $k[X_1,\ldots,X_n]/\langle f_1(a),\ldots,f_n(a)\rangle$ (cf. [9]).

Let $m^{\mathrm{c}}_{\mathscr{F}}$ be the complementary minor of $m_{\mathscr{F}}$ in M (i.e. the determinant of the square submatrix of M which is made by deleting all rows and columns that appear in $m_{\mathscr{F}}$). By the Laplace expansion of the determinant, we have that

$$\det(M) = \sum_{\mathscr{F}} s_{\mathscr{F}} \cdot m_{\mathscr{F}} \cdot m^{\mathrm{c}}_{\mathscr{F}} = \det(E_t)\left(\sum_{\mathscr{F}} s_{\mathscr{F}} \cdot m^{\mathrm{c}}_{\mathscr{F}} \cdot \Delta^t_{\mathscr{F}}\right)$$

with $s_{\mathscr{F}} = \pm 1$. Setting $m = \sum_{\mathscr{F}} s_{\mathscr{F}} \cdot m^{\mathrm{c}}_{\mathscr{F}} \cdot \Delta^t_{\mathscr{F}} \in \Lambda$, we have the desired result. □

We now give the proof of Theorem 3.2.

Proof. In [21] it is shown that $\det(E_t) \neq 0$, $\forall t \geq 0$. This implies that $\det(\mathbb{E}_t) \neq 0$. In order to prove that $\det(\mathbb{E}_t) = \det(E_t)\det(E_{t_n-t})$ divides $\det(M_t)$, we use the following trick: consider the ring $B := \mathbb{Z}[b_{\alpha_i}]_{|\alpha_i|=d_i, i=1,\ldots,n}$, where b_{α_i} are new variables, and the polynomials

$$f_{b,i} := \sum_{|\alpha_i|=d_i} b_{\alpha_i} X^{\alpha_i} \in B[X_1, \ldots, X_n].$$

Let D_t^b the matrix of the linear transformation $\psi_{2,t}^b$ determined by formula (8) but associated with the sequence $f_{b,1}, \ldots, f_{b,n}$ instead of $f_1, \ldots, f_n$. Set $\Lambda := \mathbb{Z}[a_{\alpha_i}, b_{\alpha_i}]$, and consider the matrix $M(a,b)$ with coefficients in Λ given by

$$M(a,b) = \begin{bmatrix} \Delta_t & D_t \\ {}^{\mathbf{t}}D_{t_n-t}^b & 0 \end{bmatrix}.$$

It is easy to see that $M(a,a) = M_t$, and because of Lemma 3.3, we have that $\det(E_t)$ divides $\det(M(a,b))$ in Λ. Transposing $M(a,b)$ and using a symmetry argument, again by the same lemma, we can conclude that $\det(E_{t_n-t}^b)$ divides $\det(M(a,b))$ in Λ, where $E_{t_n-t}^b$ has the obvious meaning.

The ring Λ is a factorial domain and $\det(E_t)$ and $\det(E_{t_n-t}^b)$ have no common factors in Λ because they depend on different sets of variables. So, we have

$$\det(M(a,b)) = p(a,b)\det(E_t)\det(E_{t_n-t}^b)$$

for some $p \in \Lambda$. Now, specialize $b_{\alpha_i} \mapsto a_{\alpha_i}$. The fact that $\det(M_t)$ is a multiple of the resultant has been proved in Proposition 3.1 (see also [17, Proposition 3.11.19.21]) for $0 \leq t \leq t_n$, and in [21] for $t > t_n$. On the other side, since $\operatorname{Res}_{d_1,\ldots,d_n}(f_1, \ldots, f_n)$ is irreducible and depends on all the coefficients of $f_1, \ldots, f_n$ while $\det(E_t)$ and $\det(E_{t_n-t})$ do not depend on the coefficients of f_n, we conclude that $\operatorname{Res}_{d_1,\ldots,d_n}(f_1, \ldots, f_n)$ divides $p(a,a)$. Moreover, the following lemma shows that they have the same degree. Then, their ratio is a rational number λ. We can see that $\lambda = \pm 1$, considering the specialized family $X_1^{d_1}, \ldots, X_n^{d_n}$. □

Lemma 3.4. *For each $i = 1, \ldots, n$ the degree $\deg_{(a_{\alpha_i})}(M_t)$ of M_t in the coefficients of f_i equals*

$$\begin{aligned} &\deg_{(a_{\alpha_i})}(\operatorname{Res}_{d_1,\ldots,d_n}(f_1, \ldots, f_n)) + \deg_{(a_{\alpha_i})}(E_t) + \deg_{(a_{\alpha_i})}(E_{t_n-t}) \\ &\quad = d_1 \ldots d_{i-1}.\, d_{i+1} \ldots d_n + \deg_{(a_{\alpha_i})}(E_t) + \deg_{(a_{\alpha_i})}(E_{t_n-t}). \end{aligned}$$

Proof. Set $J_u(i) := \{X^\gamma \in S_u, \gamma_i \geq d_i, \gamma_j < d_j \ \forall j \neq i\}$, $u = t, t_n - t$. From the definitions of $\psi_{2,t}$ and E_t, it is easy to check that, if δ_t is a maximal minor of D_t,

$$\deg_{(a_{\alpha_i})}(\delta_t) - \deg_{(a_{\alpha_i})}(E_t) = \#J_t(i).$$

Using Laplace expansion, it is easy to see that $\det(M_t)$ may be expanded as follows:

$$\det(M_t) = \sum_{\delta_t, \delta_{t_n-t}} s_\delta \cdot m_\delta \cdot \delta_t \cdot \delta_{t_n-t},$$

where $s_\delta = \pm 1$, δ_{t_n-t} is a maximal minor of ${}^{\mathbf{t}}D_{t_n-t}$ and m_δ is a minor of size $H_d(t)$ in Δ_t.

As each entry of Δ_t has degree 1 in the coefficients of f_i, the lemma will be proved if we show that

$$\#J_t(i) + \#J_{t_n-t}(i) + H_d(t) = d_1 \dots d_{i-1}.d_{i+1} \dots d_n. \tag{16}$$

Now, as already observed in the proof of Lemma 2.3, $H_d(t)$ can be computed as the cardinality of the following set:

$$H_{d,t} := \{X^\gamma \in S_t,\ \gamma_j < d_j\ \forall j\}, \tag{17}$$

and $d_1 \dots d_{i-1}.d_{i+1} \dots d_n$ is the cardinality of

$$\Gamma_i := \{X_1^{\gamma_1} \dots X_{i-1}^{\gamma_{i-1}} X_{i+1}^{\gamma_{i+1}} \dots X_n^{\gamma_n},\ \gamma_j < d_j\ \forall j\}.$$

In order to prove (16) it is enough to exhibit a bijection between Γ_i and the disjoint union $J_t(i) \cup J_{t_n-t}(i) \cup H_{d,t}$. This is actually a disjoint union for all t, unless $t_n - t = t$. But what follows shows that the bijection is well defined even in this case.

Let $X^{\hat{\gamma}} \in \Gamma_i$, $\hat{\gamma} = (\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n)$ with $\gamma_j < d_j\ \forall j \neq i$. If $|\hat{\gamma}| \leq t$, then there exists a unique γ_i such that $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$ verifies $|\gamma| = t$. If $\gamma_i < d_i$, then we send $X^{\hat{\gamma}}$ to $X^\gamma \in H_{d,t}$. Otherwise, we send it to $X^\gamma \in J_t(i)$.

If $|\hat{\gamma}| > t$, let $\hat{\gamma}^*$ denote the multiindex

$$(d_1 - 1 - \gamma_1, \dots, d_{i-1} - 1 - \gamma_{i-1}, d_{i+1} - 1 - \gamma_{i+1}, \dots, d_n - 1 - \gamma_n).$$

Then, $|\hat{\gamma}^*| < t_n - t$, and there exists a unique γ_i such that the multiindex γ defined by

$$(d_1 - 1 - \gamma_1, \dots, d_{i-1} - 1 - \gamma_{i-1}, \gamma_i, d_{i+1} - 1 - \gamma_{i+1}, \dots, d_n - 1 - \gamma_n)$$

has degree $t_n - t$. We can send $X^{\hat{\gamma}}$ to $X^\gamma \in J_{t_n-t}(i)$ provided that $\gamma_i \geq d_i$. Suppose this last statement does not happen, this implies that the monomial with exponent

$$\gamma^* := (\gamma_1, \dots, \gamma_{i-1}, d_i - 1 - \gamma_i, \gamma_{i+1}, \dots, \gamma_n)$$

has degree t contradicting the fact that $|\hat{\gamma}| > t$.

With these rules, it is straightforward to check that we obtain the desired bijection. □

Changing the order of the sequence $(f_1, \dots, f_n)$, and applying Theorem 3.2, we deduce that

Corollary 3.5. $\mathrm{Res}_{d_1,\dots,d_n}(f_1, \dots, f_n) = \gcd\{$*maximal minors of* $\tilde{\Psi}_t\}$.

4. Estimating the size of M_t

We have, for each integer $t \geq 0$, a matrix M_t of size $\rho(t)$, where ρ was defined in (1), whose determinant is a nontrivial multiple of the resultant, and such that, moreover, its extraneous factor is a minor of it. We want to know which is the smallest matrix we can have.

We can write ρ as

$$\rho(t) = \binom{n+t-1}{n-1} + \binom{n+t_n-t-1}{n-1} - H_d(t_n - t).$$

It is straightforward to check that $\binom{n+t-1}{n-1} + \binom{n+t_n-t-1}{n-1}$ is the restriction to the integers of a polynomial $\phi(t)$ in a real variable t, symmetric with respect to $(t_n/2)$ (i.e. for all $t \geq 0$, $\phi(t_n/2+t) = \phi(t_n/2-t)$). Moreover, ϕ reaches its minimum over $[0, t_n]$ at $t = t_n/2$. Since

$$\rho(t) = \phi(t) - H_d(t) = \phi(t_n - t) - H_d(t_n - t) = \rho(t_n - t), \tag{18}$$

in order to study the behaviour of ρ we need to understand how $H_d(t)$ varies with t. We denote as usual the integer part of a real number x by the symbol $[x]$.

Proposition 4.1. *$H_d(t)$ is non-decreasing on (the integer points of) the interval* $[0, [t_n/2]]$.

Proof. We will prove this result by induction on n. The case $n = 1$ is obvious since $t_1 = d-1$ and $H_d(t) = 1$ for any $t = 0, \dots, d-1$. Suppose then that the statement holds for n variables and set

$$\hat{d} := (d_1, \dots, d_{n+1}) \in \mathbb{N}_0^{n+1},$$

$$d := (d_1, \dots, d_n).$$

Let $t < t+1 \leq [t_{n+1}/2]$. We want to see that $\varphi(t) := H_{\hat{d}}(t+1) - H_{\hat{d}}(t)$ is non negative. Recall from (17) that, for every $t \in \mathbb{N}_0$, $H_{\hat{d}}(t)$ equals the cardinality of the set

$$\{\gamma \in \mathbb{N}_0^{n+1} \colon |\gamma| = t, \ 0 \leq \gamma_i \leq d_i - 1, \ i = 1, \dots, n+1\}.$$

Then, it can also be computed as

$$\sum_{j=0}^{d_{n+1}-1} \#\{\hat{\gamma} \in \mathbb{N}_0^n \colon |\hat{\gamma}| = t-j, \ 0 \leq \hat{\gamma}_i \leq d_i - 1, \ i = 1, \dots, n\},$$

which gives the equality $H_{\hat{d}}(t) = \sum_{j=0}^{d_{n+1}-1} H_d(t-j)$. It follows that $\varphi(t) = H_d(t+1) - H_d(t+1-d_{n+1})$.

If $t+1 \leq [t_n/2]$, we deduce that $\varphi(t) \geq 0$ by inductive hypothesis. Suppose then that $t+1$ is in the range $[t_n/2] < t+1 \leq [t_{n+1}/2]$. As $H_d(t+1) = H_d(t_n - t - 1)$, it is enough to show that $t_n - t - 1 \geq t + 1 - d_{n+1}$ and $t_n - t - 1 \leq [t_n/2]$, which can be easily checked, and the result follows again by inductive hypothesis. $\square$

Corollary 4.2. *The size $\rho(t)$ of the matrix M_t is minimal over $\mathbb{N}_0$ when $t=[t_n/2]$.*

Proof. By (18), ρ has a maximum at $[t_n/2]$ over $[0,t_n]$ because ϕ has a maximum and H_d has a minimum. If $t>t_n$, we have that $\rho(t)=\binom{n+t-1}{n-1}$. For t in this range, it is easy to check that $\rho(t_n)=\binom{n+t_n-1}{n-1}-1<\rho(t)$. Then, $\rho(t)>\rho(t_n)\geq\rho([t_n/2])$. □

Remark 4.3. Note that when t_n is odd, $\rho([t_n/2])=\rho([t_n/2]+1)$, and then the size of M_t is also minimal for $t=[t_n/2]+1$ in this case.

Denote $p:=\sum_{i=1}^n d_i/n$ the average value of the degrees, and set $q:=(p+1)/2p$. Note that except in the linear case when all $d_i=1$, it holds that $p>1$ and $q<1$.

Proposition 4.4. *Assume $p>1$. The ratio between the size of the smallest matrix M_t and the classical Macaulay matrix M_{t_n+1} can be bounded by*

$$\frac{\rho([t_n/2])}{\rho(t_n+1)}\leq 2q^{n-1}.$$

In particular, it tends to zero exponentially in n when the number of variables tends to infinity and p remains bigger that a constant $c>1$.

Proof. When t_n is even, $t_n-[t_n/2]=[t_n/2]$ and when t_n is odd, $t_n-[t_n/2]=[t_n/2]+1$. In both cases,

$$\frac{\rho([t_n/2])}{\rho(t_n+1)}\leq\frac{2\begin{pmatrix}n+[t_n/2]\\ n-1\end{pmatrix}}{\begin{pmatrix}n+t_n\\ n-1\end{pmatrix}}=2\frac{([t_n/2]+n)\dots([t_n/2]+2)}{(t_n+n)\dots(t_n+2)}$$

$$=2\left(\frac{[t_n/2]+n}{t_n+n}\right)\left(\frac{[t_n/2]+n-1}{t_n+n-1}\right)\cdots\left(\frac{[t_n/2]+2}{t_n+2}\right)$$

$$\leq 2\left(\frac{[t_n/2]+n}{t_n+n}\right)^{n-1}.$$

Since $t_n=np-n$, we deduce that

$$\frac{[t_n/2]+n}{t_n+n}\leq\frac{np/2+n/2}{np}=\frac{1}{2}+\frac{1}{2p}=q,$$

as required. □

5. Resultant complexes

In this section we consider Weyman's complexes (cf. [27,16]) and we make explicit the morphisms in these complexes, which lead to polynomial expressions for the resultant via determinantal formulas in the cases described in Lemma 5.3.

We will consider a complex which is a "coupling" of the Koszul complex $\mathbf{K}^\bullet(t; f_1,\dots,f_n)$ associated with $f_1,\dots,f_n$ in degree t and the dual of the Koszul complex $\mathbf{K}^\bullet(t_n - t, f_1,\dots,f_n)^*$ associated with $f_1,\dots,f_n$ in degree $t_n - t$. This complex arises from the spectral sequence derived from the Koszul complex of sheaves on $\mathbb{P}^{n-1}$ associated with $f_1,\dots,f_n$ twisted by $\mathcal{O}_{\mathbb{P}^{n-1}}(t)$. Here, $\mathcal{O}_{\mathbb{P}^{n-1}}(t)$ denotes as usual the t-twist of the sheaf of regular functions over the $(n-1)$-projective space $\mathbb{P}^{n-1}$ (see for instance [16, p. 34]). Its space of global sections can be identified with the space of homogeneous polynomials in n variables of degree t. We make explicit in terms of the Bezoutian the map ∂_0 (see (10) below) produced by cohomology obstructions. In fact, the non-trivial contribution is given in terms of the mapping $\psi_{1,t}$ defined in (7).

Precisely, let $\mathbf{K}^\bullet(t; f_1,\dots,f_n)$ denote the complex

$$\left\{0 \to K(t)^{-n} \xrightarrow{\delta_{-(n-1)}} \cdots \xrightarrow{\delta_{-1}} K(t)^{-1} \xrightarrow{\delta_0} K(t)^0\right\}, \tag{19}$$

where

$$K(t)^{-j} = \bigoplus_{i_1<\cdots<i_j} S_{t-d_{i_1}-\cdots-d_{i_j}}$$

and δ_{-j} are the standard Koszul morphisms.

Similarly, let $\mathbf{K}^\bullet(t_n - t; f_1,\dots,f_n)^*$ denote the complex

$$\left\{K(t_n-t)^0 \xrightarrow{\delta_0^*} K(t_n-t)^1 \xrightarrow{\delta_1^*} \cdots \xrightarrow{\delta_n^*} K(t_n-t)^n\right\}, \tag{20}$$

where

$$K(t_n-t)^j = \bigoplus_{i_1<\cdots<i_j} S^*_{t_n-t-d_{i_1}-\cdots-d_{i_j}}$$

and δ_j^* are the duals of the standard Koszul morphisms. Note that in fact $K(t_n-t)^n = 0$ for any $t \geq 0$.

Now, define $\mathbf{C}^\bullet(t; f_1,\dots,f_n)$ to be the following coupled complex:

$$\left\{0 \to C^{-n} \xrightarrow{\partial_{-(n-1)}} \cdots \xrightarrow{\partial_{-1}} C^{-1} \xrightarrow{\partial_0} C^0 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} C^{n-1} \to 0\right\}, \tag{21}$$

where

$$C^{-j} = K(t)^{-j}, \quad j = 2,\dots,n,$$

$$C^j = K(t_n-t)^{j+1}, \quad j = 1,\dots,n-1,$$

$$C^{-1} = K(t_n-t)^0 \oplus K(t)^{-1},$$

$$C^0 = K(t)^0 \oplus K(t_n-t)^1 \tag{22}$$

and the morphisms are defined by

$$\partial_{-j} = \delta_{-j}, \quad j = 2,\dots,n-1,$$

$$\partial_j = \delta_j^*, \quad j = 2,\dots,n-1,$$

$$\partial_{-1} = 0 \oplus \delta_{-1},$$
$$\partial_0 = (\psi_{1,t} + \delta_0) \oplus \delta_0^*,$$
$$\partial_1 = 0 + \delta_1^*. \tag{23}$$

More explicitly, $\partial_0(T,(g_1,\dots,g_n))=(\psi_{1,t}(T)+\delta_0(g_1,\dots,g_n),\delta_0^*(T))$ and $\partial_1(h,(T_1,\dots,T_n)) = \delta_1^*(T_1,\dots,T_n)$. Observe that ∂_0 is precisely the mapping we called $\tilde{\Psi}_t$ in the previous section.

As in the proof of Proposition 3.1, given an algebraically closed field k, and $a = (a_{\alpha_i})_{|\alpha_i|=d_i,\ i=1,\dots,n}$, a point in k^N, we denote by $f_1(a),\dots,f_n(a)$ the polynomials $\in k[X]$ obtained from $f_1,\dots,f_n$ when the coefficients are specialized to a. For any particular choice of coefficients in (21) we get a complex of k-vector spaces. We will denote the specialized modules and morphisms by $K(t)^1(a),\delta_0(a)$, etc. Let D denote the determinant (cf. [16, Appendix A], [12]) of the complex of A-modules (21) with respect to the monomial bases of the A-modules C^ℓ. This is an element in the field of fractions of A.

We now state the main result in this section.

Theorem 5.1. *The complex* (21) *is generically exact, and for each specialization of the coefficients it is exact if and only if the resultant does not vanish. For any positive integer t we have that*

$$D = \mathrm{Res}_{d_1,\dots d_n}(f_1,\dots,f_n), \tag{24}$$

and moreover, D equals the greatest common divisor of all maximal minors of a matrix representing the A-module map ∂_0.

Proof. For $t > t_n$, we get the Koszul complex in degree t, and so the specialized complex at a point $a \in k^N$ is exact if and only if $f_1(a),\dots,f_n(a)$ is a regular sequence, i.e. if and only if the resultant does not vanish. The fact that the determinant of this complex equals the resultant goes back to ideas of Cayley; for a proof see [12,16] or [8].

Suppose $0 \le t \le t_n$. Since $\delta_0 \circ \delta_{-1} = \delta_1^* \circ \delta_0^* = 0$, it is easy to see that (21) is a complex.

Set

$$U := \{a = (a_{\alpha_i}) \in k^N,\ i = 1,\dots,n, |\alpha_i| = d_i\colon \det(M_t(a)) \neq 0\}.$$

Note that the open set U is non-void because the vector of coefficients of $\{X_1^{d_1},\dots,X_n^{d_n}\}$ lies in U, since in this case $\det M_t = \pm 1$. For any choice of homogeneous polynomials $f_1(a),\dots,f_n(a) \in k[X]$ with respective degrees $d_1,\dots,d_n$ and coefficients a in U, the resultant does not vanish by Theorem 3.2 and then the specialized Koszul complexes in (19) and (20) are exact.

Then, the dimension $\dim \operatorname{Im}(\delta_0(a))$ of the image of $\delta_0(a)$ equals $i(t)=\dim\langle f_1(a),\ldots, f_n(a)\rangle_t$. Similarly, $\dim\ \ker(\delta_0^*(a)) = i(t_n - t)$. Therefore,

$$\begin{aligned}\dim\ker(\partial_0(a)) &\geq \dim\operatorname{Im}(\partial_{-1}(a)) = \dim\operatorname{Im}(\delta_{-1}(a))\\ &= \dim\ker(\delta_0(a)) = \dim K(t)^{-1}(a) - i(t).\end{aligned}$$

On the other side, the fact that $M_t(a)$ is non-singular of size $\rho(t)$ implies that

$$\begin{aligned}\dim\ker(\partial_0(a)) &\leq \dim C^{-1}(a) - \rho(t)\\ &= \dim K(t)^{-1}(a) + \dim K(t_n - t)^0(a) - \rho(t)\\ &= \operatorname{Im} K(t)^{-1}(a) + \dim S_{t_n-t}(a) - \rho(t)\\ &= \dim K(t)^{-1}(a) - i(t).\end{aligned}$$

Therefore, $\dim\operatorname{Im}(\partial_{-1}(a)) = \dim\ker(\partial_0(a))$ and the complex is exact at level -1.

In a similar way, we can check that the complex is exact at level 0, and so the full specialized complex (21) is exact when the coefficients a lie in U.

In order to compute the determinant of the complex in this case, we can make suitable choices of monomial subsets in each term of the complex starting from the index sets that define $M_t(a)$ to the left and to the right. Then,

$$D(a) = \frac{\det M_t(a)}{p_1(a)\cdot p_2(a)},$$

where $p_1(a)$ (resp. $p_2(a)$) is a quotient of product of minors of the morphisms on the left (resp. on the right).

Taking into account (19) and (20), it follows from [10] that

$$p_1(a) = \det(E_t(a)), \quad p_2(a) = \det(E_{t_n-t}(a)),$$

and so by Theorem 3.2 we have

$$\begin{aligned}D(a) &= \operatorname{Res}_{d_1,\ldots,d_n}(f_1,\ldots,f_n)(a)\frac{\det(\mathbb{E}_t)(a)}{\det(E_t(a))\det(E_{t_n-t}(a))}\\ &= \operatorname{Res}_{d_1,\ldots,d_n}(f_1,\ldots,f_n)(a)\end{aligned}$$

for all families of homogeneous polynomials with coefficients a in the dense open set U, and since D and the resultant are rational functions, this implies (24), as wanted. Moreover, it follows that the complex is exact if and only if the resultant does not vanish.

The fact that $\operatorname{Res}_{d_1,\ldots,d_n}(f_1,\ldots,f_n)$ is the greatest common divisor of all maximal minors of the matrix representing ∂_0 has been proved in Corollary 3.5. □

We remark that from the statement of Theorem 5.1 plus a close look at the map at level 0, it is not hard to deduce that for a given specialization of $f_1,\ldots,f_n$ in k with non-vanishing resultant, the specialized polynomials $\Delta_\gamma(a), |\gamma| = t_n - t$ generate the quotient of the polynomial ring $k[X]$ by the ideal $I(a) = \langle f_1(a),\ldots,f_n(a)\rangle$ in degree t.

We can instead use the known dualizing properties of the Bezoutian in case the polynomials define a regular sequence, to provide an alternative proof of Theorem 5.1. This is a consequence of Proposition 5.2 below. We refer to [17;19, Appendix F]; [25,26] for the relation between the Bezoutian and the residue (i.e. an associated trace) and we simply recall the properties that we will use.

Assume $\mathrm{Res}_{d_1,\dots,d_n}(f_1(a),\dots,f_n(a))$ is different from zero. This implies that $f_1(a),\dots,f_n(a)$ is a regular sequence and their zero locus consists of the single point $\mathbf{0}\in k^n$. Then, there exists a dualizing k-linear operator

$$R_0 : k[Y]/\langle f_1(a)(Y),\dots,f_n(a)(Y)\rangle \to k,$$

called the *residue or trace operator*, which verifies
1. $h(X) = R_0(h(Y)\,\Delta(a)(X,Y))$ in the quotient ring $k[X]/I(a)$.
2. If h is homogeneous of degree t with $t \neq t_n$, $R_0(h) = 0$

Then, for every polynomial $h(X)\in k[X]$ of degree t, it holds that

$$h(X) = \sum_{|\gamma|=t_n-t} R_0(h(Y)\,Y^{\gamma})\Delta_{\gamma}(a)(X) \bmod I(a), \tag{25}$$

where $\Delta(a)(X,Y) = \sum_{|\gamma|=t_n-t}\Delta_{\gamma}(a)(X)Y^{\gamma}$ as in (4). As a consequence, *the family* $\{\Delta_{\gamma}(a)(X)\}_{|\gamma|=t_n-t}$, *(resp.* $|\gamma|=t$*) generates the graded piece of the quotient in degree* t *(resp.* $t_n - t$*)*. Moreover, it is easy to verify that for any choice of polynomials $p_i(X,Y), q_i(X,Y)\in k[X,Y]$, $i=1,\dots,n$, the polynomial $\tilde{\Delta}_a(X,Y)$ defined by

$$\tilde{\Delta}_a(X,Y) := \Delta(a)(X,Y) + \sum_{i=1}^{n} p_i(X,Y)\,f_i(a)(X) + q_i(X,Y)f_i(a)(Y). \tag{26}$$

has the same dualizing properties as $\Delta(a)(X,Y)$.

We are ready to prove a kind of "converse" to Proposition 3.1.

Proposition 5.2. *If* $\mathrm{Res}_{d_1,\dots,d_n}(f_1(a),\dots,f_n(a)) \neq 0$, *it is possible to extract a square submatrix* M_t' of $\tilde{\Psi}_t$ *as in* (15) *such that* $\det(M_t'(a)) \neq 0$.

Proof. Since $f_1(a)(X),\dots,f_n(a)(X)$ is a regular sequence in $k[X]$, the dimensions of the graded pieces of the quotient $k[X]/I(a)$ in degrees t and $t_n - t$ are $i(t)$ and $i(t_n-t)$ respectively.

We can then choose blocks F_t and F_{t_n-t} as in (15) such that $F_t(a)$ and $F_{t_n-t}(a)$ have maximal rank. Suppose without loss of generality that the blocks F_t and F_{t_n-t} have respectively the form

$$\begin{bmatrix} Q_t \\ R_t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_{t_n-t} \\ R_{t_n-t} \end{bmatrix},$$

where $Q_t(a)$ and $Q_{t_n-t}(a)$ are square invertible matrices of maximal size. We are going to prove that, with this choice, the matrix $M_t'(a)$ is invertible.

Our specialized matrix will look as follows:

$$M_t'(a) = \begin{bmatrix} & & Q_t(a) \\ & \Delta_t(a) & R_t(a) \\ {}^{\mathbf{t}}Q_{t_n-t}(a) & {}^{\mathbf{t}}R_{t_n-t}(a) & 0 \end{bmatrix}.$$

Applying linear operations in the rows and columns of $M_t'(a)$, it can be transformed into

$$\begin{bmatrix} 0 & 0 & Q_t(a) \\ 0 & \tilde{\Delta}_{t,a} & R_t(a) \\ {}^{\mathbf{t}}Q_{t_n-t}(a) & {}^{\mathbf{t}}R_{t_n-t}(a) & 0 \end{bmatrix},$$

where the block $[\tilde{\Delta}_{t,a}]$ is square and of size $H_d(t)$.

But it is easy to check that this $\tilde{\Delta}_{t,a}$ corresponds to the components in degree t of another Bezoutian $\tilde{\Delta}_a(X,Y)$ (in the sense of (26)). This is due to the fact that each of the linear operations performed on $M_t'(a)$, when applied to the block $\Delta_{t,a}$, can be read as a polynomial combination of $f_i(a)(X)$ and $f_i(a)(Y)$ applied to the bezoutian $\Delta(a)(X,Y)$.

Using the fact that the polynomials $\tilde{\Delta}_{\gamma,a}(X)$ read in the columns of $\tilde{\Delta}_{t,a}$ generate the quotient in degree $t_n - t$ and they are as many as its dimension, we deduce that they are a basis and so

$$\det\begin{pmatrix} 0 & \tilde{\Delta}_{t,a} \\ {}^{\mathbf{t}}Q_{t_n-t}(a) & {}^{\mathbf{t}}R_{t_n-t}(a) \end{pmatrix} \neq 0,$$

which completes the proof of the claim. □

We could then avoid the consideration of the open set U in the proof of Theorem 5.1, and use Proposition 5.2 to show directly that the complex is exact outside the zero locus of the resultant. In fact, this is not surprising since for all specializations such that the resultant is non-zero, the residue operator defines a natural duality between the t-graded piece of the quotient of the ring of polynomials with coefficients in k by the ideal $I(a)$ and the $t_n - t$ graded piece of the quotient, and we can read dual residue bases in the Bezoutian.

We characterize now those data $n, d_1, \ldots, d_n$ for which we get a determinantal formula.

Lemma 5.3. *Suppose* $d_1 \le d_2 \le \cdots \le d_n$. *The determinant of the resultant complex provides a determinantal formula for the resultant* $\mathrm{Res}_{d_1,\ldots,d_n}(f_1,\ldots,f_n)$ *if and only if the following inequality is verified*:

$$d_3 + \cdots + d_n - n < d_1 + d_2 - 1. \tag{27}$$

Moreover, when (27) *holds, there exists a determinantal formula given by the resultant complex for each* t *such that*

$$d_3 + \cdots + d_n - n < t < d_1 + d_2. \tag{28}$$

Remark 5.4. When all d_i have a common value d, (27) reads

$$(n-2)d < 2d + n - 1,$$

which is true for any d for $n \le 4$, for $d = 1,2,3$ in case $n = 5$, for $d = 1,2$ in case $n = 6$, and never happens for $n \ge 7$ unless $d = 1$, as we quoted in the introduction.

Proof. The determinant of the resultant complex provides a determinantal formula for $\mathrm{Res}_{d_1,\ldots,d_n}(f_1,\ldots,f_n)$ precisely when $C^{-2} = C_1 = 0$. This is respectively equivalent to the inequalities

$$t < d_1 + d_2$$

and

$$t_n - t = d_1 + \cdots d_n - n - t < d_1 + d_2,$$

from which the lemma follows easily. We have decreased the right-hand side of (27) by a unit in order to allow for a natural number t satisfying (28). □

Corollary 5.5. *For all $n \ge 7$ there exists a determinantal formula only if $d_1 = d_2 = d_3 = 1$ and $n - 3 \le d_4 + \cdots + d_n < n$, which forces all d_i to be 1 or at most, all of them equal 1 except for two of them which equal 2, or all of them equal 1 except for one of them which equals 3.*

The proof of the corollary follows easily from inequality (27). In any case, if a determinantal formula exists, we have a determinantal formula for $t = [t_n/2]$, as the following proposition shows.

Proposition 5.6. *If a determinantal formula given by the resultant complex exists, then $M_{[t_n/2]}$ is square and of the smallest possible size $\rho([t_n/2])$.*

Proof. In order to prove that $M_{[t_n/2]}$ is square, we need to check by Lemma 5.3 that

$$d_3 + \cdots + d_n - n < \left[\frac{t_n}{2}\right] < d_1 + d_2. \tag{29}$$

If there exists a determinantal formula, then the inequality (27) holds, from which it is straightforward to verify that

$$d_3 + \cdots + d_n - n < \frac{t_n}{2} < d_1 + d_2.$$

To see that in fact (29) holds, it is enough to check that

$$d_3 + \cdots + d_n - n + 1/2 \ne \frac{t_n}{2} = \frac{d_1 + \cdots + d_n - n}{2}.$$

But if the equality holds, we would have that $d_3 + \cdots + d_n = d_1 + d_2 + n - 1$, which is a contradiction. According to Corollary 4.2, we also know that $M_{[t_n/2]}$ has the smallest possible size. □

6. Dixon formulas

We prove in this section that "affine" Dixon formulas can in fact be recovered in this setting. We first recall classical Dixon formulas to compute the resultant of three bivariate affine polynomials of degree d. We will make a slight change of notation in what follows. The input affine polynomials (having monomials of degree at most d in two variables (X_1,X_2)) will be denoted f_1,f_2,f_3 and we will use capital letters F_1,F_2,F_3 to denote the homogeneous polynomials in three variables given by their respective homogenizations (with homogeneizing variable X_3). Dixon (cf. [13]) proposed the following determinantal formula to compute the resultant $\mathrm{Res}_{d,d,d}(f_1,f_2,f_3) = \mathrm{Res}_{d,d,d}(F_1,F_2,F_3)$.

Let $\mathrm{Bez}(X_1,X_2,Y_1,Y_2)$ denote the polynomial obtained by dividing the following determinant by $(X_1-Y_1)(X_2-Y_2)$:

$$\det\begin{pmatrix} f_1(X_1,X_2) & f_2(X_1,X_2) & f_3(X_1,X_2) \\ f_1(Y_1,X_2) & f_2(Y_1,X_2) & f_3(Y_1,X_2) \\ f_1(Y_1,Y_2) & f_2(Y_1,Y_2) & f_3(Y_1,Y_2) \end{pmatrix}.$$

Note that by performing row operations we have that $\mathrm{Bez}(X_1,X_2,Y_1,Y_2)$ equals the determinant of the matrix

$$\det\begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ f_1(Y_1,Y_2) & f_2(Y_1,Y_2) & f_3(Y_1,Y_2) \end{pmatrix},$$

where Δ_{ij} are as in (3). Write

$$\mathrm{Bez}(X_1,X_2,Y_1,Y_2) = \sum_{|\beta|\le 2d-2} B_\beta(X_1,X_2)Y_1^{\beta_1}Y_2^{\beta_2}.$$

Set $A := \mathbb{Z}[a]$, where a denotes one indeterminate for each coefficient of f_1,f_2,f_3. Let S denote the free module over A with basis $\mathscr{B}$ given by all monomials in two variables of degree less or equal than $d-2$, which has an obvious isomorphism with the free module S' over A with basis $\mathscr{B}'$ given by all monomials in three variables of degree equal to $d-2$. The monomial basis of all polynomials in two variables of degree less or equal than $2d-2$ will be denoted by $\mathscr{C}$.

Let M be the square matrix of size $2d^2-d$ whose columns are indexed by $\mathscr{C}$ and whose rows contain consecutively the expansion in the basis $\mathscr{C}$ of $m\cdot f_1$, of $m\cdot f_2$, and of $m\cdot f_3$, where m runs in the three cases over $\mathscr{B}$, and finally, the expansion in the basis $\mathscr{C}$ of all B_β, $|\beta|\le d-1$. Then, Dixon's formula says that

$$\mathrm{Res}_{d,d,d}(f_1,f_2,f_3) = \pm\det M.$$

Here, $d_1=d_2=d_3=d$ and $n=3$, so that (27) holds and by (28) there is a determinantal formula for each t such that $d-3<t<2d$. So, one possible choice is $t=2d-2$. Then, $t_3-t=d-1<d$, which implies $\langle F_1,F_2,F_3\rangle_{t_3-t}=0$. Also, $t-d=d-2<d$, and therefore $S^{t,i}=S'$, for all $i=1,2,3$.

Let $\Delta(X_1,X_2,X_3,Y_1,Y_2,Y_3) = \sum_{|\gamma| \leq 3d-3} \Delta_\gamma(X)Y^\gamma$ be the Bezoutian associated with the homogeneous polynomials F_1,F_2,F_3. We know that $\mathrm{Res}_{d,d,d}(F_1,F_2,F_3) = \pm\det M_{2d-2}$. In this case, the transposed matrix ${}^{\mathbf{t}}M_{2d-2}$ is a square matrix of the same size as M, and it is obvious that their $3d(d-1)/2$ first rows coincide (if the columns are ordered conveniently). According to (7), the last $(d+1)d/2$ rows of ${}^{\mathbf{t}}M_{2d-2}$ contain the expansion in the basis $\mathscr{B}'$ of all Δ_γ, $|\gamma| = d-1$.

Proposition 6.1. *The "affine" matrix M and the "homogeneous" matrix ${}^{\mathbf{t}}M_{2d-2}$ coincide.*

Proof. Denote $P(X_1,X_2,X_3,Y_1,Y_2,t)$ the homogeneous polynomial of degree $3d-2$ in 6 variables obtained by dividing the following determinant by $(X_1-Y_1)(X_2-Y_2)$:

$$\det\begin{pmatrix} \Delta_{1,1}(F) & \Delta_{2,1}(F) & \Delta_{3,1}(F) \\ \Delta_{1,2}(F) & \Delta_{2,2}(F) & \Delta_{3,2}(F) \\ F_1(Y_1,Y_2,t) & F_2(Y_1,Y_2,t) & F_3(Y_1,Y_2,t) \end{pmatrix},$$

where

$$\Delta_{i,1}(F) := F_i(X_1,X_2,X_3) - F_i(Y_1,Y_2,X_3), \quad i=1,2,3$$

and

$$\Delta_{i,2}(F) := F_i(Y_1,X_2,X_3) - F_i(Y_1,Y_2,X_3), \quad i=1,2,3.$$

It is easy to check that

$$(X_3-Y_3)\Delta(X,Y) = P(X_1,X_2,X_3,Y_1,Y_2,X_3) - P(X_1,X_2,X_3,Y_1,Y_2,Y_3) \tag{30}$$

and that

$$P(X_1,X_2,1,Y_1,Y_2,1) = \mathrm{Bez}(X_1,X_2,Y_1,Y_2). \tag{31}$$

We are looking for the elements in $\mathrm{Bez}(X_1,X_2,Y_1,Y_2)$ of degree less or equal than $d-1$ in the variables Y_1, Y_2. But it is easy to check that $\deg_y(P(X_1,X_2,X_3,Y_1,Y_2,Y_3)) \geq d$. This, combined with the equality given in (30), implies that, for each $1 \leq j \leq d-1$:

$$X_3 \sum_{|\gamma|=j} \Delta_\gamma(X)Y^\gamma - Y_3 \sum_{|\gamma|=j-1} \Delta_\gamma(X)Y^\gamma$$

is equal to the piece of degree j in the variables Y_i of the polynomial $P(X_1,X_2,X_3,Y_1,Y_2,X_3)$.

Besides, this polynomial does not depend on Y_3, so the following formula holds for every pair γ, $\tilde{\gamma}$ such that $\gamma = \tilde{\gamma} + (0,0,k), |\gamma| = j$:

$$X_3^k \Delta_\gamma(X) = \Delta_{\tilde{\gamma}}(X). \tag{32}$$

This allows us to compute $\Delta_\gamma(X)$ for every $|\gamma| = d-1$, in terms of the homogeneization of $B_{(\gamma_1,\gamma_2)}$. From Eq. (32), the claim follows straightforwardly. □

We conclude that Dixon's formula can be viewed as a particular case of the determinantal expressions that we addressed. Moreover, Proposition 6.1 can be extended to any number of variables and all Dixon matrices as in [14, Section 3.5] can be recovered in degrees t such that $\psi^*_{2,t_n-t}=0$, i.e. such that $t_n \geq t > t_n - \min\{d_1,\dots,d_n\}$. As we have seen, all one can hope in general is the explicit quotient formula we give in Theorem 3.2. In fact, we have the following consequence of Lemma 5.3

Lemma 6.2. *There exists a determinantal Dixon formula if and only if $n=2$, or $n=3$ and $d_1=d_2=d_3$, i.e. in the case considered by Dixon.*

Proof. Assume $d_1 \leq d_2 \leq \cdots \leq d_n$. If inequality (28) is verified for $t > t_n - d_1$, we deduce that

$$(n-2)d_1 - n \leq d_3 + \cdots + d_n - n < d_1 - 2, \tag{33}$$

and so $(n-3)d_1 < n-2$. This equality cannot hold for any natural number d_1 unless $n \leq 3$. It is easy to check that for $n=2$ there exist a determinantal Dixon formula for any value of d_1, d_2. In case $n=3$, (33) implies that $d_3 < d_1 + 1$. Then, $d_1=d_2=d_3$, as claimed. □

7. Other known formulas and some extensions

We can recognize other well-known determinantal formulas for resultants in this setting.

7.1. Polynomials in one variable

Let

$$f_1(x) = \sum_{j=0}^{d_1} a_j x^j, \quad f_2(x) = \sum_{j=0}^{d_2} b_j x^j$$

be generic univariate polynomials (or their homogenizations in two variables) of degrees $d_1 \leq d_2$. In this case, inequality (28) is verified for all $t = 0,\dots,d_1+d_2-1$ and so we have a determinantal formula for all such t. Here, $t_2 = d_1+d_2-2$. When $t = d_1+d_2-1 = t_2+1$ we have the classical Sylvester formula.

Assume $d_1=d_2=d$ and write

$$\frac{f_1(x)f_2(y)-f_1(y)f_2(x)}{x-y} = \sum_{i,j=0}^{d-1} c_{ij} x^i y^j.$$

Then, the classical *Bézout* formula for the resultant between f_1 and f_2 says that

$$\mathrm{Res}_{d,d}(f_1,f_2) = \det(c_{ij}).$$

It is easy to see that we obtain precisely this formulation for $t=d-1$. For other values of t we get formulas interpolating between Sylvester and Bézout as in [16, Chapter 12], even in case $d_1 \neq d_2$. It is easy to check that the smallest possible matrix has size d_2.

Suppose for example that $d_1 = 1, d_2 = 2$. In this case, $[t_2/2] = [1/2] = 0$, and M_0 is a 2×2 matrix representing a map from S_1^* to $S_0 \oplus S_0^*$, whose determinant equals the resultant

$$\mathrm{Res}_{1,2}(f_1, f_2) = a_1^2 b_0 - a_0 a_1 b_1 + b_2 a_0^2.$$

If we write $f_1(x) = 0x^2 + a_1 x + a_0$ and we use the classical Bezout formula for $d = 2$, we would also get a 2×2 matrix but whose determinant equals $b_2 \cdot \mathrm{Res}_{1,2}(f_1, f_2)$. The exponent 1 in b_2 is precisely the difference $d_2 - d_1$.

7.2. Sylvester formula for three ternary quadrics

Suppose that $n=3$, $d_1=d_2=d_3=2$ and $2 \neq 0$. Let J denote the Jacobian determinant associated with the homogeneous polynomials f_1, f_2, f_3. A beautiful classical formula due to Sylvester says that the resultant $\mathrm{Res}_{2,2,2}(f_1, f_2, f_3)$ can be obtained as $1/512$ times the determinant of the 6×6 matrix whose columns are indexed by the monomials in 3 variables of degree 2 and whose rows correspond to the expansion in this monomial basis of $f_1, f_2, f_3, \partial J/\partial X_1, \partial J/\partial X_2$ and $\partial J/\partial X_3$. In this case, $[t_3/2] = [3/2] = 1$, and by Lemma 5.3 we have a determinantal formula in this degree since $2 - 3 < 1 < 4$. From Euler equations

$$2 f_i(X) = \sum_{j=1}^{3} X_j \frac{\partial f_i(X)}{\partial X_j},$$

we can write

$$\begin{aligned} 2(f_i(X) - f_i(Y)) &= \sum_{j=1}^{3} \left(X_j \frac{\partial f_i(X)}{\partial X_j} - Y_j \frac{\partial f_i(Y)}{\partial Y_j} \right) \\ &= \sum_{j=1}^{3} (X_j - Y_j) \frac{\partial f_i(X)}{\partial X_j} + Y_j \left(\frac{\partial f_i(X)}{\partial X_j} - \frac{\partial f_i(Y)}{\partial Y_j} \right) \\ &= \sum_{j=1}^{3} \left((X_j - Y_j) \frac{\partial f_i(X)}{\partial X_j} + Y_j \sum_{l=1}^{3} \frac{\partial^2 f_i(X)}{\partial X_j \partial X_l} (X_l - Y_l) \right). \end{aligned}$$

Because of (26), we can compute the Bezoutian using

$$\Delta_{ij}(X, Y) := \frac{1}{2} \left(\frac{\partial f_i(X)}{\partial X_j} + \sum_{l=1}^{3} \frac{\partial^2 f_i(X)}{\partial X_l \partial X_j} Y_l \right).$$

With this formulation, it is not difficult to see that we can recover Sylvester formula from the equality $\mathrm{Res}_{2,2,2}(f_1, f_2, f_3) = \det M_1$.

7.3. Jacobian formulations

When $t = t_n$, one has $H_d(t) = 1$, and via the canonical identification of S_0^* with A, the complex (22) reduces to the following modified Koszul complex:

$$0 \to K(t)^{-n} \xrightarrow{\delta_{-(n-1)}} \cdots \xrightarrow{\delta_{-1}} A \oplus K(t)^{-1} \xrightarrow{\delta_0} K(t)^0 \to 0, \tag{34}$$

where δ_0 is the following map:

$$A \oplus S_{t_n-d_1} \oplus \cdots \oplus S_{t_n-d_n} \to S_{t_n}$$

$$(\lambda, g_1, \ldots, g_n) \mapsto \lambda\Delta_0 + \sum_{i=1}^{n} g_i f_i,$$

and $\Delta_0 := \Delta(X, 0)$. As a corollary of Theorem 5.1 we get that, for every specialization of the coefficients, Δ_0 is a non-zero element of the quotient if the resultant does not vanish.

Assume that the characteristic of k does not divide the product $d_1 \ldots d_n$. It is a well-known fact that the jacobian determinant J of the sequence $(f_1, \ldots, f_n)$ is another non-zero element of degree t_n, which is a non-zero element of the quotient whenever the resultant does not vanish (cf. for instance [25]). In fact, one can easily check that

$$J = d_1 \ldots d_n \Delta_0 \operatorname{mod} \langle f_1, \ldots, f_n \rangle. \tag{35}$$

In [6], the same complex is considered in a more general toric setting, but using J instead of Δ_0. Because of (35), we can recover their results in the homogeneous case.

Theorem 7.1. *Consider the modified complex* (34) *with J instead of Δ_0. Then, for every specialization of the coefficients, the complex is exact if and only if the resultant does not vanish. Moreover, the determinant of the complex equals* $d_1 \ldots d_n$ $\mathrm{Res}_{d_1,\ldots,d_n}(f_1, \ldots, f_n)$.

We can also replace Δ_0 by J in Macaulay's formula (Theorem 3.2), and have the following result:

Theorem 7.2. *Consider the square submatrix $\tilde{M}_{t_n}$ which is extracted from the matrix of δ_0 in the monomial bases, choosing the same rows and columns of M_{t_n}. Then,* $\det(\tilde{M}_{t_n}) \neq 0$, *and we have the following formula à la Macaulay*:

$$d_1 \ldots d_n \mathrm{Res}_{d_1,\ldots,d_n}(f_1, \ldots, f_n) = \frac{\det(\tilde{M}_{t_n})}{\det(E_{t_n})}.$$

We end the paper by addressing two natural questions that arise:

7.4. Different choices of monomial bases

Following Macaulay's original ideas, one can show that there is some flexibility in the choice of the monomial bases defining $S^{t,i}$ in order to get other non-zero minors, of $\tilde{\Psi}_t$, i.e different square matrices M'_t whose determinants are non-zero multiples of $\mathrm{Res}_{d_1,\ldots,d_n}(f_1, \ldots, f_n)$ with different extraneous factors $\det(E'_t), \det(E'_{t_n-t})$, for appropiate square submatrices E'_t, E'_{t_n-t} of M'_t. Besides the obvious choices coming from a permutation in the indices of the variables, other choices can be made as follows.

For any $i=1,\dots,n$, set $\hat{d}_i := (d_1,\dots,d_{i-1},d_{i+1},\dots,d_n)$ and define $H_{\hat{d}_i}(t)$ for any positive integer t by the equality

$$\frac{\prod_{j\neq i}\left(1-Y^{d_j}\right)}{(1-Y)^{n-1}} = \sum_{t=0}^{\infty} H_{\hat{d}_i}(t)\cdot Y^t.$$

For each $t\in\mathbb{N}_0$, set also $\Lambda_t := \{X^\gamma \in S_t:\ \gamma_j < d_j,\ j=1,\dots,n\}$.

We then have the following result:

Proposition 7.3. *Let M_t' a square submatrix of $\tilde{\Psi}_t$ of size $\rho(t)$. Denote its blocks as in* (15). *Suppose that, for each $i=1,\dots,n$, the block F_t has* exactly $H_{\hat{d}_i}(t-d_i)$ *of its columns corresponding to f_i in common with the matrix D_t defined in* (8) *and, also, the block F_{t_n-t} shares exactly $H_{\hat{d}_i}(t_n-t-d_i)$ columns corresponding to f_i* with D_{t_n-t}. *Then, if* $\det(M_t')$ *is not identically zero, the resultant* $\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n)$ *can be computed as the ratio* $\det(M_t')/\det(\mathbb{E}_t')$, *where $\mathbb{E}_t'$ is made by joining two submatrices E_t' of F_t and E_{t_n-t}' of F_{t_n-t}. These submatrices are obtained by omitting the columns in common with D_t (resp. D_{t_n-t}) and the rows indexed by all common monomials in D_t (resp. D_{t_n-t}) and all monomials in Λ_t (resp. Λ_{t_n-t}).*

We omit the proof which is rather technical, and based in [21, 6a], and [9,10].

7.5. Zeroes at infinity

Given a non-homogeneous system of polynomial equations $\tilde{f}_1,\dots,\tilde{f}_n$ in $n-1$ variables with respective degrees $d_1,\dots,d_n$, we can homogenize these polynomials and consider the resultant $\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n)$ associated with their respective homogenizations $f_1,\dots,f_n$. However, this resultant may vanish due to common zeros of $f_1,\dots,f_n$ at infinity in projective space $\mathbb{P}^{n-1}$ even when there is no affine common root to $\tilde{f}_1 = \cdots = \tilde{f}_n = 0$. We can in this case extend Canny's construction [4] of the generalised characteristic polynomial (GCP) for classical Macaulay's matrices to the matrices M_t for any natural number t. In fact, when we specialize f_i to $X_i^{d_i}$ for all $i=1,\dots,n$, the Bezoutian is given by

$$\sum_{j_1=0}^{d_1-1}\cdots\sum_{j_n=0}^{d_n-1} X_1^{d_1-1-j_1}\cdots X_n^{d_n-1-j_n} Y_1^{j_1}\cdots Y_n^{j_n},$$

and the specialized matrix $M_t(e)$ of M_t has a single non zero entry on each row and column which is equal to 1, so that $\det(M_t(e))=\pm 1$. We order the columns in such a way that $M_t(e)$ is the identity matrix. With this convention, define the polynomial $C_t(s)$ by

$$C_t(s) := \frac{\mathrm{Charpoly}(M_t)(s)}{\mathrm{Charpoly}(\mathbb{E}_t)(s)},$$

where s denotes a new variable and Charpoly means characteristic polynomial. We then have by the previous observation that

$$C_t(s) = \mathrm{Res}_{d_1,\dots,d_n}(f_1 - s x_1^{d_1},\dots,f_n - s x_1^{d_n}).$$

Moreover, this implies that $C_t(s)$ coincides with Canny's GCP $C(s)$, but involves matrices of smaller size. Canny's considerations on how to compute more efficiently the GCP also hold in this case. Of course, it is in general much better to find a way to construct "tailored" resultants for polynomials with a generic structure which is not dense, as in the case of sparse polynomial systems [14,16].

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