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Multiple solutions for a system of equations with p-Laplacian ☆

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Abstract

In the paper, by using of the Limit Index, we prove a theorem applying to get multiple critical values of some strongly indefinite nonsmooth functionals, and then we apply it to a system of equations involving the p-Laplacian.

Keywords: Limit Index; Indefinite functional; Nonsmooth; Multiple solutions; p-Laplacian

1. Introduction

In this paper, we study the existence of multiple critical points of strongly indefinite nonsmooth functionals. There are many indefinite functionals which correspond to some Hamiltonian systems, nonlinear wave equations coming from physical situations. Some of them have symmetries with respect to the action of some Lie group, such as $\mathbb{Z}_2$, S^1, and the corresponding variational problems are expected to have multiple solutions. There are some papers concerned with the topic. For example, see [1,6,10–12,15].

In [1], Benci assumed X is a Hilbert space, f satisfies $(PS)_c$ and has the form

$$f(u) = \frac{1}{2}\langle Lu, u\rangle + \Phi(u),$$

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where L is bounded self-adjoint operator and Φ' is compact. In [12], Li established a Limit Index Theory, and applied it to estimate the number of solutions of the boundary value problem

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = F_u(x,u,v), & \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \\ -\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = F_v(x,u,v), & \text{in } \Omega, \quad v|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $u, v \in W_0^{1,p}(\Omega)$, Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < N$. In [10], the authors applied Limit Index Theory to get solutions of a strongly indefinite functional in $\mathbb{R}^N$. However, in those papers, functionals are of class C^1 at least. For the nonsmooth functionals, Chang [3] considered locally Lipschitz continuous functionals in Banach spaces, and it was done in some other papers [5,7,8].

This paper is devoted to deal with the corresponding functional of (1.1) which may not locally Lipschitz continuous in Banach spaces.

The paper is organized as follows.

In Section 2, we recall some facts concerning Limit Index. In Section 3, first we introduce weak slope and the corresponding $(PS)_c$ and $(PS)_c^*$, then we prove a deformation lemma for continuous functionals satisfying $(PS)_c$ and $(PS)_c^*$. In Section 4, we establish some abstract critical point theorems which are applied to continuous functionals. In Section 5, we get solutions of (1.1). Although a similar result may be found in [12], the assumptions we put on $F(x,s,t)$ here are different.

Notation. $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. $|\cdot|_{L^p}$ is the usual norm in $L^p(\Omega)$, $L_2^p(\Omega) = L^p(\Omega) \times L^p(\Omega)$ with the norm $\|(u,v)\|_p := (|u|_{L^p}^p + |v|_{L^p}^p)^{1/p}$. Weak (respectively, strong) convergence is denoted by $\rightharpoonup$ (respectively, $\to$). C will denote constant and C may be different in different places.

2. Some facts concerning the Limit Index

Definition 2.1. (See [10, Definition 2.1].) Let G be a compact Lie group, X be a Banach space. The action of a topological group G on a normed space X is a continuous map

$$G \times X \to X : [g,x] \mapsto gx$$

such that

$$1 \cdot x = x, \quad (gh)x = g(hx), \quad x \mapsto gx \quad \text{is linear for every } g,h \in G.$$

The action is isometric if

$$\|gz\| = \|z\|.$$

Denote the fixed point set of G by

$$\operatorname{Fix} G := \{x \in X;\ gx = x,\ \forall g \in G\}.$$

A set $A \subset X$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : X \to \mathbb{R}$ is invariant if $\varphi(g(x)) = \varphi(x)$ for every $g \in G$, $x \in X$. A map $f : X \to X$ is equivariant if $f(g(x)) = g(f(x))$ for every $g \in G$, $x \in X$.

Assume X is a G-Banach space, that is, there is an isometric action G on X. Let

$$\Sigma = \{A \subset X;\ A \text{ is closed and } gA = A,\ \forall g \in G\}$$

be the family of all G-invariant closed subsets of X. Let

$$\Gamma = \bigl\{h \in C^0(X, X);\ h(gx) = g\bigl(h(x)\bigr),\ \forall g \in G\bigr\}$$

be the class of all G-equivariant mappings of X.

Definition 2.2. (See [13, p. 207].) An index for (G, Σ, Γ) is a mapping $i : \Sigma \to \mathbb{Z}_+ \cup \{+\infty\}$ (where $\mathbb{Z}_+$ is the set of all nonnegative integers), such that for all $A, B \in \Sigma$, $h \in \Gamma$ the following conditions are satisfied:

(a) $i(A) = 0 \Leftrightarrow A = \emptyset$,
(b) (*Monotonicity*) $A \subset B \Rightarrow i(A) \leqslant i(B)$, $\forall A, B \in \Sigma$,
(c) (*Subadditivity*) $i(A \cup B) \leqslant i(A) + i(B)$, $\forall A, B \in \Sigma$,
(d) (*Supervariance*) $i(A) \leqslant i(\overline{h(A)})$,
(e) (*Continuity*) if A is compact and $A \cap \operatorname{Fix} G = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighborhood N of A such that $i(N) = i(A)$.

Definition 2.3. (See [1, Definition 2.1].) An index i is said to satisfy the d-dimension property if there is a positive integer d such that

$$i\bigl(V^{dk} \cap S_1\bigr) = k,$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \operatorname{Fix} G = \{0\}$ (recall that S_1 is the unit sphere in X).

Now we recall the Limit Index introduced in [12].

Assume $U, V \in \Sigma$ such that $X = U \oplus V$, where V is infinite-dimensional and $V = \overline{\bigcup_{j=1}^{\infty} V_j}$, where V_j is a dn_j-dimensional G-invariant subspace of V, $j = 1, 2, \ldots$, and $V_1 \subset V_2 \subset \cdots$.

Let $X_j = U \oplus V_j$ and $\forall A \in \Sigma$, let $A_j = A \cap X_j$.

Definition 2.4. (See [12, Definition 2.4].) Let i be an index satisfying the d-dimension property. A limit index with respect to (X_j) induced by i is a mapping

$$i^\infty : \Sigma \to \mathbb{Z} \cup \{-\infty, +\infty\}$$

given by $i^\infty(A) = \limsup_{j\to\infty}(i(A_j) - n_j)$.

i^∞ has the following properties:

Proposition 2.5. *(See [12, Proposition 2.5].) Let $A, B \in \Sigma$, then i^∞ satisfies*

(a) $A = \emptyset \Rightarrow i^\infty(A) = -\infty$,
(b) (*Monotonicity*) *if $A \subset B$, then $i^\infty(A) \leqslant i^\infty(B)$*,

(c) (*Subadditivity*) $i^\infty(A \cup B) \leqslant i^\infty(A) + i(B)$,
(d) *if* $V \cap \operatorname{Fix} G = \{0\}$, *then* $i^\infty(S_\rho \cap V) = 0$, *where* $S_\rho = \{x \in X;\ \|x\| = \rho\}$,
(e) *if* Y_0 *and* $\tilde{Y}_0$ *are* G*-invariant closed subspaces of* V *such that* $V = Y_0 \oplus \tilde{Y}_0$, $\tilde{Y}_0 \subset V_{j_0}$ *for some* j_0 *and* $\dim \tilde{Y}_0 = dm$, *then* $i^\infty(Y_0 \cap S_\rho) \geqslant -m$.

3. Deformation properties for continuous functionals

X is a Banach space, which has the decomposition $X = \overline{\bigcup_{n=1}^\infty X_n}$, where $X_1 \subset X_2 \subset \cdots$, and X_n $(n = 1, 2, \ldots)$ are closed subspaces of X. So X_n $(n = 1, 2, \ldots)$ is also Banach space. $\|\cdot\|$ is denoted the norm of X.

Set $\mathcal{D}(f) = \{u \in X;\ f(u) < +\infty\}$. Let $f_n = f|_{X_n}$.

In the following, we recall from [5,7] some basic facts concerning the notion of the weak slope and the deformation properties for continuous functionals.

Definition 3.1. (See [5, Definition 2.1].) Let $f : X \to \mathbb{R}$ be a continuous function and $u \in X$. We denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty]$ such that there exist $\delta > 0$ and $H : B_\delta(u) \times [0, \delta] \to X$ continuous with

$$\forall v \in B_\delta(u),\ \forall t \in [0, \delta]:\quad \big\|H(v,t) - v\big\| \leqslant t, \tag{3.1}$$

$$\forall v \in B_\delta(u),\ \forall t \in [0, \delta]:\quad f\big(H(v,t)\big) \leqslant f(v) - \sigma t. \tag{3.2}$$

The extended real number $|df|(u)$ is called the weak slope of f at u.

Remark 3.2. When we replace $B_\delta(u)$ in Definition 3.1 by $B_\delta(u) \cap X_n$ and replace X by X_n, we can get the weak slope of f_n at $u \in X_n$, denoted by $|df_n|(u)$.

Definition 3.3. (See [8, Definition 2.1].) We say that $u \in \mathcal{D}(f)$ is a critical point of f, if $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a critical value of f, if there exists a critical point $u \in \mathcal{D}(f)$ of f with $f(u) = c$.

Then we let

$$\mathbb{K}_c = \big\{u \in \mathcal{D}(f);\ f(u) = c,\ |df|(u) = 0\big\},$$
$$\mathbb{K}_c^{(n)} = \big\{u \in \mathcal{D}(f_n) \cap X_n;\ f_n(u) = c,\ |df_n|(u) = 0\big\}.$$

Definition 3.4. (See [7, Definition 2.2].) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We define the function

$$\mathcal{G}_f : \operatorname{epi}(f) \to \mathbb{R}$$

putting

$$\operatorname{epi}(f) = \big\{(u, \xi) \in X \times \mathbb{R}:\ f(u) \leqslant \xi\big\} \quad \text{and} \quad \mathcal{G}_f(u, \xi) = \xi.$$

In the following $\operatorname{epi}(f)$ will be endowed with the metric

$$d\big((u,\xi),(v,\mu)\big)=\big(\|u-v\|^2+(\xi-\mu)^2\big)^{1/2}.$$

From [7, p. 75] we can see $\operatorname{epi}(f)$ is closed in $X\times\mathbb{R}$ and $\mathcal{G}_f$ is Lipschitz continuous with constant 1, $|d\mathcal{G}_f|(u,\xi)\leqslant 1$ for every $(u,\xi)\in\operatorname{epi}(f)$.

Then we present the relationship between $|df|$ and $|d\mathcal{G}_f|$.

Proposition 3.5. *(See [7, Proposition 2.3].) Let $f:X\to\mathbb{R}$ be a continuous function and let $(u,\xi)\in\operatorname{epi}(f)$. Then*

$$|d\mathcal{G}_f|(u,\xi)=\begin{cases}\frac{|df|(u)}{\sqrt{1+|df|(u)^2}}, & \text{if } f(u)=\xi \text{ and } |df|(u)<+\infty,\\ 1, & \text{if } f(u)<\xi \text{ or } |df|(u)=+\infty.\end{cases}$$

Definition 3.6. (See [8, Definition 2.3].) The functional $f\in C(X,\mathbb{R})$ is said to satisfy the Palais–Smale condition at level $c\in\mathbb{R}$ ($(PS)_c$ for short) if every sequence $\{u_n\}\subset X$, satisfying $f(u_n)\to c$, $|df|(u_n)\to 0$, $n\to\infty$, possesses a subsequence which converges in X.

Definition 3.7. The functional $f\in C(X,\mathbb{R})$ is said to satisfy Palais–Smale condition with respect to (X_n) at level $c\in\mathbb{R}$ ($(PS)_c^*$ with respect to X_n) if every sequence $\{u_{n_k}\}$ satisfying $u_{n_k}\in X_{n_k}$, $f(u_{n_k})\to c$, $|df_{n_k}|(u_{n_k})\to 0$ and $n_k\to\infty$ as $k\to\infty$ possesses a subsequence which converges in X to a critical point of f.

A condition similar to $(PS)_c^*$ above may be found in [4,9], in fact, it is a particular case of [4].

Next we recall some properties of the weak slope.

Proposition 3.8. *(See [7, Proposition 2.6].) Let $f:X\to\mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function. Then, for every sequence $\{u_n\}$ in $\mathcal{D}(f)$ converging to u with $f(u_n)$ converging to $f(u)$, we have $|df|(u)\leqslant\liminf_n|df|(u_n)$.*

If f satisfies $(PS)_c$ and f is continuous, then we can see $\mathbb{K}_c$ are compact. In fact, for any $\{u_n\}\subset\mathbb{K}_c$, $f(u_n)=c$, $|df|(u_n)=0$. Since f satisfies $(PS)_c$, $\{u_n\}$ has a subsequence converging in X. Let $u_n\to u$ after passing to a subsequence, so by f is continuous we have $f(u_n)\to f(u)$ and $f(u)=c$. Then by Proposition 3.8, $|df|(u)\leqslant\liminf_n|df|(u_n)\to 0$. So $u\in\mathbb{K}_c$ and $\mathbb{K}_c$ is compact.

Lemma 3.9. *(See [7, Theorem 2.11].) Let $f:X\to\mathbb{R}$ be a continuous function, M a closed subset of X and δ, $\sigma>0$ such that*

$$\operatorname{dist}(u,M)\leqslant\delta \quad\Rightarrow\quad |df|(u)>\sigma.$$

Then there exists a continuous map $\eta:X\times[0,\delta]\to X$ such that

$$\big\|\eta(u,t)-u\big\|\leqslant t,$$
$$f\big(\eta(u,t)\big)\leqslant f(u),$$

$$\operatorname{dist}(u, M) \geqslant \delta \quad \Rightarrow \quad \eta(u,t) = u,$$
$$u \in M \quad \Rightarrow \quad f\big(\eta(u,t)\big) \leqslant f(u) - \sigma t.$$

Proposition 3.10. *Let $f : X \to \mathbb{R}$ be a continuous function, and let $c \in \mathbb{R}$. Assume f satisfies $(PS)^*_c$ and $(PS)_c$. Then, given $\bar{\epsilon} > 0$, N a neighborhood of $\mathbb{K}_c$ and $\lambda > 0$, there exist $\epsilon_0 > 0$, $n_0 > 0$ and $\eta_n : X_n \times [0,1] \to X_n$, for any $0 < \epsilon < \epsilon_0$, $n > n_0$, η_n continuous having properties*

(a) $\|\eta_n(u,t) - u\| \leqslant \lambda t$,
(b) $f_n(\eta_n(u,t)) \leqslant f_n(u)$,
(c) $f_n(u) \notin [c - \bar{\epsilon}, c + \bar{\epsilon}] \Rightarrow \eta_n(u,t) = u$,
(d) $\eta_n(f_n^{c+\epsilon} \setminus N, 1) \subseteq f_n^{c-\epsilon}$.

Proof. First, we suppose that f is Lipschitz continuous with constant 1.

Since f satisfies $(PS)_c$, $\mathbb{K}_c$ is compact, so we can find $\delta > 0$ and $N_{2\delta} = \{u \in X, \operatorname{dist}(x, \mathbb{K}_c) < 2\delta\}$ such that $N_{2\delta} \subset N$.

Let $U^c_{n,\sigma} = \{u \in X_n;\ |f_n(u) - c| \leqslant 2\sigma,\ |df_n|(u) \leqslant \sigma\}$. It is clear that $\mathbb{K}_c \subset U^c_{n,\sigma}$.

There exist $0 < \sigma_0 < \bar{\epsilon}/2$ and $n_0 > 0$ such that whenever $n > n_0$, then $U^c_{n,\sigma_0} \subset N_\delta$. In fact, if this does not hold, then for every $k > 0$ there exists $n_k \geqslant k$ and

$$u_{n_k} \in U^c_{n_k,1/k}, \quad u_{n_k} \notin N_\delta. \tag{3.3}$$

It follows that $f(u_{n_k}) \to c$ and $|df_{n_k}|(u_{n_k}) \to 0$ as $k \to \infty$. By $(PS)^*_c$, after passing to a subsequence, we have

$$u_{n_k} \to \bar{u} \in \mathbb{K}_c.$$

This contradicts (3.3). Choose $\delta_0 = \min\{\sigma_0, \delta\}$, such that there exists $m \geqslant 1$, $m\delta_0 = \delta$, therefore, $U^c_{n,\sigma_0} \subset N_{m\delta_0}$.

In the following we assume $n > n_0$.

Let

$$M_n = \big\{u \in X_n,\ c - \sigma_0 \leqslant f_n(u) \leqslant c + \sigma_0,\ u \notin N_{2\delta}\big\}.$$

It is easy to see that M_n is the closed subsequence of X_n.

Since f is Lipschitz continuous with constant 1, we have

$$\operatorname{dist}(u, M_n) \leqslant \delta_0 \quad \Rightarrow \quad |df_n|(u) \geqslant \sigma_0.$$

Let $\eta'_n : X_n \times [0, \delta_0] \to X_n$ be a continuous map as in Lemma 3.9. We can assume, without loss of generality, $\lambda \leqslant \delta_0$ and define $\eta_n : X_n \times [0,1] \to X_n$ by $\eta_n(u,t) = \eta'_n(u, \lambda t)$.

Properties (a) and (b) are obvious. Because f is Lipschitz with constant 1, $\forall u \in X_n$, $f_n(u) \notin [c - \bar{\epsilon}, c + \bar{\epsilon}]$ implies $\operatorname{dist}(u, M_n) \geqslant \delta_0$, hence $\eta_n(u,t) = u$. Finally, set $\epsilon_0 = \min\{\sigma_0\lambda/2, \sigma_0\}$, whenever, $0 < \epsilon < \epsilon_0$, if $u \in f_n^{c+\epsilon} \setminus N$ and $f_n(u) \geqslant c - \epsilon \geqslant c - \sigma_0$, it follows $u \in M_n$, hence

$$f_n\big(\eta_n(u,1)\big) = f_n\big(\eta'_n(u,\lambda)\big) \leqslant f_n(u) - \sigma_0\lambda \leqslant c + \epsilon - \sigma_0\lambda \leqslant c - \epsilon.$$

If $u \in f_n^{c+\epsilon} \setminus N$ and $f_n(u) \leqslant c - \epsilon$, we deduce from (b) that $f_n(\eta_n(u,1)) \leqslant c - \epsilon$.

Now let us consider the general case. Being closed in $X_n \times \mathbb{R}$, $\mathrm{epi}(f_n)$ is complete. Let us denote by $\tilde{K}_c$ the set of critical points of $\mathcal{G}_f$ at level c. By Proposition 3.5, the function $\mathcal{G}_f$ satisfies $(PS)_c$ and $(PS)_c^*$ with respect to $(\mathrm{epi}(f_n))$. Moreover, $(N \times \mathbb{R}) \cap \mathrm{epi}(f)$ is a neighborhood of $\tilde{K}_c$ and $\mathcal{G}_f$ is Lipschitz continuous with constant 1.

By the previous step, we can find $\epsilon_0 > 0$, $n_0 > 0$, when $0 < \epsilon < \epsilon_0$ and $n > n_0$, there exist continuous maps

$$\tilde{\eta}_n = \left(\tilde{\eta}_n^{(1)}, \tilde{\eta}_n^{(2)}\right) : \mathrm{epi}(f_n) \times [0,1] \to \mathrm{epi}(f_n)$$

such that

$$\begin{gathered} d\left(\tilde{\eta}_n\left((u,\xi),t\right),(u,\xi)\right) \leqslant \lambda t, \\ \tilde{\eta}_n^{(2)}\left((u,\xi),t\right) \leqslant \xi, \\ \xi \notin [c-\bar{\epsilon}, c+\bar{\epsilon}] \quad \Rightarrow \quad \tilde{\eta}_n\left((u,\xi),t\right) = (u,\xi), \\ \xi \leqslant c+\epsilon, \quad u \notin N \quad \Rightarrow \quad \tilde{\eta}_n^{(2)}\left((u,\xi),1\right) \leqslant c-\epsilon. \end{gathered}$$

Let us define $\eta_n : X_n \times [0,1] \to X_n$ by $\eta_n(u,t) = \tilde{\eta}_n^{(1)}((u, f_n(u)),t)$. Since $\tilde{\eta}_n$ takes its values in $\mathrm{epi}(f_n)$, we have

$$f_n\left(\tilde{\eta}_n^{(1)}\left(\left(u, f_n(u)\right),t\right)\right) \leqslant \tilde{\eta}_n^{(2)}\left(\left(u, f_n(u)\right),t\right).$$

Then (a), (b), (c) and (d) easily follow. □

Now we define a group action of $G = \{1, \tau_1\} \cong \mathbb{Z}_2$ by setting $\tau_1(u,v) = (-u,-v)$. Then we have the following deformation property.

Proposition 3.11. *Suppose that all the assumptions of Proposition* 3.10 *hold, f is G-invariant and $|df_n|(0) = 0$. Then, given $\bar{\epsilon} > 0$, N a neighborhood of $\mathbb{K}_c$ and $\lambda > 0$, there exist $\epsilon_0 > 0$, $n_0 > 0$, $\eta_n : X_n \times [0,1] \to X_n$ continuous, such that whenever $0 < \epsilon < \epsilon_0$ and $n > n_0$, having properties* (a) *to* (d) *of Proposition* 3.10 *and*

(e) *$\eta_n(\cdot,t) : X_n \to X_n$ is G-equivariant for each $t \in [0,1]$.*

Proof. Since f is G-invariant, f_n is also G-invariant. So $|df_n|(-u) = |df_n|(u)$, for all $u \in X_n$.

For any $n \in \mathbb{N}$, if $|df_n|(u) > \sigma$, let $\delta_n > 0$ and $H_n : B_{\delta_n}(u) \times [0,\delta_n] \to X_n$ be a continuous map satisfying (3.1) and (3.2). Since $|df_n|(0) = 0$, we can suppose $\delta_n < \|u\|$.

Then

$$\tilde{H}_n : \left(\left(B_{\delta_n}(u) \cap X_n\right) \cup \left(B_{\delta_n}(-u) \cap X_n\right)\right) \times [0,\delta_n] \to X_n$$

defined by

$$\tilde{H}_n(v,t) = \begin{cases} H(v,t), & \text{if } v \in B_{\delta_n}(u) \cap X_n, \\ -H(-v,t), & \text{if } v \in B_{\delta_n}(-u) \cap X_n, \end{cases}$$

is continuous, G-invariant with respect to the first variable and satisfies (3.1) and (3.2).

Then, all the constructions of Theorem 2.8 in [7], Lemma 3.9 and Proposition 3.10 can be repeated in a symmetric way, yielding the result. □

Proposition 3.12. *Let X and X_n $(n = 1, 2, \ldots)$ be G-spaces. Suppose $f \in C(X, \mathbb{R})$, satisfies $(PS)_c$ and $(PS)_c^*$ with respect to (X_n) and $|df_n|(0) = 0$. Assume that f is G-invariant and N is a G-invariant closed neighborhood of $\mathbb{K}_c$. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,*

$$i^\infty\big(f^{c+\epsilon}\big) \leqslant i^\infty\big(f^{c-\epsilon}\big) + i(N).$$

Proof. From Proposition 3.11, $\exists \epsilon_0 > 0$, $\exists n_0 > 0$, for $\forall \epsilon \in (0, \epsilon_0)$ and $\forall n > n_0$, there exist η_n satisfying

$$\eta_n\big(f_n^{c+\epsilon} \setminus N, 1\big) \subset f_n^{c-\epsilon} \quad \text{and} \quad \eta_n(\cdot, 1) \text{ is } G\text{-equivariant.}$$

By Definition 2.2(b), (d), we have

$$i\big(\overline{f_n^{c+\epsilon} \setminus N}\big) \leqslant i\big(\overline{\eta_n\big(\overline{f_n^{c+\epsilon} \setminus N}, 1\big)}\big) \leqslant i\big(f_n^{c-\epsilon}\big).$$

Therefore, by the definition of the Limit Index, we get

$$i^\infty\big(\overline{f_n^{c+\epsilon} \setminus N}\big) \leqslant i^\infty\big(f_n^{c-\epsilon}\big).$$

According to the subadditivity of the Limit Index,

$$i^\infty\big(f^{c+\epsilon}\big) \leqslant i^\infty\big(f^{c-\epsilon}\big) + i(N). \qquad \square$$

4. Abstract critical point theorem

In this section, by using the Limit Index, we present critical point theorems. First we make the following assumptions, where (f_2)–(f_4) and (f_6)–(f_8) appeared in [12]:

- (f_1) $f \in C(X, \mathbb{R})$ is G-invariant,
- (f_2) there are G-invariant closed subspaces U and V such that V is infinite-dimensional and $X = U \oplus V$,
- (f_3) there is a sequence of G-invariant finite-dimensional subspaces $V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots$, $\dim V_j = dn_j$, such that $V = \overline{\bigcup_{j=1}^\infty V_j}$,
- (f_4) there is an index i on X satisfying the d-dimension property,
- (f_5) let $X_n = U \oplus V_n$, $f_n = f|_{X_n}$, such that $|df_n|(0) = 0$,
- (f_6) there are G-invariant subspaces Y_0, $\tilde{Y}_0$, Y_1 of V such that $V = Y_0 \oplus \tilde{Y}_0$, $Y_1, \tilde{Y}_0 \subset V_{j_0}$ for some j_0 and $md = \dim \tilde{Y}_0 < kd = \dim Y_1$,
- (f_7) there are ξ_1 and ξ_2, $\xi_1 < \xi_2$ such that f satisfies $(PS)_c$ and $(PS)_c^*$ with respect to (X_n), $\forall c \in [\xi_1, \xi_2]$,

$$(f_8) \qquad \begin{cases} \text{(a) } \operatorname{Fix} G \subset U \oplus Y_1 \text{ or } \operatorname{Fix} G \cap V = \{0\}, \\ \text{(b) there is } \rho > 0 \text{ such that } f(x) \geqslant \xi_1,\ \forall x \in Y_0 \cap S_\rho, \\ \text{(c) } f(x) \leqslant \xi_2,\ \forall x \in U \oplus Y_1. \end{cases}$$

Then we have the following abstract critical point theorems, which are similar with Theorems 4.1 and 4.2 in [12].

Theorem 4.1. *Assume* (f_1)–(f_5)*, let* i^∞ *be a Limit Index corresponding to* i*. Set*

$$c_k = \inf_{i^\infty(A) \geqslant k} \sup_{x \in A} f(x),$$

where $A \in \Sigma$*. Suppose* f *satisfies* $(PS)_c^*$ *with respect to* (X_n) *and* $(PS)_c$*. If* $c = c_k$ *is finite, it is a critical value of* f*. Moreover, if* $c = c_k = \cdots = c_{k+p}$*, for some* $p \geqslant 0$*, then* $i(\mathbb{K}_c) \geqslant p + 1$*.*

The proof is similar to that of Theorem 4.1 in [12]. So we omit it.

Theorem 4.2. *Assume* (f_1)–(f_8)*. If* i^∞ *is a limit index corresponding to* i*, then the numbers*

$$c_j = \inf_{i^\infty(A) \geqslant j} \sup_{x \in A} f(x), \quad -k + 1 \leqslant j \leqslant -m,$$

are critical values of f *and* $\xi_1 \leqslant c_{-k+1} \leqslant \cdots \leqslant c_{-m} \leqslant \xi_2$*. Moreover, if* $c = c_l = \cdots = c_{l+r}$*,* $r \geqslant 0$*, then* $i(\mathbb{K}_c) \geqslant p + 1$*.*

The proof is the same as [12, Theorem 4.2] and we omit it.

5. An application

Let $Z = W_0^{1,p}(\Omega)$, $p > 1$, the usual Sobolev space (of real-valued functions) with the norm $\|u\| = (\int_\Omega |\nabla u|^p \, dx)^{1/p}$.

In this section we consider the system of equations

$$(S) \qquad \begin{cases} \Delta_p u = F_s(x, u, v) & \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \\ -\Delta_p v = F_t(x, u, v) & \text{in } \Omega, \quad v|_{\partial\Omega} = 0 \end{cases}$$

(where $F = F(x, s, t)$, $F_s = \partial F/\partial s$, $F_t = \partial F/\partial t$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$). Assume

(F_1) $F \in C(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$, F_s, F_t exist, $F_s(x, s, t)$ is continuous with respect to (s, t), $F_t(x, s, t)$ is continuous with respect to t,

(F_2) $F(x, s, t)$ is convex with respect to (s, t),

(F_3) there exist $a_1, a_2 \in \mathbb{R}^+$, such that

$$\left|F_s(x, s, t)\right| + \left|F_t(x, s, t)\right| \leqslant a_1\left(|s|^r + |t|^r\right) + a_2,$$

where $1 \leqslant r < Np/(2(N - p))$ if $N > p$, $1 \leqslant r < \infty$, otherwise.

(F_4) $\exists M > 0$ and $\mu > p$ such that

$$0 < \mu F(x, s, t) \leqslant s F_s(x, s, t) + t F_t(x, s, t), \quad \forall (x, s, t) \in \overline{\Omega} \times \mathbb{R}^2 \text{ with } s^2 + t^2 \geqslant M^2,$$

(F_5) $F(x, -s, -t) = F(x, s, t)$.

Observe that (F_4) implies that [14, p. 88]

$$F(x,s,t) \geqslant b_1\big(|s|^\mu + |t|^\mu\big) - b_2, \tag{5.1}$$

where $b_1 > 0$. And from (F_3)–(F_4) we can get when $s^2+t^2 \geqslant M^2$, without loss of generality, we can assume $|s| \geqslant 1$, $|t| \geqslant 1$,

$$\begin{aligned} F(x,s,t) &\leqslant \frac{1}{\mu}(sF_s + tF_t) \\ &\leqslant \frac{1}{\mu}\big(|s|+|t|\big)(F_s+F_t) \\ &\leqslant \frac{1}{\mu}a\big(|s|^r+|t|^r\big)\big(|s|+|t|\big) \\ &\leqslant C\big(|s|^{r+1}+|t|^{r+1}\big). \end{aligned}$$

When $s^2+t^2 \leqslant M^2$, because $F(x,s,t)$ is continuous, we have

$$F(x,s,t) \leqslant C.$$

In a word, for all (s,t), there exists $C>0$, such that

$$F(x,s,t) \leqslant C\big(|s|^{r+1}+|t|^{r+1}+1\big). \tag{5.2}$$

Define

$$\begin{aligned} I(u,v) &= -\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \int_\Omega F(x,u,v)\,dx \\ &= I_1(u,v) + I_2(u,v), \end{aligned}$$

where

$$\begin{aligned} I_1(u,v) &= -\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{p}\int_\Omega |\nabla v|^p\,dx, \\ I_2(u,v) &= -\int_\Omega F(x,u,v)\,dx. \end{aligned}$$

From (F_2) we can see I_2 is convex with respect to (u,v).

Lemma 5.1. *Assume that $1 \leqslant m,n < \infty$, $f \in C(\overline{\Omega}\times\mathbb{R}^2)$ and $f(x,u,v) \leqslant C(|u|^{m/n} + |v|^{m/n}+1)$. Then, for every $(u,v) \in L_2^m(\Omega)$ the operator $T_1 : (u,v) \mapsto f(x,u,v)$ is continuous from $L_2^m(\Omega)$ to $L_2^n(\Omega)$.*

The proof is similar to [17, Theorem A.2]. We omit it.

According to Lemma 5.1 and (5.2), we can see that $I_2(u,v) \in C(Z\times Z,\mathbb{R})$.

Now we recall some definitions and their properties.

Definition 5.2. (See [2, Definition 2.26].) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function, $\forall u_0 \in \operatorname{int}(\mathcal{D}(f))$, $\alpha \in X^*$ is called a subgradient of f at u_0 if

$$f(u_0) + \langle \alpha, u - u_0 \rangle \leqslant f(u), \quad \forall u \in X.$$

The set of all subgradients at u_0 is called the subdifferential of f at u_0 and is denoted by $\partial f(u_0)$.

Proposition 5.3. *(See [8, Proposition 3.1].) Let $f_1 : X \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function, $f_2 : X \to \mathbb{R}$ a function of class C^1 and let $f = f_1 + f_2$. Then for every $u \in \mathcal{D}(f)$,*

$$|df|(u) = \begin{cases} \min\{\|f_1^* + f_2'(u)\|:\ f_1^* \in \partial f_1(u)\}, & \text{if } \partial f_1(u) \neq \emptyset, \\ +\infty, & \text{if } \partial f_1(u) = \emptyset. \end{cases}$$

Proposition 5.4. *(See [2, pp. 84–85].)*

(1) *If $u_0 \in \operatorname{int}(\mathcal{D}(f))$, then $\partial f(u_0) \neq \emptyset$.*
(2) *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex and is G-differentiable at a point $u_0 \in \operatorname{int}(\mathcal{D}(f))$, then $\partial f(u_0)$ is a single point α satisfying $\langle \alpha, h \rangle = df(u_0, h)$.*

Then we have the following proposition.

Proposition 5.5. *Assume F satisfies* (F_1)–(F_5). *Set $\mathcal{D}(I) = \{(u, v) \in E;\ I(u, v) < +\infty\}$. Then for $\forall (u, v) \in \operatorname{int}(\mathcal{D}(I))$, $\forall (\tilde{u}, \tilde{v}) \in Z \times Z$, $|dI|(u, v) = \|\alpha(u, v)\| = \|I_1'(u, v) + \alpha_2(u, v)\|$, where $\langle \alpha_2(u, v), (\tilde{u}, \tilde{v}) \rangle = -\int_\Omega F_s(x, u, v)\tilde{u}\, dx - \int_\Omega F_t(x, u, v)\tilde{v}\, dx$.*

Proof. $I_1(u, v)$ is Fréchet differentiable at (u, v) and for any $\tilde{u}, \tilde{v} \in Z$,

$$\big\langle I_1'(u, v), (\tilde{u}, \tilde{v}) \big\rangle = -\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \tilde{u}\, dx + \int_\Omega |\nabla v|^{p-2} \nabla v \nabla \tilde{v}\, dx.$$

From [18, Proposition 26.12] we have $I_1'(u, v)$ is continuous.

(1) First we want to prove that the Gâteaux differential of $I_2(u, v)$ at $(\tilde{u}, \tilde{v})$ exists:

$$\begin{aligned} dI_2\big((u, v), (\tilde{u}, \tilde{v})\big) &= \lim_{\lambda \to 0} \frac{I_2(u + \lambda \tilde{u}, v + \lambda \tilde{v}) - I_2(u, v)}{\lambda} \\ &= \lim_{\lambda \to 0} -\frac{\int_\Omega [F(x, u + \lambda \tilde{u}, v + \lambda \tilde{v}) - F(x, u, v)]\, dx}{\lambda} \\ &\overset{\theta_1, \theta_2 \in (0,1)}{=} -\lim_{\lambda \to 0} \left[\int_\Omega \big(F_s(x, u + \lambda\theta_1 \tilde{u}, v + \lambda \tilde{v})\tilde{u} + F_t(x, u, v + \lambda\theta_2 \tilde{v})\tilde{v} \big)\, dx \right]. \end{aligned}$$

If λ is small enough, from (F_3) we have

$$\begin{aligned}
&\left|F_s(x,u+\lambda\theta\tilde{u},v+\lambda\tilde{v})\tilde{u}+F_t(x,u,v+\lambda\theta_2\tilde{v})\tilde{v}\right|\\
&\quad\leqslant a\left[\left(|u|+|\tilde{u}|\right)^r+\left(|v|+|\tilde{v}|\right)^r\right]|\tilde{u}|+a\left[|u|^r+\left(|v|+|\tilde{v}|\right)^r\right]|\tilde{v}|+C\\
&\quad\leqslant C_1|u|^r|\tilde{u}|+C_2|\tilde{u}|^{r+1}+C_3|v|^r|\tilde{u}|+C_4|\tilde{v}|^r|\tilde{u}|+C_5|u|^r|\tilde{v}|\\
&\qquad+C_6|v|^r|\tilde{v}|+C_7|\tilde{v}|^{r+1}+C_8.
\end{aligned}$$

Since $r<\frac{Np}{2(N-p)}$, by Hölder's inequality, the right-hand side above is of class L^1, so it follows from Lebesgue dominated convergence theorem that

$$dI_2\big((u,v),(\tilde{u},\tilde{v})\big)=-\int_\Omega\left[F_s(x,u,v)\tilde{u}+F_t(x,u,v)\tilde{v}\right]dx.$$

Then by Proposition 5.4,

$$\partial I_2(u,v)=\left\{\alpha_2(u,v)\right\},$$

where $\langle\alpha_2(u,v),(\tilde{u},\tilde{v})\rangle=-\int_\Omega[F_s(x,u,v)\tilde{u}+F_t(x,u,v)\tilde{v}]\,dx$.

(2) Because I_2 is a convex continuous function and I_1 is of class C^1, from Proposition 5.5, $\forall(u,v)\in\operatorname{int}(\mathcal{D}(I))$,

$$\begin{aligned}
|dI|(u,v)&=\min\left\{\left\|I_1'(u,v)+\alpha_2(u,v)\right\|:\ \alpha_2(u,v)\in\partial I_2(u,v)\right\}\\
&=\left\|I_1'(u,v)+\alpha_2(u,v)\right\|.\qquad\square
\end{aligned}$$

Now in order to prove that I satisfies $(PS)_c^*$ and $(PS)_c$, we recall some properties of the Banach space Z. According to [16, Section 4.9.4] there exists a Schauder basis $\{e_n\}_{n=1}^\infty$. Denote

$$Z_n=\operatorname{span}\{e_1,\dots,e_n\},\qquad Z_n^\perp=\overline{\operatorname{span}}\{e_{n+1},\dots\}.$$

Let $P_n:Z\to Z_n$ be the projector corresponding to the decomposition $Z=Z_n\oplus Z_n^\perp$. Then we set $E=Z\times Z$, $E_n=Z\times Z_n$.

Note that E_n is a closed subspace of E, so E_n $(n=1,2,\dots)$ is a Banach space. Set $\mathcal{D}(I_n)=\{(u,v)\in E_n;\ I(u,v)<+\infty\}$. Then it is easy to see that for every $(u,v)\in\mathcal{D}(I_n)$, $|dI_n|(u,v)=\|\alpha_n(u,v)\|=\|{I_n^{(1)}}'(u,v)+\alpha_n^{(2)}(u,v)\|$, where $I_n^{(1)}(u,v)=-\frac{1}{p}\int_\Omega|\nabla u|^p\,dx+\frac{1}{p}\int_\Omega|\nabla v|^p\,dx$ and $\langle\alpha_n^{(2)}(u,v),(\tilde{u},\tilde{v})\rangle=-\int_\Omega F_s(x,u,v)\tilde{u}\,dx-\int_\Omega F_t(x,u,v)\tilde{v}\,dx$, where $(\tilde{u},\tilde{v})\in E_n$.

Let $\Sigma=\{A\subset E;\ A$ is closed and $(u,v)\in A\Rightarrow(-u,-v)\in A\}$.

Define an index γ on Σ by:

$$\gamma(A)=\begin{cases}\min\{n\in\mathbb{Z}_+;\ \exists h\in C(A,\mathbb{R}^n\setminus\{0\})\text{ such that }h(-u,-v)=h(u,v)\},\\ 0,\quad\text{if }A=\emptyset,\\ \infty,\quad\text{if such }h\text{ does not exist.}\end{cases}$$

The index γ satisfies the one-dimension property. γ^∞ is well defined with respect to E_n from γ [10, Remark 3.7].

Next we will prove $I(u,v)$ satisfies $(PS)_c$ and $(PS)_c^*$.

Proposition 5.6. *Assume that F satisfies* (F_1)–(F_4) *with $\mu = r + 1$. For every $(u_n, v_n) \in E$, if $I(u_n, v_n) \to c$, $|dI|(u_n, v_n) \to 0$, $n \to \infty$, then $\{(u_n, v_n)\}$ has a convergence subsequence.*

Proof. First we shall prove $\{(u_n, v_n)\}$ is bounded in E.

Because $I(u_n, v_n) \to c$, without loss of generality, we can assume $(u_n, v_n) \in \operatorname{int}(\mathcal{D}(I))$.

By $|dI|(u_n, v_n) \to 0$ and $|dI|(u_n, v_n) = \|\alpha(u_n, v_n)\| = \|I_1'(u_n, v_n) + \alpha_2(u_n, v_n)\|$, we can see $\alpha(u_n, v_n) = I_1'(u_n, v_n) + \alpha_2(u_n, v_n) \to 0$.

(F_3)–(F_4) imply that

$$\begin{aligned}
C + \|u_n\| + \|v_n\| &\geqslant I(u_n, v_n) - \frac{1}{p}\big\langle \alpha(u_n, v_n), (u_n, v_n)\big\rangle \\
&= \frac{1}{p}\int_\Omega F_s(x, u_n, v_n)u_n\,dx + \frac{1}{p}\int_\Omega F_t(x, u_n, v_n)v_n\,dx \\
&\quad - \frac{1}{p}\int_\Omega F(x, u_n, v_n)\,dx \\
&\geqslant \left(\frac{\mu}{p} - 1\right)\int_\Omega F(x, u_n, v_n)\,dx \\
&\geqslant \left(\frac{\mu}{p} - 1\right)b_1\int_\Omega \big(|u_n|^\mu + |v_n|^\mu\big)\,dx + C \\
&= \left(\frac{\mu}{p} - 1\right)b_1\big(\|u_n\|_{L^\mu}^\mu + \|v_n\|_{L^\mu}^\mu\big) + C. \qquad (5.3)
\end{aligned}$$

On the other hand,

$$\|u_n\| \geqslant \big\langle -\alpha(u_n, v_n), (u_n, 0)\big\rangle = \|u_n\|^p + \int_\Omega F_s(x, u_n, v_n)u_n\,dx,$$

$$\|v_n\| \geqslant \big\langle \alpha(u_n, v_n), (0, v_n)\big\rangle = \|v_n\|^p - \int_\Omega F_t(x, u_n, v_n)v_n\,dx.$$

Using (F_3), we obtain

$$\begin{aligned}
\|u_n\|^p + \|v_n\|^p &\leqslant \int_\Omega \big(-F_s(x, u_n, v_n)u_n + F_t(x, u_n, v_n)v_n\big)\,dx + \|u_n\| + \|v_n\| \\
&\leqslant C_1\int_\Omega \big(|u_n|^{r+1} + |v_n|^{r+1}\big)\,dx + \|u_n\| + \|v_n\| + C_2 \\
&= C_1\big(\|u_n\|_{L^{r+1}}^{r+1} + \|u_n\|_{L^{r+1}}^{r+1}\big) + \|u_n\| + \|v_n\| + C_2 \\
&\leqslant C_1\big(\|u_n\|_{L^\mu}^{r+1} + \|v_n\|_{L^\mu}^{r+1}\big) + \|u_n\| + \|v_n\| + C_2. \qquad (5.4)
\end{aligned}$$

From (5.3), (5.4), we can get

$$\|u_n\|^p + \|v_n\|^p \leqslant C_1\big(\|u_n\| + \|v_n\|\big) + C_2.$$

And so $\{\|u_n\| + \|v_n\|\}$ is bounded because $p > 1$.

Going if necessary to a subsequence, we can assume $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$. Then we need to prove $u_n \to u$, $v_n \to v$ in $W_0^{1,p}(\Omega)$.

According to Sobolev embedding theorem, we have

$$\begin{aligned} &u_n \to u, \quad \text{in } L^q(\Omega), \\ &v_n \to v, \quad \text{in } L^q(\Omega), \quad 1 \leqslant q \leqslant Np/(N-p). \end{aligned}$$

From (F$_3$), $|F_s(x,u,v)| \leqslant a_1(|u|^r + |v|^r) + a_2 = a_1(|u|^{2r/2} + |v|^{2r/2}) + a_2$.

Since $1 \leqslant r < Np/(2(N-p))$, by Lemma 5.1, we can see

$$F_s(x,u_n,v_n) \to F_s(x,u,v) \quad \text{in } L^2(\Omega).$$

And therefore,

$$\begin{aligned} &\left|\int_\Omega F_s(x,u_n,v_n)(u_n-u)\,dx\right| \\ &\quad \leqslant \left(\int_\Omega \big(F_s(x,u_n,v_n)\big)^2\,dx\right)^{1/2}\left(\int_\Omega |u_n-u|^2\,dx\right)^{1/2} \to 0, \\ &\left|\int_\Omega F_t(x,u_n,v_n)(v_n-v)\,dx\right| \\ &\quad \leqslant \left(\int_\Omega \big(F_s(x,u_n,v_n)\big)^2\,dx\right)^{1/2}\left(\int_\Omega |v_n-v|^2\,dx\right)^{1/2} \to 0. \end{aligned}$$

Note that

$$\begin{aligned} \big\langle -\alpha(u_n,v_n),(u_n-u,0)\big\rangle &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \\ &\quad + \int_\Omega F_s(x,u_n,v_n)(u_n-u)\,dx \\ &= o(1). \end{aligned}$$

So

$$\langle -\Delta_p u_n, u_n - u\rangle = -\int_\Omega F_s(x,u_n,v_n)(u_n-u)\,dx + o(1) \to 0. \tag{5.5}$$

Let $\varphi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx$, $u \in W_0^{1,p}(\Omega)$. Observe that

$$\varphi'(u_n) = -\Delta_p u_n \in W^{-1,p'}(\Omega).$$

$\varphi(u)$ being convex on $W_0^{1,p}(\Omega)$, we have

$$\varphi(u) - \varphi(u_n) \geqslant \langle \varphi'(u_n), u - u_n \rangle = \langle -\Delta_p u_n, u - u_n \rangle.$$

Combining this inequality and (5.5), one refers

$$\liminf_{n\to\infty}\bigl(\varphi(u) - \varphi(u_n)\bigr) \geqslant 0. \tag{5.6}$$

On the other hand, the convexity of φ and $u_n \rightharpoonup u$ imply

$$\liminf_{n\to\infty} \varphi(u_n) \geqslant \varphi(u). \tag{5.7}$$

Then (5.6), (5.7) force $\varphi(u) = \lim_{n\to\infty}\varphi(u_n)$, as it is, $|\nabla u_n|_{L^p} \to |\nabla u|_{L^p}$.

Finally, since the norm on $W_0^{1,p}(\Omega)$ is uniformly convex, we deduce from $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $|\nabla u_n|_{L^p} \to |\nabla u|_{L^p}$ that $u_n \to u$ in $W_0^{1,p}(\Omega)$.

The same argument can be took to show $v_n \to v$ in $W_0^{1,p}(\Omega)$. □

Proposition 5.7. *Assume that F satisfies* (F$_1$)–(F$_4$) *with $\mu = r+1$. For every $(u_{n_k}, v_{n_k}) \in E_{n_k}$, if $I(u_{n_k}, v_{n_k}) \to c$, $|dI_{n_k}|(u_{n_k}, v_{n_k}) \to 0$, $n_k \to \infty$ as $k \to \infty$, then $\{(u_{n_k}, v_{n_k})\}$ has a convergence subsequence.*

Proof. From the same way as Proposition 5.6, we can get $\{(u_{n_k}, v_{n_k})\}$ is bounded in E. Then going if necessary to a subsequence, we can assume $u_{n_k} \rightharpoonup u$, $v_{n_k} \rightharpoonup v$ in $W_0^{1,p}(\Omega)$.

Note that

$$\begin{aligned}\bigl\langle \alpha_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k} - v)\bigr\rangle &= \bigl\langle \alpha_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k} - P_{n_k}v)\bigr\rangle \\ &\quad + \bigl\langle \alpha_{n_k}(u_{n_k}, v_{n_k}), (0, P_{n_k}v - v)\bigr\rangle\end{aligned}$$

and $P_{n_k}v \to v$ as $n_k \to \infty$.

Since $\alpha_{n_k}(u_{n_k}, v_{n_k}) \to 0$, $\{\alpha_{n_k}(u_{n_k}, v_{n_k})\}$ is a bounded set. And so

$$\bigl\langle \alpha_{n_k}(u_{n_k}, v_{n_k}), (0, P_{n_k}v - v)\bigr\rangle \to 0.$$

Finally, we can obtain $u_{n_k} \to u$, $v_{n_k} \to v$ in $W_0^{1,p}(\Omega)$ by the same argument as Proposition 5.6. □

The main result of this section is the following.

Theorem 5.8. *Suppose that F satisfies* (F$_1$)–(F$_4$) *with $\mu = r+1$, then the equations (S) possess an unbounded (in E and $L^\infty(\Omega) \times L^\infty(\Omega)$) sequence of weak solutions.*

Proof. We shall verify that I satisfies the hypotheses of Theorem 4.2.

(f_1) It is obviously that I is invariant with respect to the action of G.

(f_2) Set

$$E = U \oplus V, \qquad U = X \times \{0\}, \qquad V = \{0\} \times X,$$

U and V are G-invariant closed subspaces.

(f_3) $Z = \overline{\bigcup_{n=1}^{\infty} Z_n}$, $V = \overline{\bigcup_{n=1}^{\infty}(\{0\} \times Z_n)}$, $\{0\} \times Z_n$ $(n = 1, 2, \ldots)$ is G-invariant closed subspaces.

(f_4) γ satisfies one-dimension property.

(f_5) The weak slope of $I|_{E_n}$ at $(0,0)$ is 0, that is, $|dI_n|(0,0) = 0$.

(f_6) Set

$$Y_0 = \{0\} \times Z_m^{\perp}, \qquad Y_1 = \{0\} \times Z_k,$$

where m and k are to be determined. It is clear that Y_0, Y_1 are G-invariant and $\operatorname{codim}_V Y_0 = m$, $\dim Y_1 = k$.

(f_7) According to Propositions 5.6 and 5.7, we can see I satisfies $(PS)_c$ and $(PS)_c^*$ with respect to (E_n).

(f_8) (a) $\operatorname{Fix} G = \{0\}$, $\operatorname{Fix} G \cap V = \{0\}$.

(b) Note that

$$\begin{aligned} I(u,0) &= -\frac{1}{p}\|u\|^p - \int_\Omega F(x,u,0)\,dx \\ &\leqslant -\frac{1}{p}\|u\|^p - b_1 \int_\Omega |u|^\mu\,dx + C \quad \text{from } (F_4). \end{aligned}$$

Therefore, $\sup_{u\in Z} I(u,0) < +\infty$. Choose ξ_1 such that $\xi_1 > \sup_{u\in Z} I(u,0)$.

If $(0,0) \in Y_0 \cap S_\rho$ (where ρ is to be determined), then

$$\begin{aligned} I(u,0) &= \frac{1}{p}\|v\|^p - \int_\Omega F(x,0,v)\,dx \\ &= \frac{1}{p}\rho^p - \int_\Omega F(x,0,v)\,dx \\ &\geqslant \frac{1}{p}\rho^p - C_1 \int_\Omega |v|^{r+1}\,dx + C_2. \end{aligned} \tag{5.8}$$

For each ρ fixed, $\forall (0,v) \in S_\rho \cap V$ we have $(\mathrm{id} - P_m)v \to 0$, as $m \to \infty$ by the properties of Schauder bases.

Since the embedding $Z \hookrightarrow L^{r+1}$ is compact, $S_\rho \cap V$ is a compact set in the L^{r+1}-topology and $|(\mathrm{id} - P_m)v|_{L^{r+1}} \to 0$ uniformly in $v \in S_\rho \cap V$.

Hence $\forall\epsilon > 0$, $\exists m_0 > 0$ such that whenever $m > m_0$, we have

$$\left|(\mathrm{id} - P_m)v\right|_{L^{r+1}} < \epsilon.$$

Thus we can choose first ρ and then m such that $\int_\Omega |v|^{r+1}\,dx$ in (5.8) is small enough and $I|_{Y_0\cap S_\rho} \geqslant \xi_1$.

(c) For each $(u, v) \in U \oplus Y_1$,

$$\begin{aligned} I(u, v) &\leqslant \frac{1}{p}\|u\|^p + \frac{1}{p}\|v\|^p - b_1 \int_\Omega \left(|u|^\mu + |v|^\mu\right) dx + C \\ &\leqslant \frac{1}{p}\|v\|^p - b_1 \int_\Omega |v|^\mu\,dx + C. \end{aligned}$$

Since all norms are equivalent in the finite-dimensional space Y_1, we obtain

$$I(u, v) \leqslant \frac{1}{p}\|v\|^p - C_1\|v\|^\mu + C_2, \quad \text{where } C_1 > 0.$$

Since $\mu > p$, we have $\sup I|_{U\oplus Y_1} < +\infty$. Thus we can choose $k > m$ and $\xi_2 > \xi_1$ such that $f|_{U\oplus Y_1} \leqslant \xi_2$. According to Theorem 4.2, set

$$c_j = \inf_{\gamma^\infty(A)\geqslant j}\ \sup_{(u,v)\in A} I(u, v), \quad -k + 1 \leqslant j \leqslant -m,$$

and $\xi_1 \leqslant c_j \leqslant \xi_2$, we obtain a critical value $c_j \geqslant \xi_1$.

Because ξ_1 can be chosen arbitrarily large, I has a sequence of critical values $c_n \to \infty$. Finally, let (u_n, v_n) be a critical point corresponding to c_n, then

$$c_n = I(u_n, v_n) - \frac{1}{p}\big\langle \alpha(u_n, v_n), (u_n, v_n)\big\rangle.$$

Furthermore, it follows from (5.2) that F is bounded below, so we have

$$\begin{aligned} c_n &\leqslant \frac{1}{p}\int_\Omega \big(F_s(x, u_n, v_n)u_n + F_t(x, u_n, v_n)v_n\big)\,dx + C \\ &\leqslant C_1 \int_\Omega \left(|u_n|^{r+1} + |v_n|^{r+1}\right) dx + C_2 \\ &= C_1\left(|u_n|^{r+1}_{L^{r+1}} + |v_n|^{r+1}_{L^{r+1}}\right) + C_2, \quad \text{where } C_1 > 0. \end{aligned}$$

Therefore, $\{(u_n, v_n)\}$ is unbounded in E and in $L^\infty(\Omega) \times L^\infty(\Omega)$. Since each of the critical points of I is a weak solution of (S), the conclusion follows. □

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