NORTH-HOLLAND

Some Solutions to the Matrix Equation for Three Point Nevanlinna-Pick Interpolation on the Bidisc

Linda J. Patton
Department of Mathematics,
California Polytechnic State University,
San Luis Obispo, California 93407

ABSTRACT

Some explicit solutions to the 3×3 case of Agler's matrix equation for Nevanlinna-Pick interpolation on the bidisc are provided. Agler showed there exists a holomorphic function bounded by 1 on the bidisc $\mathbf{D}^2$ which maps n prescribed points in $\mathbf{D}^2$ to n prescribed points in $\mathbf{D}$ if and only if there exists a pair of $n \times n$ positive semidefinite matrices satisfying a certain matrix equation. We show there exists a solution to Agler's equation in which one matrix has a row and column of zeros if and only if an explicit set of inequalities that depend on the data are satisfied. A solution of this form is also equivalent to the existence of an interpolating function which is constant with respect to one coordinate of one bidisc data point. The best possible bounds on diagonal elements of solutions to Agler's 2×2 matrix equation are also provided.

1. INTRODUCTION

The classical Nevanlinna-Pick interpolation theorem [3, 4] states that there exists a holomorphic function bounded by 1 on the open unit disc $\mathbf{D}$ which maps n prescribed data points in $\mathbf{D}$ to n prescribed points in $\mathbf{D}$ if and only if a particular $n \times n$ matrix with entries that depend on the data is positive semidefinite. Agler's generalization of the Nevanlinna-Pick theorem to the bidisc $\mathbf{D}^2$ [1] states that there exists a holomorphic function bounded by 1 on the bidisc $\mathbf{D}^2$ which maps n prescribed points in $\mathbf{D}^2$ to n prescribed points in $\mathbf{D}$ if and only if there exist two $n \times n$ positive semidefinite matrices which satisfy a certain matrix equation; the matrix equation depends on the interpolation data. In general, it is difficult to determine whether or not two

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such matrices exist. When $n = 2$, it is well known that a solution exists if and only if an explicit set of inequalities that depend on the data are satisfied. For $n \geqslant 3$, the generalization of these inequalities is sufficient but not necessary for a solution.

In this paper, we find (for $n = 3$) an explicit set of inequalities which are satisfied precisely when there exists a solution to Agler's equation in which one matrix has a row and column of zeros. A solution of this form corresponds to the existence of an interpolating function which is constant with respect to one coordinate of one interpolation point in $\mathbf{D}^2$.

Bounds on the diagonal entries of solutions to the matrix equation when $n = 2$ are also provided. These bounds are achieved; in addition, any $n \times n$ matrix which is part of a solution to the equation for $n \geqslant 3$ will inherit these bounds from its 2×2 principal submatrices.

First we present some matrix and function theoretic notation which is used throughout the paper. Whenever an uppercase letter denotes a matrix, the row i column j element of the matrix will be denoted by the corresponding lowercase letter with subscript ij.

We will write $M \geqslant 0$ to denote that M is positive semidefinite and $M > 0$ to denote that M is positive definite.

If M is an $n \times n$ matrix, then

$$M_{[i_1, \dots, i_m]}$$

denotes the $m \times m$ submatrix of M obtained by deleting all rows and columns of M which are not indexed by an element of $\{i_1, \dots, i_m\}$.

If A and B are $n \times n$ matrices, then $A * B$ denotes the Schur or Hadamard product of A and B; that is, the (i, j) entry of $A * B$ is $a_{ij} b_{ij}$.

The norm on functions defined on both $\mathbf{D}$ and $\mathbf{D}^2$ is the supremum norm; thus for either $k = 1$ or $k = 2$,

$$\|F\|_\infty = \sup_{\lambda \in \mathbf{D}^k} |F(\lambda)|.$$

Finally, $H^\infty(\mathbf{D}^k)$ denotes the set of all bounded holomorphic functions on $\mathbf{D}^k$ and

$$\text{ball } H^\infty(\mathbf{D}^k) = \left\{F \in H^\infty(\mathbf{D}^k) \,\middle|\, \|F\|_\infty \leqslant 1\right\}.$$

Pick's version of the classical Nevanlinna-Pick theorem is stated below.

THEOREM 1.1 (Nevanlinna, Pick). *Let* $\lambda_1, \ldots, \lambda_n \in \mathbf{D}$ *and let* $z_1, \ldots, z_n \in \mathbf{D}$. *There exists a function* $f \in$ ball $H^\infty(\mathbf{D})$ *such that*

$$f(\lambda_i) = z_i \qquad \textit{for} \quad 1 \leqslant i \leqslant n$$

if and only if

$$M \geqslant 0,$$

where M *is the* $n \times n$ *matrix with* (i, j) *entry*

$$m_{ij} = \frac{1 - \bar{z}_i z_j}{1 - \bar{\lambda}_i \lambda_j} \qquad \textit{for} \quad 1 \leqslant i, j \leqslant n. \tag{1}$$

Agler used operator theory to generalize Theorem 1.1 to the polydisc. We now state the special case of this generalization where the domain of the interpolating function is the bidisc. Another proof of Theorem 1.2 is due to Cole and Wermer [2].

THEOREM 1.2 (Agler). *Let* $\lambda_1, \ldots, \lambda_n \in \mathbf{D}^2$ *and let* $z_1, \ldots, z_n \in \mathbf{D}$. *There exists* $F \in$ ball $H^\infty(\mathbf{D}^2)$ *such that*

$$F(\lambda_i) = z_i \qquad \textit{for} \quad 1 \leqslant i \leqslant n$$

if and only if there exist two $n \times n$ *positive semidefinite matrices* $A^{(1)}$ *and* $A^{(2)}$ *such that*

$$1 - \bar{z}_i z_j = \left(1 - \bar{\lambda}_{i1} \lambda_{j1}\right) a_{ij}^{(1)} + \left(1 - \bar{\lambda}_{i2} \lambda_{j2}\right) a_{ij}^{(2)} \qquad \textit{for} \quad 1 \leqslant i, j \leqslant n. \tag{2}$$

Given interpolation data $\lambda_1, \ldots, \lambda_n \in \mathbf{D}^2$ and $z_1, \ldots, z_n \in \mathbf{D}$, Agler's theorem reduces the problem of determining whether there exists an interpolating function which is holomorphic and bounded by 1 on $\mathbf{D}^2$ to the problem of determining whether the matrix equation (2) has a solution (in the form of a pair of positive semidefinite matrices). In the classical Nevanlinna-Pick theorem on $\mathbf{D}$, one can simply check to see whether the matrix M whose entries are uniquely defined by (1) is positive semidefinite to determine whether or not there exists an interpolating function. However, when Agler's condition holds, the pair of matrices which satisfy (2) is not necessarily unique and may be difficult to construct. An explicit test (in terms of the interpolation data) for when such a solution exists for general n is an open question.

Some explicit matrix pairs for which it is easy to check whether (2) holds depend on the matrices which are bidisc analogs of the one variable Pick matrix M defined by (1). Therefore we present some notation to simplify discussion of these matrices.

For $k = 1$ and $k = 2$, define the entries of $P^{(k)}$ by

$$p_{ij}^{(k)} = \frac{1 - \bar{z}_i z_j}{1 - \bar{\lambda}_{ik} \lambda_{jk}} \qquad \text{for} \quad 1 \leqslant i, j \leqslant n. \tag{3}$$

That is, $P^{(k)}$ is the Pick matrix as in Theorem 1.1 that depends on the kth coordinate of the λ_i data. Define the entries of $C^{(k)}$ for $k = 1, 2$ by

$$c_{ij}^{(1)} = \frac{1 - \bar{\lambda}_{i2} \lambda_{j2}}{1 - \bar{\lambda}_{i1} \lambda_{j1}} \qquad \text{for} \quad 1 \leqslant i, j \leqslant n, \tag{4}$$

and

$$c_{ij}^{(2)} = \frac{1 - \bar{\lambda}_{i1} \lambda_{j1}}{1 - \bar{\lambda}_{i2} \lambda_{j2}} \qquad \text{for} \quad 1 \leqslant i, j \leqslant n. \tag{5}$$

Note that

$$c_{ij}^{(2)} = \frac{1}{c_{ij}^{(1)}}, \tag{6}$$

and that Equation (2) can be rewritten in terms of these matrices as

$$A^{(2)} = C^{(2)} * (P^{(1)} - A^{(1)}). \tag{7}$$

Equation (2) and Equation (7) will be used interchangeably.

Observe that the simplest algebraic solution to (7) corresponds to the simplest interpolating function. That is, if $P^{(1)} \geqslant 0$, then

$$A^{(1)} = P^{(1)} \quad \text{and} \quad A^{(2)} = 0$$

is a solution to (7). However, $P^{(1)} \geqslant 0$ also implies, via Theorem 1.1, that there exists a holomorphic function f which is bounded by 1 on $\mathbf{D}$ and satisfies

$$f(\lambda_{i1}) = z_i \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

A corresponding interpolating function on $\mathbf{D}^2$ which satisfies the conditions of Theorem 1.2 can be defined by

$$F(\lambda_{(1)}, \lambda_{(2)}) = f(\lambda_{(1)}) \qquad \text{for} \quad (\lambda_{(1)}, \lambda_{(2)}) \in \mathbf{D}^2.$$

This solution of (7) and the analogous solution that corresponds to the case where $P^{(2)} \geqslant 0$ are described in the following proposition; we wish to analyze when (or to what extent) the converse holds.

PROPOSITION 1.3. *Let $\lambda_1, \ldots, \lambda_n \in \mathbf{D}^2$ and let $z_1, \ldots, z_n \in \mathbf{D}$. Let $P^{(k)}$ and $C^{(k)}$ be defined as in* (3), (4), *and* (5). *If $P^{(1)} \geqslant 0$ or $P^{(2)} \geqslant 0$, then there exist two $n \times n$ positive semidefinite matrices $A^{(1)}$ and $A^{(2)}$ such that*

$$1 - \bar{z}_i z_j = \left(1 - \bar{\lambda}_{i1}\lambda_{j1}\right)a_{ij}^{(1)} + \left(1 - \bar{\lambda}_{i2}\lambda_{j2}\right)A_{ij}^{(2)} \qquad \textit{for} \quad 1 \leqslant i, j \leqslant n.$$

The converse of Proposition 1.3 does not hold for $n \geqslant 3$; see the counterexample below. However, when $n = 2$ the converse is true: the matrices $P^{(k)}$ and $C^{(k)}$ (for $k = 1, 2$) are all 2×2; when this fact is combined with the relationship (6) between $C^{(1)}$ and $C^{(2)}$, it is easy to see that either $C^{(1)} \geqslant 0$ or $C^{(2)} \geqslant 0$. If $C^{(1)} \geqslant 0$ and there is a solution $(A^{(1)}, A^{(2)})$ to (7), then the Schur product theorem can be used to show $P^{(1)} \geqslant 0$. Similarly, if there is a solution $(A^{(1)}, A^{(2)})$ and $C^{(2)} \geqslant 0$, then $P^{(2)} \geqslant 0$. Thus the converse of Proposition 1.3 holds for $n = 2$.

When $n = 2$ in the case of one complex variable the Nevanlinna-Pick interpolation theorem is equivalent to an invariant form of the Schwarz lemma. Similarly, the well-known generalization of the Schwarz lemma to the bidisc can be derived from Proposition 1.3 and its converse. We state this generalization, which can also be derived from the one variable Schwarz lemma, in a form which will be useful in the remainder of the paper.

PROPOSITION 1.4 (Schwarz lemma on bidisc). *Let $\lambda_1, \lambda_2 \in \mathbf{D}$ and let $w_1, w_2 \in \mathbf{D}$. There exists $\varphi \in$* ball $H^\infty(\mathbf{D}^2)$ *such that*

$$\varphi(\lambda_i) = w_i, \qquad i = 1, 2, \tag{8}$$

if and only if

$$\frac{\left(1-|w_1|^2\right)\left(1-|w_2|^2\right)}{|1-\bar{w}_1 w_2|^2} \geqslant \min_{k=1,2} \frac{\left(1-|\lambda_{1k}|^2\right)\left(1-|\lambda_{2k}|^2\right)}{|1-\bar{\lambda}_{1k}\lambda_{2k}|^2}. \tag{9}$$

Note that Proposition 1.4 follows from Proposition 1.3 and its converse because the inequality (9) is equivalent to

$$\max_{k=1,2} \det P^{(k)} \geqslant 0.$$

The following counterexample shows that the converse of Proposition 1.3 does not hold for $n \geqslant 3$. Let $\lambda_1 = (0,0)$, $\lambda_2 = (\lambda_{21}, 0)$, $\lambda_3 = (0, \lambda_{32})$ be in $\mathbf{D}^2$; let $z_1 = 0$, $z_2 = \lambda_{21}/2$, and $z_3 = \lambda_{32}/2$. Interpolation will be achieved by the function $F \in \text{ball } H^\infty(\mathbf{D}^2)$ defined by

$$F(w_{(1)}, w_{(2)}) = \frac{w_{(1)} + w_{(2)}}{2} \qquad \text{for} \quad (w_{(1)}, w_{(2)}) \in \mathbf{D}^2.$$

However, both $P^{(1)}$ and $P^{(2)}$ have 2×2 submatrices which are not positive semidefinite and therefore cannot be positive semidefinite themselves.

Since the converse to Proposition 1.3 does not hold for $n \geqslant 3$, our goal was to find a condition that is equivalent to the existence of a solution to the matrix equation in (2) and that can be explicitly checked for any set of three interpolation data pairs $\lambda_1, \lambda_2, \lambda_3$ and z_1, z_2, z_3. The main result in this paper, Theorem 1.5 below, describes when there exists a pair of 3×3 positive semidefinite matrices that satisfy (1) and also satisfy the additional condition of having zeros in the first row and column of one matrix in the solution pair. With this additional condition, the existence of a solution can be checked in terms of the data [condition (III) below], and the existence of an interpolating function which is constant with respect to the one coordinate of one of the domain points is guaranteed.

THEOREM 1.5. *Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{D}^2$ and let $z_1, z_2, z_3 \in \mathbf{D}$. The following are equivalent*:

(I) *There exist two 3×3 matrices $A^{(1)}$ and $A^{(2)}$ such that*

(a) $$A^{(k)} \geqslant 0, \qquad k = 1, 2,$$

(b)

$$1 - \bar{z}_i z_j = \left(1 - \bar{\lambda}_{i1}\lambda_{j1}\right)a_{ij}^{(1)} + \left(1 - \bar{\lambda}_{i2}\lambda_{j2}\right)a_{ij}^{(2)} \qquad \textit{for} \quad 1 \leqslant i, j \leqslant 3,$$

and

(c) $$a_{11}^{(1)} = \frac{1 - |z_1|^2}{1 - |\lambda_{11}|^2}.$$

(II) *There exists* $F \in$ ball $H^\infty(\mathbf{D}^2)$ *such that*

(a) $$F(\lambda_{i1}, \lambda_{i2}) = z_i \qquad \textit{for} \quad i = 2, 3,$$

and

(b) $$F(\lambda_{11}, \omega) = z_1 \qquad \textit{for all} \quad \omega \in \mathbf{D}.$$

(III) $P^{(1)} \geqslant 0$ *or* $Q^{(2)} \geqslant 0$, *where*

$$P^{(1)} = \left[\frac{1 - \bar{z}_i z_j}{1 - \bar{\lambda}_{i1} \lambda_{j1}} \right]_{i,j=1}^{3}$$

and

$$Q^{(2)} = \left[\left(\frac{1 - \bar{z}_i z_j}{1 - \bar{\lambda}_{i2} \lambda_{j2}} \right) - \left(\frac{1 - \bar{\lambda}_{i1} \lambda_{j1}}{1 - \bar{\lambda}_{i2} \lambda_{j2}} \right) \left(\frac{1 - \bar{z}_i z_1}{1 - \bar{\lambda}_{i1} \lambda_{11}} \right) \left(\frac{1 - \bar{z}_1 z_j}{1 - \bar{\lambda}_{11} \lambda_{j1}} \right) \left(\frac{1 - |\lambda_{11}|^2}{1 - |z_1|^2} \right) \right]_{i,j=2}^{3}.$$

Note that the definition of $P^{(1)}$ in Theorem 1.5 is consistent with the 3×3 case of Equation (3); in the notation from (3), (4), and (5), $Q^{(2)}$ could be written

$$Q^{(2)} = \begin{pmatrix} c_{22}^{(2)} \left(p_{22}^{(1)} - \dfrac{|p_{12}^{(1)}|^2}{p_{11}^{(1)}} \right) & c_{23}^{(2)} \left(p_{23}^{(1)} - \dfrac{\bar{p}_{12}^{(1)} p_{13}^{(1)}}{p_{11}^{(1)}} \right) \\ \bar{c}_{23}^{(2)} \left(\bar{p}_{23}^{(1)} - \dfrac{p_{12}^{(1)} \bar{p}_{13}^{(1)}}{p_{11}^{(1)}} \right) & c_{33}^{(2)} \left(p_{33}^{(1)} - \dfrac{|p_{13}^{(1)}|^2}{p_{11}^{(1)}} \right) \end{pmatrix}. \qquad (10)$$

In the remainder of the paper, $P^{(k)}$, $C^{(k)}$, and their entries refer to 3×3 matrices defined by (3), (4), and (5).

REMARK. If conditions (I)(a), (b), and (c) hold, then the entire first row and column of both $A^{(1)}$ and $A^{(2)}$ are determined. For, if $a_{11}^{(1)} = p_{11}^{(1)}$, then condition (I)(b) implies $a_{11}^{(2)} = 0$, which in turn implies the entries of the entire first row and columns of $A^{(2)}$ are zeros. Again using condition (I)(b), we conclude that the first-row entries of $A^{(1)}$ are

$$a_{1j}^{(1)} = p_{1j}^{(1)}, \qquad 1 \leqslant j \leqslant 3.$$

The first column of $A^{(1)}$ is obtained by conjugating the first row.

The equivalence of (II) and (III) can be obtained using the Schur algorithm and one variable function theory; the author thanks Don Marshall for pointing out a function theoretic proof that (III) implies (II). However, the connection with condition (I) clarifies what must be done to find a general explicit condition which is equivalent to the existence of a solution to Equation (2) in Theorem 1.2. That is, since condition (I)(c) could be written in terms of any of the three diagonal entries of either $A^{(1)}$ or $A^{(2)}$, this theorem can be used to determine whether or not there exists a positive semidefinite solution $(A^{(1)}, A^{(2)})$ to (2) with a zero on any of the diagonals. Thus it remains to determine an explicit condition equivalent to the existence of a solution when the minimum which can be achieved by each diagonal entry is strictly positive.

We compare this situation with the 2×2 case. Since the converse of Proposition 1.3 hold for the case when $n = 2$, there is a solution to the 2×2 matrix equation (7) if and only if there exists a solution with either $A^{(1)} = 0$ or $A^{(2)} = 0$. Therefore it is impossible to have a solution to the 2×2 equation unless there exists a solution with two diagonal entries equal to zero. However, we can analyze the case when a particular diagonal entry cannot be zero. If $P^{(1)} \geqslant 0$, then a solution with $a_{11}^{(2)} = 0$ exists. Otherwise we obtain the lower bound (which is achieved) described in Theorem 1.6.

THEOREM 1.6. *Let* $\lambda_1, \lambda_2 \in \mathbf{D}^2$ *and* $z_1, z_2 \in \mathbf{D}$, *and let* $P^{(k)}$ *and* $C^{(k)}$ $(k = 1, 2)$ *be* 2×2 *matrices defined by* (3), (4), *and* (5). *Let* Λ *denote the set of all* 2×2 *matrix pairs* $(A^{(1)}, A^{(2)})$ *such that*

$$A^{(2)} = C^{(2)} * (P^{(1)} - A^{(1)})$$

and

$$A^{(k)} \geqslant 0, \qquad k = 1, 2.$$

If $P^{(1)} \not\geqslant 0$, $P^{(2)} \geqslant 0$, *and* $(A^{(1)}, A^{(2)}) \in \Lambda$, *then*

$$a_{11}^{(2)} \geqslant \frac{\det P^{(1)}}{p_{22}^{(2)} \det C^{(1)}}. \tag{11}$$

Furthermore, if $\tilde{A}^{(1)}$ *and* $\tilde{A}^{(2)}$ *are defined by*

$$\tilde{A}^{(1)} = \frac{\det P^{(2)}}{\det C^{(2)}} \begin{pmatrix} \dfrac{1}{p_{22}^{(1)}} & \dfrac{1}{\bar{p}_{12}^{(1)}} \\ \dfrac{1}{p_{12}^{(1)}} & \dfrac{p_{22}^{(1)}}{|p_{12}^{(1)}|^2} \end{pmatrix}$$

and

$$\tilde{A}^{(2)} = \frac{\det P^{(1)}}{\det C^{(1)}} \begin{pmatrix} \dfrac{1}{p_{22}^{(2)}} & \dfrac{1}{\bar{p}_{12}^{(2)}} \\ \dfrac{1}{p_{12}^{(2)}} & \dfrac{p_{22}^{(2)}}{|p_{12}^{(2)}|^2} \end{pmatrix}$$

then $(\tilde{A}^{(1)}, \tilde{A}^{(2)}) \in \Lambda$ *and equality holds in* (11).

Since $a_{11}^{(1)} = c_{11}^{(1)}(p_{11}^{(2)} - a_{11}^{(2)})$, the (1, 1) entry of $\tilde{A}^{(1)}$ is an upper bound for $a_{11}^{(1)}$. The natural symmetry of this theorem also provides a lower bound for $a_{22}^{(2)}$ and an upper bound for $a_{22}^{(1)}$ if the roles of the subscript indices are exchanged.

The details of the proof of Theorem 1.6 are straightforward, so only an outline will be provided. The inequalities obtained from $\det A^{(1)} \geqslant 0$ and $\det A^{(2)} \geqslant 0$ can both be written in terms of the entries of $A^{(2)}$, $P^{(2)}$, and $C^{(1)}$. There is a simultaneous solution if and only if two discs intersect; $a_{12}^{(2)}$ can be eliminated from the inequality which holds if this intersection is nonempty. This inequality can be rewritten as an equality which is quadratic in $a_{11}^{(2)}$ and $a_{22}^{(2)}$. The inequality (11) is equivalent to the condition that the discriminant of the quadratic equations in $a_{22}^{(2)}$ is nonnegative. It is straightforward to check that $(\tilde{A}^{(1)}, \tilde{A}^{(2)}) \in \Lambda$.

Theorem 1.6 is related to the solution of the 3×3 equation (7). When $(A^{(1)}, A^{(2)})$ is a 3×3 solution to (7), Theorem 1.6 can be applied to all principal submatrices of $A^{(1)}$ and $A^{(2)}$ when appropriate hypotheses hold for corresponding submatrices of $P^{(k)}$ and $C^{(k)}$. Therefore the diagonal entries of solutions to the 3×3 equation will inherit any bounds on diagonals of 2×2 submatrices. These bounds will not necessarily be achieved by any 3×3 solution; however, any algorithm for finding a solution to the 3×3 equation can be restricted to checking solutions with diagonals within the bounds inherited by each 2×2 submatrix.

Furthermore, if the technique of minimizing a diagonal entry can be extended to the 3×3 case, it is possible that the extremal solution matrix will contain an explicit condition which is equivalent to the existence of a solution in general. Theorem 1.5 describes an explicit condition in the case that the minimum value of $a_{11}^{(2)}$ is 0.

Section 2 of the paper contains the proof of Theorem 1.5, an outline of which follows. That (I) implies (II) is proved by inductively augmenting $A^{(1)}$ and $A^{(2)}$ and taking a limit of the resulting interpolating functions, which are guaranteed to exist by Theorem 1.2. That (II) implies (III) is proved by applying the Schwarz lemma on the bidisc to a composition of F with Möbius transformation. The resulting inequality provides the determinant conditions require to prove (III). Finally, (III) easily implies (I), since $P^{(1)} \geqslant 0$ implies a solution as in Proposition 1.3, and $Q^{(2)} \geqslant 0$ implies $A^{(2)} \geqslant 0$ when $A^{(1)}$ is a rank 1 matrix which satisfies (I)(c).

2. PROOF OF THEOREM 1.5

Lemma 2.1 below will be used to prove that (I) implies (II) in Theorem 1.5. The lemma states that if a positive semidefinite matrix is augmented with its first row and column, then the augmented matrix is also positive semidefinite. The proof is a straightforward induction argument and is therefore omitted.

LEMMA 2.1. *Let $A = [a_{ij}]$ be an $n \times n$ matrix, and let $\mathbf{a}_1$ denote the first row of A. Define the $(n+1) \times (n+1)$ matrix B by*

$$B = \begin{pmatrix} a_{11} & \mathbf{a}_1 \\ \mathbf{a}_1^* & A \end{pmatrix}.$$

If $A \geqslant 0$, then $B \geqslant 0$.

We are now prepared to prove that (I) implies (II). Assume there exist two 3×3 matrices $A^{(1)}$ and $A^{(2)}$ which satisfy (I)(a), (b), and (c). An interpolating function $F \in \operatorname{ball} H^\infty(\mathbf{D}^2)$ which satisfies (II)(a) and (b) will be obtained by taking the limit of a sequence of interpolating functions $F_n \in$ ball $H^\infty(\mathbf{D}^2)$. The properties of each F_n will be established, via Theorem 1.2, by augmenting $A^{(1)}$ and $A^{(2)}$ into $(n+3) \times (n+3)$ matrices.

Let $\{\omega_l\}_{l=1}^\infty$ be a dense subset of distinct elements of $\mathbf{D}$ which does not contain λ_{12}. Define $\gamma_l \in \mathbf{D}^2$ by

$$\gamma_l = (\lambda_{11}, \omega_l), \qquad l = 1, 2, 3, \ldots.$$

Fix a positive integer n. We wish to apply Theorem 1.2 with domain data $\gamma_1, \ldots, \gamma_n, \lambda_1, \lambda_2, \lambda_3 \in \mathbf{D}^2$ and range data $z_1, \ldots, z_1, z_1, z_2, z_3 \in \mathbf{D}$, where z_1 is the range value corresponding to $\gamma_1, \ldots, \gamma_n$ and λ_1. This will be achieved by constructing $(n+3) \times (n+3)$ positive semidefinite matrices which satisfy Equation (2) of Theorem 1.2 with these data.

Accordingly, define $A^{(1)}(n)$ as the partitioned matrix

$$A^{(1)}(n) = \begin{pmatrix} T^{(1)}(n) & B^{(1)}(n) \\ B^{(1)}(n)^* & A^{(1)} \end{pmatrix}, \tag{12}$$

where the blocks of $A^{(1)}(n)$ are defined in the following discussion.

The block $T^{(1)}(n)$ is a constant $n \times n$ matrix with each entry equal to $(1 - |z_1|^2)/(1 - |\lambda_{11}|^2)$; recall from Equation (3) that

$$p_{11}^{(1)} = \frac{1 - |z_1|^2}{1 - |\lambda_{11}|^2}.$$

Thus,

$$T^{(1)}(n) = \begin{pmatrix} p_{11}^{(1)} & p_{11}^{(1)} & \cdots & p_{11}^{(1)} \\ p_{11}^{(1)} & p_{11}^{(1)} & \cdots & p_{11}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{11}^{(1)} & p_{11}^{(1)} & \cdots & p_{11}^{(1)} \end{pmatrix}.$$

The block $B^{(1)}(n)$ is an $n \times 3$ matrix with (i, j) entry defined by

$$B_{ij}^{(1)}(n) = \frac{1 - \bar{z}_1 z_j}{1 - \bar{\lambda}_{11} \lambda_{j'}} \qquad \text{for} \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant 3.$$

Thus, recalling Equation (3) again,

$$B^{(1)}(n) = \begin{pmatrix} p_{11}^{(1)} & p_{12}^{(1)} & p_{13}^{(1)} \\ p_{11}^{(1)} & p_{12}^{(1)} & p_{13}^{(1)} \\ \vdots & \vdots & \vdots \\ p_{11}^{(1)} & p_{12}^{(1)} & p_{13}^{(1)} \end{pmatrix}.$$

Finally, the block $A^{(1)}$ is the 3×3 matrix which exists and has properties as described in condition (I) by hypothesis.

Now define $A^{(2)}(n)$ to be the $(n+3) \times (n+3)$ partitioned matrix

$$A^{(2)}(n) = \begin{pmatrix} 0 & 0 \\ 0 & A^{(2)} \end{pmatrix},$$

where $A^{(2)}$ is the 3×3 matrix assumed by condition (I), the upper left block is the $n \times n$ zero matrix, and the off-diagonal blocks are $n \times 3$ and $3 \times n$ zero matrices. For convenience in an induction proof later, define $A^{(k)}(0) = A^{(k)}$ for $k = 1$ and $k = 2$.

To check that $A^{(1)}(n)$ and $A^{(2)}(n)$ satisfy the hypotheses of Theorem 1.2 (and thus imply the existence of an interpolating function with the desired properties), we must check that the entries of $A^{(1)}(n)$ and $A^{(2)}(n)$ satisfy Equation (2) and that $A^{(1)}(n)$ and $A^{(2)}(n)$ are both positive semidefinite.

We will check that Equation (2) is satisfied block by block.

First, the upper left blocks correspond to domain data $(\lambda_{11}, \omega_1), \dots, (\lambda_{11}, \omega_n)$ and range data $z_1, \dots, z_1$. Therefore, substituting the appropriate entries of $A^{(1)}(n)$ and $A^{(2)}(n)$ into Equation (2) results in

$$1 - |z_1|^2 = \left(1 - |\lambda_{11}|^2\right) p_{11}^{(1)} + \left(1 - |\omega_i|^2\right) \cdot 0 \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

By definition of $p_{11}^{(1)}$, this equation is clearly satisfied.

Next, the lower right blocks correspond to domain data $\lambda_1, \lambda_2, \lambda_3$ and range data z_1, z_3, z_3. These blocks are the 3×3 matrices $A^{(1)}$ and $A^{(2)}$; therefore Equation (2) is equivalent to condition (I)(b), which holds by hypothesis.

Finally, to check the off-diagonal blocks, we must check the cross terms of the data for the diagonal blocks. For instance, in the upper right blocks, we conjugate the data used for the upper left diagonal blocks (since these correspond to the rows) but do not conjugate the data from the lower right blocks (the column data). Therefore, Equation (2) becomes

$$1 - \bar{z}_1 z_j = \left(1 - \bar{\lambda}_{11}\lambda_{j1}\right)p_{1j}^{(1)} + \left(1 - \bar{\omega}_i \lambda_{j2}\right)\cdot 0$$

$$\text{for} \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant 3.$$

By definition of $p_{ij}^{(1)}$, this equation is clearly satisfied; thus Equation (2) holds for all entries of $A^{(1)}(n)$ and $A^{(2)}(n)$.

It is easy to see that $A^{(2)}(n)$ is positive semidefinite, since $A^{(2)} \geqslant 0$ by (I)(a). Therefore, the only remaining condition to check in Theorem 1.2 is that $A^{(1)}(n)$ is positive semidefinite. This will be established by induction; Lemma 2.1 will be used in the proof.

INDUCTION HYPOTHESIS 2.2. *For all integers $n \geqslant 0$,*

$$A^{(1)}(n) \geqslant 0, \tag{13}$$

and

$$\textit{the first row of } A^{(1)}(n) \textit{ is } \left[\begin{array}{ccc|ccc} p_{11}^{(1)} & \cdots & p_{11}^{(1)} & p_{11}^{(1)} & p_{12}^{(1)} & p_{13}^{(1)} \end{array}\right], \tag{14}$$

where n copies of $p_{11}^{(1)}$ precede the partition.

To prove Induction Hypothesis 2.2, first note that $A^{(1)}(0) = A^{(1)}$, which is positive semidefinite by condition (I)(a). By the remark immediately following Theorem 1.5, the first row of $A^{(1)}(0)$ is

$$\left[\begin{array}{ccc} p_{11}^{(1)} & p_{12}^{(1)} & p_{13}^{(1)} \end{array}\right].$$

Since 0 copies of $p_{11}^{(1)}$ precede the partition, the induction hypothesis holds for $n = 0$.

Now assume (13) and (14) hold when $n = k$. By the general definition of $A^{(1)}(n)$ in Equation (12), the entries of the block $T^{(1)}(n)$ and the rows of $B^{(1)}(n)$ are constant; only the dimensions change. That is, when $n = k + 1$,

$$A^{(1)}(k+1) = \begin{pmatrix} T^{(1)}(k+1) & B^{(1)}(k+1) \\ B^{(1)}(k+1)^* & A^{(1)} \end{pmatrix},$$

where

$$T^{(1)}(k+1) = \begin{pmatrix} [p_{11}^{(1)}] & [p_{11}^{(1)} \quad \cdots \quad p_{11}^{(1)}] \\ \begin{bmatrix} p_{11}^{(1)} \\ \vdots \\ p_{11}^{(1)} \end{bmatrix} & T^{(1)}(k) \end{pmatrix} \tag{15}$$

and

$$B^{(1)}(k+1) = \begin{pmatrix} p_{11}^{(1)} & p_{12}^{(1)} & p_{13}^{(1)} \\ & B^{(1)}(k) & \end{pmatrix}.$$

In (15), the vector $[p_{11}^{(1)} \ \ldots \ p_{11}^{(1)}]$ is k-dimensional. Thus, the top row of $A^{(1)}(k+1)$ is of the form $[p_{11}^{(1)} \ \ldots \ p_{11}^{(1)} | p_{11}^{(1)} \ p_{12}^{(1)} \ p_{13}^{(1)}]$ with $k+1$ copies of $p_{11}^{(1)}$ preceding the partition. This shows that $A^{(1)}(k+1)$ satisfies Equation (14) of the induction hypothesis. In addition, since $A^{(1)}(k)$ satisfies both (13) and (14), we see that the manner in which $A^{(1)}(k+1)$ augments $A^{(1)}(k)$ is consistent with the hypotheses of Lemma 2.1, and we may therefore conclude that $A^{(1)}(k+1) \geqslant 0$. This concludes the induction proof.

We will now use the matrices just constructed to prove an interpolating function satisfying (II)(a) and (II)(b) exists. For each positive integer n, $A^{(1)}(n)$ and $A^{(2)}(n)$ satisfy the hypotheses of Theorem 1.2 corresponding to the original three interpolation data point pairs and the first n points of the augmented data. We may conclude that for each integer $n \geqslant 1$ there exists a function $F_n \in$ ball $H^\infty(\mathbf{D}^2)$ such that

$$F_n(\lambda_{i1}\ \lambda_{i2}) = z_i \qquad \text{for} \quad 1 \leqslant i \leqslant 3$$

and

$$F_n(\lambda_{11}, \omega_l) = z_1 \qquad \text{for} \quad 1 \leqslant l \leqslant n.$$

The family $\{F_n\}_{n=1}^\infty$ is normal and thus a subsequence converges uniformly on compact subsets of $\mathbf{D}^2$ to a limit function $F \in$ ball $H^\infty(\mathbf{D}^2)$. Therefore

$$F(\lambda_{i1}, \lambda_{i2}) = z_i \qquad z_i \qquad \text{for} \quad 1 \leqslant i \leqslant 3$$

and

$$F(\lambda_{11}, \omega_l) = z_1 \qquad \text{for} \quad l = 1, 2, 3, \ldots.$$

Since $\{\omega_l\}_{l=1}^\infty$ is dense in $\mathbf{D}$, the latter set of equalities is equivalent to property (II)(b). This concludes the proof that (I) implies (II).

To show (II) implies (III), we assume the existence of a function F satisfying (II) and will show either $P^{(1)} \geqslant 0$ or $Q^{(2)} \geqslant 0$ as in (III).

The proof will be established in separate cases which depend on the determinants of the 2×2 submatrices of $P^{(1)}$. In each case, the Schwarz lemma on the bidisc (Proposition 1.4) will be applied to F or a rational function of F. The inequality (9) in the Schwarz lemma will be equivalent to $P^{(1)} \geqslant 0$ or $Q^{(2)} \geqslant 0$.

First note that by applying the one variable Schwarz lemma to the function f defined by $f(\lambda) = F(\lambda, \lambda_{22})$ for $\lambda \in \mathbf{D}$, we obtain $P^{(1)}_{[1,2]} \geqslant 0$. Similarly, $P^{(1)}_{[1,3]} \geqslant 0$.

Case 1: Assume $\det P^{(1)}_{[1,2]} > 0$ *and* $\det P^{(1)}_{[1,3]} > 0$

These assumptions are algebraically equivalent to

$$|w_i| < 1 \qquad \text{for} \quad i = 2, 3,$$

where

$$w_i = \frac{(z_1 - z_i)/(1 - \bar{z}_1 z_i)}{(\lambda_{11} - \lambda_{i1})/(1 - \bar{\lambda}_{11}\lambda_{i1})} \qquad \text{for} \quad i = 2, 3.$$

Now define

$$\varphi(\lambda_{(1)}, \lambda_{(2)}) = \frac{[z_1 - F(\lambda_{(1)}, \lambda_{(2)})]/[1 - \bar{z}_1 F(\lambda_{(1)}, \lambda_{(2)})]}{(\lambda_{11} - \lambda_{i1})/(1 - \bar{\lambda}_{11}\lambda_{i1})}$$

for $\lambda = (\lambda_{(1)}, \lambda_{(2)}) \in \mathbf{D}^2$, $\lambda_{(1)} \neq \lambda_{11}$. The holomorphic extension of φ is in ball $H^\infty(\mathbf{D}^2)$. We may therefore apply Proposition 1.4 to φ with domain points λ_2 and λ_3 and range points $w_2 = \varphi(\lambda_2)$ and $w_3 = \varphi(\lambda_3)$. The inequality (9) translates to

$$\frac{(1 - |w_2|^2)(1 - |w_3|^2)}{|1 - \bar{w}_2 w_3|^2} \geqslant \min_{k=1,2} \frac{(1 - |\lambda_{2k}|^2)(1 - |\lambda_{3k}|^2)}{|1 - \bar{\lambda}_{2k}\lambda_{3k}|^2}. \qquad (16)$$

If the minimum in (16) is achieved when $k = 1$, then a straightforward computation shows that (16) is equivalent to $\det P^{(1)} \geqslant 0$. When combined with our assumption that $\det P^{(1)}_{[1,2]} > 0$ and the fact that $p^{(1)}_{11} > 0$, this implies

$$P^{(1)} \geqslant 0.$$

If the minimum in (16) is achieved when $k = 2$, then (16) is equivalent to det $Q^{(2)} \geqslant 0$. This inequality and det $P^{(1)}_{[1,2]} > 0$ imply

$$Q^{(2)} \geqslant 0.$$

In either case, (III) holds.

Case 2: Assume det $P^{(1)}_{[1,2]} = 0$ *or* det $P^{(1)}_{[1,3]} = 0$
By symmetry, it suffices to show det $P^{(1)}_{[1,2]} = 0$ implies $P^{(1)} \geqslant 0$. Assumptions (II)(a) and (b) imply that

$$F(\lambda_{i1}, \lambda_{i2}) = z_i \qquad \text{for} \quad i = 2, 3$$

and

$$F(\lambda_{11}, \lambda_{22}) = z_1.$$

An application of Theorem 1.2 to F and these three pairs of data points establishes the existence of two 3×3 positive semidefinite matrices $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ that satisfy

$$\tilde{A}^{(1)} = P^{(1)} - \tilde{C}^{(1)} * \tilde{A}^{(2)}, \tag{17}$$

where

$$\tilde{C}^{(1)} = \begin{pmatrix} \dfrac{1 - |\lambda_{22}|^2}{1 - |\lambda_{11}|^2} & \dfrac{1 - |\lambda_{22}|^2}{1 - \bar{\lambda}_{11}\lambda_{21}} & \dfrac{1 - \bar{\lambda}_{22}\lambda_{32}}{1 - \bar{\lambda}_{11}\lambda_{31}} \\ \dfrac{1 - |\lambda_{22}|^2}{1 - \bar{\lambda}_{21}\lambda_{11}} & \dfrac{1 - |\lambda_{22}|^2}{1 - |\lambda_{21}|^2} & \dfrac{1 - \bar{\lambda}_{22}\lambda_{32}}{1 - \bar{\lambda}_{21}\lambda_{31}} \\ \dfrac{1 - \bar{\lambda}_{32}\lambda_{22}}{1 - \bar{\lambda}_{31}\lambda_{11}} & \dfrac{1 - \bar{\lambda}_{32}\lambda_{22}}{1 - \bar{\lambda}_{31}\lambda_{21}} & \dfrac{1 - |\lambda_{32}|^2}{1 - |\lambda_{31}|^2} \end{pmatrix}.$$

By considering upper left 2×2 submatrices in (17), we obtain

$$\tilde{A}^{(1)}_{[1,2]} = P^{(1)}_{[1,2]} - \tilde{C}^{(1)}_{[1,2]} * \tilde{A}^{(2)}_{[1,2]}.$$

Since $P^{(1)}_{[1,2]}$ is a rank 1 positive semidefinite matrix, one of the following occurs:

$$\tilde{A}^{(1)}_{[1,2]} = P^{(1)}_{[1,2]} \quad \text{and} \quad \tilde{A}^{(2)}_{[1,2]} = 0, \tag{18}$$

or

$$z_1 = z_2 \quad \text{and} \quad \lambda_{11} = \lambda_{21}. \tag{19}$$

Either (18) or (19) can be used to prove $P^{(1)} \geqslant 0$. This concludes the proof of case 2; thus we have concluded the proof that (II) implies (III).

Finally, to prove that condition (III) implies condition (I), we must show that $P^{(1)} \geqslant 0$ and $Q^{(2)} \geqslant 0$ each imply there exist $A^{(1)}$ and $A^{(2)}$ satisfying (I)(a), (b), and (c).

As discussed in the introduction, if $P^{(1)} \geqslant 0$ then

$$A^{(1)} = P^{(1)} \quad \text{and} \quad A^{(2)} = 0$$

provide a solution to (I)(a) and (b) which also clearly satisfies (I)(c). Therefore it remains to check (I) when $Q^{(2)} \geqslant 0$.

Accordingly, assume $Q^{(2)} \geqslant 0$ and define

$$A^{(1)} = \begin{pmatrix} p^{(1)}_{11} & p^{(1)}_{12} & p^{(1)}_{13} \\ \bar{p}^{(1)}_{12} & \dfrac{|p^{(1)}_{12}|^2}{p^{(1)}_{11}} & \dfrac{\bar{p}^{(1)}_{12} p^{(1)}_{13}}{p^{(1)}_{11}} \\ \bar{p}^{(1)}_{13} & \dfrac{p^{(1)}_{12} \bar{p}^{(1)}_{13}}{p^{(1)}_{11}} & \dfrac{|p^{(1)}_{13}|^2}{p^{(1)}_{11}} \end{pmatrix}$$

and

$$A^{(2)} = \begin{pmatrix} 0 & 0 \quad 0 \\ \begin{matrix} 0 \\ 0 \end{matrix} & Q^{(2)} \end{pmatrix}.$$

Note $A^{(1)}$ is the rank 1 matrix defined by

$$A^{(1)} = pp^*$$

with

$$p = \frac{1}{\sqrt{p_{11}^{(1)}}} \begin{pmatrix} p_{11}^{(1)} \\ \bar{p}_{12}^{(1)} \\ \bar{p}_{13}^{(1)} \end{pmatrix}.$$

Thus $A^{(1)} \geqslant 0$.

The matrix $A^{(2)}$ is a block diagonal with diagonal blocks consisting of 0 and the matrix $Q^{(2)}$, which is positive semidefinite by assumption. Therefore $A^{(2)}$ is also positive semidefinite, and it follows that (I)(a) holds.

To see that (I)(b) holds, note that the first row and column of $A^{(1)}$ and $A^{(2)}$ satisfy (I)(b) as in the case where $P^{(1)} \geqslant 0$. Since (I)(b) is equivalent to Equation (7), it is easy to see the lower right 2×2 submatrices of $A^{(1)}$ and $A^{(2)}$ satisfy (I)(b) where $Q^{(2)}$ is written as in (10).

Finally, (I)(c) holds by definition of $A^{(1)}$. This concludes the proof that (III) implies (I) and the proof of the theorem.

REFERENCES

1 J. Agler, Some interpolation theorems of Nevanlinna-Pick type, *J. Funct. Anal.*, to appear.

2 B. Cole and J. Wermer, Pick interpolation, von Neumann inequalities, and hyperconvex sets, in *Complex Potential Theory* (P. M. Gauthier, Ed.), NATO Adv. Sci. Inst. Ser. C 439, Kluwer, Dordrecht, 1994, pp. 89–120.

3 R. Nevanlinna, Über beschränkte Funktionen, die in gegebenen Punkten vorgeschriebene Werte annehmen, *Ann. Acad. Sci. Fenn. Ser. A.* 13:1–71 (1919).

4 G. Pick, Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden, *Math. Ann.* 77:7–23 (1916).

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