Discrete Applied Mathematics 157 (2009) 2918–2927

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Kirchhoff index of composite graphs ☆

Heping Zhang *, Yujun Yang, Chuanwen Li

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, PR China

ARTICLE INFO

Article history:
Received 5 January 2008
Received in revised form 25 February 2009
Accepted 15 March 2009
Available online 15 April 2009

Keywords:
Resistance distance
Kirchhoff index
Laplacian spectrum
Corona
Cluster

ABSTRACT

Let $G_1 + G_2$, $G_1 \circ G_2$ and $G_1\{G_2\}$ be the join, corona and cluster of graphs G_1 and G_2, respectively. In this paper, Kirchhoff index formulae of these composite graphs are given.

1. Introduction

Let G be a graph with vertices labeled $1, 2, \ldots, n$. In 1993, Klein and Randić [15] defined a new distance function named resistance distance framed in terms of electrical network theory. If G is connected, then the resistance distance between vertices i and j, denoted by r_{ij} (if more than one graph is considered, we use $r_{ij}(G)$ to avoid confusion), is defined to be the effective electrical resistance between them when unit resistors are placed on every edge of G. Define

$$Kf(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij},$$

known as the Kirchhoff index of G [15]. If G is disconnected, then by setting $r_{ij} = \infty$ for i and j in different components and $Kf(G) = \infty$, the above definitions remain applicable and the results obtained in this paper still hold. In what follows, we restrict our consideration to connected graphs.

The original index based on distances in a graph G is the famous Wiener index $W(G)$ [24], which counts the sum of distances between pairs of vertices in G, that is

$$W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij},$$

where d_{ij} is the distance between vertices i and j. Klein and Randić [15] proved that $Kf(G) \leq W(G)$ with equality, if and only if G is a tree.

☆ This work is supported by NSFC (grant no. 10831001).
* Corresponding author.
E-mail addresses: zhanghp@lzu.edu.cn (H. Zhang), yangyj03@lzu.cn (Y. Yang), licw05@lzu.cn (C. Li).

0166-218X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2009.03.007

Like the Wiener index, the Kirchhoff index is a structure descriptor [25]. But it is difficult to implement some algorithms [2,4,15,19,32] to compute resistance distances and the Kirchhoff index of a graph. Hence it makes sense to find closed-form formulae for the Kirchhoff index. To this end, on the one hand, closed-form formulae for the Kirchhoff index have been given for some classes of graphs, such as cycles [13,16], complete graphs [16], geodetic graphs [20], distance-transitive graphs [20], circulant graphs [31], linear hexagonal chains [29], etc. [2,11,19–21]; on the other hand, numerical values of the Kirchhoff index have been computed for platonic solids [16,17], some fullerene graphs including buckminsterfullerene [2,3,8], etc. [6, 11,14]. Besides, some efforts were also made to determine bounds for some classes of graphs and characterize extremal graphs as well, such as unicyclic graphs [28].

It is of interest to study some topological indices of certain composite operations between two graphs, such as product, composition, join, corona, cluster, etc. In [30], Yeh and Gutman computed the Wiener index of these composite graphs. As a generalization of their results, Sagan et al. [22] and Stevanović [23] computed the Hosoya (Wiener) polynomial of these graphs. Xu [26] computed the Kirchhoff index of product and composition (called lexicographic product in that article) of graphs. In the present work, the Kirchhoff index of join, corona and cluster of graphs are computed.

Composite graphs under consideration are defined in the following [30]:

(i) The (Cartesian) *product* $G_1 \times G_2$:

$$V(G_1 \times G_2) = V(G_1) \times V(G_2);$$

the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1 \times G_2$ are adjacent if and only if $[u_1 = v_1, (u_2, v_2) \in E(G_2)]$ or $[u_2 = v_2, (u_1, v_1) \in E(G_1)]$.

(ii) The *composition* $G_1[G_2]$:

$$V(G_1[G_2]) = V(G_1) \times V(G_2);$$

the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $G_1[G_2]$ are adjacent if and only if $[u_1 = v_1, (u_2, v_2) \in E(G_2)]$ or $[(u_1, v_1) \in E(G_1)]$.

(iii) The *join* $G_1 + G_2$:

$$V(G_1 + G_2) = V(G_1) \cup V(G_2);$$
$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(u_1, u_2) | u_1 \in V(G_1), u_2 \in V(G_2)\}.$$

(iv) The *corona* $G_1 \circ G_2$ is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and by joining each vertex of the ith copy of G_2 to the ith vertex of G_1, $i = 1, 2, \ldots, |V(G_1)|$.

(v) The *cluster* $G_1\{G_2\}$ is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of a rooted graph G_2, and by identifying the root of the ith copy of G_2 with the ith vertex of G_1, $i = 1, 2, \ldots, |V(G_1)|$.

2. Resistance distance and Kirchhoff index

The adjacency matrix $A(G)$ of G is an $n \times n$ matrix with the (i, j)-entry equal to 1 if vertices i and j are adjacent and 0 otherwise. Let $D(G) = \mathrm{diag}\,(d_G(1), d_G(2), \ldots, d_G(n))$ be the diagonal matrix of vertex degrees, where $d_G(i)$ is the degree of i in G for $1 \le i \le n$. The Laplacian matrix of G is $L(G) = D(G) - A(G)$.

The Laplacian matrix plays an extremely important role in the computation of resistance distance and the Kirchhoff index. In this section, we only list some important formulae which will be used later.

In the following, for a matrix C, we use $C(i)$ and $C(i, j)$ to denote the submatrix obtained from C by deleting the ith row and column, and by deleting the ith and jth rows and columns, respectively. Here for convenience, we abbreviate $L(G)(i)$ and $L(G)(i, j)$ to $L(i)$ and $L(i, j)$, respectively. Then we state the following theorem.

Theorem 2.1 ([4]). *Let G be a connected graph on n vertices, $n \ge 3$, and $1 \le i \ne j \le n$. Then*

$$r_{ij} = \frac{\det L(i, j)}{\det L(i)}. \tag{2.1}$$

According to the famous matrix-tree theorem [10] for any graph G and for any $i = 1, 2, \ldots, n$,

$$\det L(i) = t(G),$$

where $t(G)$ is the number of spanning trees of G. Hence, Eq. (2.1) can also be written as

$$r_{ij} = \frac{\det L(i, j)}{t(G)}. \tag{2.2}$$

It is well known that $L(G)$ is singular and has no inverse. However, $L(G)$ has a generalized inverse $\Gamma(G)$, which is 0 on the null space (0-eigenvalue eigenspace) of $L(G)$ and is the inverse to $L(G)$ on the orthogonal complement to this null space. A fundamental formula for computing the Kirchhoff index is given below.

Theorem 2.2 (*[15]*). *For any connected n-vertex graph G* ($n \geq 2$),

$$Kf(G) = n\,\mathrm{Tr}(\Gamma(G)), \tag{2.3}$$

where $\mathrm{Tr}(\Gamma(G))$ *denotes the trace of* $\Gamma(G)$.

The Laplacian polynomial $L(G, \lambda)$ is the characteristic polynomial of $L(G)$:

$$L(G, \lambda) = \det(\lambda I_n - L(G)),$$

where I_n is the identity matrix of order n. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $L(G)$ (also roots of $L(G, \lambda)$), called the Laplacian eigenvalues of G. Suppose that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then $\lambda_1 = 0$ and $\lambda_2 > 0$ if and only if G is connected [18] and the Laplacian spectrum of G is

$$S(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

From Theorem 2.2, a nice relationship between the Laplacian spectrum and the Kirchhoff index has been established as stated below:

Theorem 2.3 (*[9,12]*). *For any connected n-vertex graph G* ($n \geq 2$),

$$Kf(G) = n \sum_{k=2}^{n} \frac{1}{\lambda_k}. \tag{2.4}$$

3. Main results

Let G_1 and G_2 be graphs on n and m vertices, respectively. Briefly, let L_1 and L_2 denote the Laplacian matrices of G_1 and G_2 respectively and suppose that $S(G_1) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $S(G_2) = (\mu_1, \mu_2, \ldots, \mu_m)$ such that $\lambda_1 = \mu_1 = 0$. In this section, the Kirchhoff indices of $G_1 + G_2$, $G_1 \circ G_2$ and $G_1\{G_2\}$ are computed.

3.1. Kirchhoff index of $G_1 + G_2$

Proposition 3.1 (*[18]*). The Laplacian eigenvalues of $G_1 + G_2$ are

$$0, m+n, m+\lambda_i, n+\mu_j \quad i = 2, 3, \ldots, n; j = 2, 3, \ldots, m.$$

Theorem 3.2. *For any two graphs G_1 and G_2,*

$$Kf(G_1 + G_2) = 1 + (m+n)\left(\sum_{i=2}^{n} \frac{1}{m+\lambda_i} + \sum_{j=2}^{m} \frac{1}{n+\mu_j}\right). \tag{3.1}$$

Proof. Theorem 2.3 and Proposition 3.1 yield the result. □

Let K_n and $\overline{K_n}$ denote the complete graph and empty graph on n vertices, respectively. Let $K_{m,n}$ denote the complete bipartite graph such that one class has n vertices and the other has m vertices. It is interesting that $Kf(G_1 + G_2)$ can not only be completely determined by the Laplacian spectra of G_1 and G_2, but also expressed as a linear combination of Kirchhoff indices of its three spanning subgraphs.

Theorem 3.3. *For any two graphs G_1 and G_2,*

$$Kf(G_1 + G_2) = Kf(G_1 + \overline{K_m}) + Kf(G_2 + \overline{K_n}) - Kf(K_{m,n}). \tag{3.2}$$

Proof. We can easily verify that

$$S(\overline{K_n}) = (0, 0, \ldots, 0).$$

By Proposition 3.1, we have

$$S(G_1 + \overline{K_m}) = (0, m+n, \underbrace{n, \ldots, n}_{m-1}, \lambda_2 + m, \ldots, \lambda_n + m),$$

$$S(G_2 + \overline{K_n}) = (0, m+n, \underbrace{m, \ldots, m}_{n-1}, \mu_2 + n, \ldots, \mu_m + n),$$

$$S(K_{m,n}) = (0, m+n, \underbrace{n, \ldots, n}_{m-1}, \underbrace{m, \ldots, m}_{n-1}).$$

Therefore by Theorem 2.3,

$$Kf(G_1+\overline{K_m}) = 1 + \frac{(m+n)(m-1)}{n} + (m+n)\sum_{i=2}^{n}\frac{1}{m+\lambda_i},$$
$$Kf(G_2+\overline{K_n}) = 1 + \frac{(m+n)(n-1)}{m} + (m+n)\sum_{i=2}^{m}\frac{1}{n+\mu_i},$$
$$Kf(K_{m,n}) = 1 + \frac{(m+n)(m-1)}{n} + \frac{(m+n)(n-1)}{m}.$$

Hence

$$Kf(G_1+G_2) = Kf(G_1+\overline{K_m}) + Kf(G_2+\overline{K_n}) - Kf(K_{m,n}). \quad \square$$

Though it is difficult to find a relationship between $Kf(G_1+G_2)$ and $Kf(G_1)$ and $Kf(G_2)$, we can obtain an upper bound to $Kf(G_1+G_2)$ in terms of $Kf(G_1)$ and $Kf(G_2)$.

Proposition 3.4. Let G_1 and G_2 be connected graphs. Then

$$Kf(G_1+G_2) \le Kf(G_1) + Kf(G_2) + 1,$$

with equality if and only if G_1+G_2 is complete.

Proof. By Theorems 2.3 and 3.2,

$$\begin{aligned} Kf(G_1)+Kf(G_2)+1-Kf(G_1+G_2) &= n\sum_{i=2}^{n}\frac{1}{\lambda_i} + m\sum_{j=2}^{m}\frac{1}{\mu_j} + 1 - \left(1+(m+n)\left(\sum_{i=2}^{n}\frac{1}{m+\lambda_i} + \sum_{j=2}^{m}\frac{1}{n+\mu_j}\right)\right) \\ &= \sum_{i=2}^{n}\left(\frac{n}{\lambda_i} - \frac{m+n}{m+\lambda_i}\right) + \sum_{j=2}^{m}\left(\frac{m}{\mu_j} - \frac{m+n}{n+\mu_j}\right) \\ &= \sum_{i=2}^{n}\frac{m(n-\lambda_i)}{\lambda_i(m+\lambda_i)} + \sum_{j=2}^{m}\frac{n(m-\mu_j)}{\mu_j(n+\mu_j)}, \end{aligned}$$

hence the desired inequality holds since $\lambda_i \le n$ and $\mu_j \le m$ for $i = 2, \ldots, n$ and $j = 2, \ldots, m$. The equality holds if and only if $\lambda_i = n$ and $\mu_j = m$ for each i and j $(2 \le i \le n, 2 \le j \le m)$. This means that G_1 and G_2 are complete graphs, so also is G_1+G_2. $\square$

3.2. Kirchhoff index of $G_1 \circ G_2$

Complete information about the Laplacian spectrum of $G_1 \circ G_2$ is provided by Barik et al. [5] in terms of the Laplacian spectra of G_1 and G_2.

Lemma 3.5 ([5]). *Let G_1, G_2 be any graphs and $G = G_1 \circ G_2$. Then*

(a) $\frac{\lambda_i+m+1\pm\sqrt{(m+1)^2-4\lambda_i}}{2} \in S(G)$ *with multiplicity* 1 *for* $i = 1, \ldots, n$ *and*
(b) $\mu_j + 1 \in S(G)$ *with multiplicity* n *for* $j = 2, \ldots, m$.

If G_1 is connected, then $G_1 \circ G_2$ is connected. Since $\frac{\lambda_1+m+1-\sqrt{(m+1)^2-4\lambda_1}}{2} = 0$, combining Lemma 3.5 with Theorem 2.3, we readily have the following theorem.

Theorem 3.6. *Let G_1 be a connected graph. Then*

$$Kf(G_1 \circ G_2) = n + 2(m+1)n\sum_{i=2}^{n}\frac{1}{\lambda_i+m+1\pm\sqrt{(m+1)^2-4\lambda_i}} + (m+1)n^2\sum_{j=2}^{m}\frac{1}{\mu_j+1}. \tag{3.3}$$

The expression given in Theorem 3.6 is somewhat complicated. In the following, by computing $Kf(G_1 \circ G_2)$ in a different way we obtain a much simpler expression. Furthermore, we show that $Kf(G_1 \circ G_2)$ can be expressed as a linear combination of $Kf(G_1)$ and $Kf(G_2+K_1)$.

Recall that L_1 is the Laplacian matrix of G_1. Xu [27] obtained the following result:

Lemma 3.7. *The Laplacian polynomial of* $G_1 \circ G_2$ *can be expressed as follows*

$$L(G_1 \circ G_2, \lambda) = \left(\prod_{i=2}^{m}(\lambda - 1 - \mu_i)^n\right) \begin{vmatrix} -L_1 & -(\lambda - m - 1)I_n \\ \lambda I_n & (\lambda - 1)I_n \end{vmatrix}. \tag{3.4}$$

For convenience, we employ some notations. Given a matrix C, let $C^*(i)$ and $C^\dagger(i)$ denote the submatrices obtained from C by deleting the ith row, and ith column, respectively. Similarly, let $C^*(i, j)$ and $C^\dagger(i, j)$ denote the submatrices obtained from C by deleting the ith and jth rows, and ith and jth columns, respectively.

Theorem 3.8. *Let* G_1 *be a connected graph. Then*

$$Kf(G_1 \circ G_2) = (m + 1)n^2 - mn + (m + 1)^2 Kf(G_1) + (m + 1)n^2 \sum_{i=2}^{m} \frac{1}{\mu_i + 1}. \tag{3.5}$$

Proof. If $n = 1$, $Kf(K_1 \circ G_2) = Kf(K_1 + G_2)$ and the result holds by Theorem 3.2. Now we assume that $n \geq 2$.
Let

$$f(\lambda) := \begin{vmatrix} -L_1 & -(\lambda - m - 1)I_n \\ \lambda I_n & (\lambda - 1)I_n \end{vmatrix} = \sum_{i=0}^{2n} a_i \lambda^i.$$

Claim 1.

$$Kf(G_1 \circ G_2) = n(m + 1)\left(\sum_{i=2}^{m} \frac{n}{1 + \mu_i} - \frac{a_2}{a_1}\right). \tag{3.6}$$

Proof. By Lemma 3.7,

$$L(G_1 \circ G_2, \lambda) = f(\lambda)\left(\prod_{i=2}^{m}(\lambda - 1 - \mu_i)^n\right).$$

Since $1 + \mu_i > 0$, $f(\lambda)$ can be written as

$$f(\lambda) = \lambda(\lambda - b_1)(\lambda - b_2)\ldots(\lambda - b_{2n-1})$$

such that $b_i > 0$ for $1 \leq i \leq 2n - 1$. Thus,

$$\sum_{i=1}^{2n-1} \frac{1}{b_i} = \frac{\sum_{i=1}^{2n-1} \prod_{\substack{j=1 \\ j \neq i}}^{2n-1} b_j}{\prod_{i=1}^{2n-1} b_i} = \frac{a_2}{-a_1} = -\frac{a_2}{a_1}.$$

Hence Claim 1 holds by Theorem 2.3. □

The only thing left is the computation of a_1 and a_2, which is given in the following two claims. Let 0_n denote the zero matrix of order n. For $1 \leq i \leq n$, let A_i be the $n \times n$ matrix whose (i, i)-entry is $m + 1$ and whose other entries are all zero.
Before stating the claims, we make some elementary transformations to the determinant in Eq. (3.4).

$$\begin{aligned} f(\lambda) &= \begin{vmatrix} -L_1 & -(\lambda - m - 1)I_n \\ \lambda I_n & (\lambda - 1)I_n \end{vmatrix} = \left| \begin{bmatrix} -L_1 & -(\lambda - m - 1)I_n \\ \lambda I_n & (\lambda - 1)I_n \end{bmatrix} \begin{bmatrix} I_n & 0_n \\ -I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & I_n \\ 0_n & I_n \end{bmatrix} \right| \\ &= \begin{vmatrix} (\lambda - m - 1)I_n - L_1 & -L_1 \\ I_n & \lambda I_n \end{vmatrix}. \end{aligned}$$

Let $M := \begin{bmatrix} -(m+1)I_n - L_1 & -L_1 \\ I_n & 0_n \end{bmatrix}$. Note that $f(0) = \det M$.

Claim 2.

$$a_1 = -n(m + 1)t(G_1).$$

Proof. It is easy to observe that

$$a_1 = \sum_{i=1}^{2n} \det M(i). \tag{3.7}$$

For $1 \le i \le n$, $\det M(i) = 0$ since the $(n-1+i)$th row has all zero entries. Hence

$$\begin{aligned} a_1 &= \sum_{i=n+1}^{2n} \det M(i) = \sum_{i=1}^{n} \begin{vmatrix} -(m+1)I_n - L_1 & -L_1^\dagger(i) \\ I_n^*(i) & 0_{n-1} \end{vmatrix} \\ &= (-1)^n \sum_{i=1}^{n} \begin{vmatrix} (m+1)I_n + L_1 & L_1^\dagger(i) \\ I_n^*(i) & 0_{n-1} \end{vmatrix}. \end{aligned}$$

By consecutively interchanging the ith column with the $(i+1)$th, $(i+2)$th, … and $(n+i-1)$th columns in the last determinant, we obtain

$$\begin{aligned} a_1 &= (-1)^n(-1)^{n-1} \sum_{i=1}^{n} \begin{vmatrix} (m+1)I_n^\dagger(i) + L_1^\dagger(i) & L_1 + A_i \\ I_{n-1} & 0_n^*(i) \end{vmatrix} \\ &= (-1)(-1)^{2n-2} \sum_{i=1}^{n} \begin{vmatrix} L_1 + A_i & (m+1)I_n^\dagger(i) + L_1^\dagger(i) \\ 0_n^*(i) & I_{n-1} \end{vmatrix} \\ &= -\sum_{i=1}^{n} \det(L_1 + A_i) = -\sum_{i=1}^{n} (\det L_1 + (m+1)\det L_1(i)) \\ &= -(m-1)\sum_{i=1}^{n} t(G_1) = -n(m+1)t(G_1). \quad \square \end{aligned}$$

Claim 3.

$$a_2 = (n + n(m+1))t(G_1) + (m+1)^2 \sum_{1 \le i < j \le n} L_1(i,j). \tag{3.8}$$

Proof. First observe that

$$a_2 = \sum_{1 \le i < j \le 2n} \det M(i,j).$$

If $1 \le i < j \le n$, then $\det M(i,j) = 0$ since all elements of the $(n-2+i)$th and $(n-2+j)$th rows are zero. Also, if $1 \le i \le n, j > n$ and $j \ne n+i$, $\det M(i,j) = 0$ since all elements of the $(n-1+i)$th or $(n-2+i)$th row are zero. Hence

$$a_2 = \sum_{i=1}^{n} \det M(i, n+i) + \sum_{n \le i < j \le 2n} \det M(i,j).$$

On the one hand,

$$\begin{aligned} \sum_{i=1}^{n} \det M(i, n+i) &= \sum_{i=1}^{n} \begin{vmatrix} -(m+1)I_{n-1} - L_1(i) & -L_1(i) \\ I_{n-1} & 0_{n-1} \end{vmatrix} \\ &= (-1)^{n-1} \sum_{i=1}^{n} \det(-L_1(i)) = nt(G_1), \end{aligned}$$

while on the other hand,

$$\sum_{n \le i < j \le 2n} \det M(i,j) = \sum_{1 \le i < j \le n} \begin{vmatrix} -(m+1)I_n - L_1 & -L_1^\dagger(i,j) \\ I_n^*(i,j) & 0_{n-2} \end{vmatrix}.$$

By consecutively, first interchanging the ith column with the $(i+1)$th, $(i+2)$th, …, $(n+i-1)$th columns in the last determinant, and then interchanging the jth column with the $(j+1)$th, $(j+2)$th, …, $(n+j-1)$th columns in the resulting determinant, we have

$$\begin{aligned}\sum_{n\le i<j\le 2n}\det M(i,j) &= (-1)^{2n-2}\sum_{1\le i<j\le n}\begin{vmatrix}-(m+1)I_n^{\dagger}(i,j)-L_1^{\dagger}(i,j) & -L_1-A_i-A_j\\ I_{n-2} & 0_n^*(i,j)\end{vmatrix}\\ &= (-1)^{n-2}\sum_{1\le i<j\le n}\begin{vmatrix}-L_1-A_i-A_j & -(m+1)I_n^{\dagger}(i,j)-L_1^{\dagger}(i,j)\\ 0_n^*(i,j) & I_{n-2}\end{vmatrix}\\ &= (-1)^{n-2}(-1)^n\sum_{1\le i<j\le n}\det(L_1+A_i+A_j)\\ &= \sum_{1\le i<j\le n}((m+1)\det L_1(i)+(m+1)\det L_1(j)+(m+1)^2\det L_1(i,j))\\ &= \sum_{1\le i<j\le n}(2(m+1)t(G_1)+(m+1)^2\det L_1(i,j))\\ &= n(n-1)(m+1)t(G_1)+\frac{1}{2}n(n-1)(m+1)^2\det L_1(i,j).\end{aligned}$$

Hence a_2 is obtained by summing $\sum_{i=1}^{n}\det M(i,n+i)$ and $\sum_{n\le i<j\le 2n}\det M(i,j)$. □

Note by Eq. (2.2) that

$$\frac{\sum_{1\le i<j\le n}L_1(i,j)}{t(G)}=\sum_{1\le i<j\le n}r_{ij}(G_1)=Kf(G_1).$$

We complete the proof by substituting Eqs. (3.7) and (3.8) back into Eq. (3.6). □

Theorem 3.9.

$$Kf(G_1\circ G_2)=mn^2-mn+(m+1)^2Kf(G_1)+n^2Kf(G_2+K_1). \tag{3.9}$$

Proof. By Proposition 3.1,

$$S(G_2+K_1)=(0,m+1,1+\mu_2,1+\mu_3,\ldots,1+\mu_m).$$

Then by Theorem 2.3,

$$Kf(G_2+K_1)=(m+1)\left(\frac{1}{m+1}+\sum_{i=2}^{n}\frac{1}{1+\mu_i}\right)=1+(m+1)\sum_{i=2}^{n}\frac{1}{1+\mu_i}.$$

Hence the result follows by Theorem 3.8. □

3.3. *Kirchhoff index of $G_1\{G_2\}$*

Since the root-vertex of each copy of G_2 is a cut-vertex in $G_1\{G_2\}$, we will compute the Kirchhoff index of $G_1\{G_2\}$ by means of the following cut-vertex property.

Proposition 3.10 ([15]). Let k be a cut-vertex of a graph, and let i and j be vertices occurring in different components which arise upon deletion of k. Then

$$r_{ij}=r_{ik}+r_{kj}.$$

Let G be a connected graph of order n. For convenience, for $i\in V(G)$, define

$$Kf_i(G)=\sum_{j=1}^{n}r_{ij}(G).$$

Then obviously $Kf(G)$ can also be written as

$$Kf(G)=1/2\sum_{i=1}^{n}Kf_i(G).$$

Theorem 3.11. *Let G_1 and G_2 be connected graphs. Let r be the root-vertex of G_2. Then*

$$Kf(G_1\{G_2\})=m^2Kf(G_1)+nKf(G_2)+m(n^2-n)Kf_r(G_2). \tag{3.10}$$

Proof. If two vertices u and v belong to the same copy of G_2, then it is obvious that

$$r_{uv}(G_2) = r_{uv}(G_1\{G_2\}).$$

The corresponding contribution to $Kf(G_1\{G_2\})$ is

$$A = nKf(G_2).$$

If the vertices u and v belong to different copies of G_2, then by Proposition 3.10,

$$r_{uv}(G_1\{G_2\}) = r_{ur}(G_2) + r_{ij}(G_1) + r_{rv}(G_2),$$

where i and j denote the vertices of G_1 to which the copies of G_2 are attached. For each fixed pair i, j, there are m^2 such pairs u, v and their contribution to $Kf(G_1\{G_2\})$ amounts to $2mKf_r(G_2) + m^2 r_{ij}(G_1)$. These contributions over all the distinct pairs i, j are

$$B = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (2mKf_r(G_2) + m^2 r_{ij}(G_1)) = m(n^2 - n)Kf_r(G_2) + m^2 Kf(G_1).$$

Therefore the proof is completed by summing A and B. □

In particular, if G_2 is vertex-transitive, then vertices in G_2 are identical. Hence

$$Kf_r(G_2) = \frac{2}{m} Kf(G_2)$$

and we immediately arrive at the following corollary.

Corollary 3.12. *If G_2 is vertex-transitive, then*

$$Kf(G_1\{G_2\}) = m^2 Kf(G_1) + (2n^2 - n)Kf(G_2). \tag{3.11}$$

3.4. Additional result

Notice that $G_1 \circ G_2 = G_1\{G_2 + K_1\}$, where the root of $G_2 + K_1$ is chosen to be the vertex belonging to K_1. By comparing Theorems 3.9 and 3.11, we can obtain the following result:

Theorem 3.13. *Let G be a connected graph on n vertices. If $i \in V(G)$ satisfies $d_G(i) = n - 1$, then*

$$Kf(G) = nKf_i(G) - n + 1. \tag{3.12}$$

In addition, if $G - i$ is vertex-transitive, then for any $j \in G - i$,

$$Kf(G) = (n - 1)(nr_{ij} - 1). \tag{3.13}$$

Proof. By Theorem 3.11, we have

$$Kf(G_1\{G_2 + K_1\}) = (m + 1)^2 Kf(G_1) + nKf(G_2) + (m + 1)(n^2 - n)Kf_r(G_2), \tag{3.14}$$

where r is chosen to be the vertex of K_1, in other words, $d_{G_2+K_1}(r) = m$. Since $G_1 \circ G_2 = G_1\{G_2 + K_1\}$, comparing Eq. (3.14) with Eq. (3.9), we have

$$Kf(G_2 + K_1) = (m + 1)Kf_r(G_2 + K_1) - m,$$

which yields Eq. (3.12).

If $G - i$ is vertex-transitive, then for any $j \in G - i$,

$$Kf_i(G) = (n - 1)r_{ij}. \tag{3.15}$$

Hence Eq. (3.13) is derived by substituting Eq. (3.15) into Eq. (3.12). □

For example, we compute the Kirchhoff (Wiener) index of the n-vertex star S_n. Suppose that v is the center of S_n. Then $d_{S_n}(v) = n - 1$ and obviously $S_n - v$ is vertex-transitive. Hence by Eq. (3.13),

$$Kf(S_n) = W(S_n) = (n - 1)(n - 1) = (n - 1)^2.$$

This agrees with the result obtained by Entringer et al. [7].

3.5. Some examples

Let C_n and P_n denote the cycle and path of order n, respectively. The Laplacian spectrum of C_n and P_n [1] are

$$S(C_n) = \left(0, 4\sin^2\frac{\pi}{n}, 4\sin^2\frac{2\pi}{n}, \dots, 4\sin^2\frac{(n-1)\pi}{n}\right),$$
$$S(P_n) = \left(0, 4\sin^2\frac{\pi}{2n}, 4\sin^2\frac{2\pi}{2n}, \dots, 4\sin^2\frac{(n-1)\pi}{2n}\right),$$

and it is well known [7,13] that

$$Kf(C_n) = \frac{n^3-n}{12},$$
$$Kf(P_n) = \frac{n^3-n}{6}.$$

The cone graph $C_{m,n}$ and fan graph $F_{m,n}$ are defined as $C_m + \overline{K_n}$ and $\overline{K_m} + P_n$, respectively. Hence by Theorem 3.2, we have

$$Kf(C_{m,n}) = \frac{mn + (n-1)n}{m} + (m+n)\sum_{k=1}^{m-1}\frac{1}{n + 4\sin^2\frac{k\pi}{m}}, \tag{3.16}$$

$$Kf(F_{m,n}) = \frac{(m-1)m + mn}{n} + (m+n)\sum_{k=1}^{n-1}\frac{1}{m + 4\sin^2\frac{k\pi}{2n}}. \tag{3.17}$$

In particular, for the wheel graph W_n and fan F_n, which are defined as $C_{n-1,1}$ and $F_{1,n-1}$, we have

$$Kf(W_n) = 1 + \sum_{k=1}^{n-2}\frac{1}{1 + 4\sin^2\frac{k\pi}{n-1}}, \tag{3.18}$$

$$Kf(F_n) = 1 + \sum_{k=1}^{n-2}\frac{1}{1 + 4\sin^2\frac{k\pi}{2(n-1)}}. \tag{3.19}$$

The r-corona graph of a graph G, denoted by $I_r(G)$, is defined as $G \circ \overline{K_r}$. Since $\overline{K_r} + K_1 = S_{r+1}$, by Theorem 3.9,

$$Kf(I_r(G)) = (r+1)^2 Kf(G) + n^2r^2 + n^2r - nr, \tag{3.20}$$

where n is the vertex number of G.

The sun graph $\mathrm{Sun}_{m,n} = C_m\{P_{n+1}\}$, such that P_{n+1} is rooted at a vertex of degree one. Then by Theorem 3.11, we have

$$Kf(\mathrm{Sun}_{m,n}) = \frac{m(n+1)}{12}((m-1)(n+1)(m+6n+1) + 2n(n+2)). \tag{3.21}$$

4. Concluding Remarks

In this paper, we study the Kirchhoff index of three types of composite graphs. Combining the results obtained in the present paper with those in [26], formulae for computing the Kirchhoff index of the product, composition, join, corona and cluster of graphs are obtained. Along this line, we may consider the Kirchhoff index of other composite graphs, such as the tensor product, symmetric difference of graphs, etc. Furthermore, to gain a more intuitive understanding of the Kirchhoff index of composite graphs, finding possible relations between the Kirchhoff indices of original graphs and those of their composite graphs, such as a linear combination between them, may be of interest. Besides, it is worth considering the Kirchhoff index of graphs derived from a single graph, such as the line graph, the subdivision graph, the total graph, etc.

References

[1] W.N. Anderson, T.D. Morley, Eigenvalues of the Laplacian of a graph, Linear Multilinear Algebra 18 (1985) 141–145.
[2] D. Babić, D.J. Klein, I. Lukovits, S. Nikolić, N. Trinajstić, Resistance-distance matrix: A computational algorithm and its application, Int. J. Quantum Chem. 90 (2002) 166–176.
[3] A.T. Balaban, X. Liu, D.J. Klein, D. Babic, T.G. Schmalz, W.A. Seitz, M. Randić, Graph invariants for fullerenes, J. Chem. Inf. Comput. Sci. 35 (1995) 396–404.
[4] R.B. Bapat, I. Gutman, W.J. Xiao, A simple method for computing resistance distance, Z. Naturforsch. 58a (2003) 494–498.
[5] S. Barik, S. Pati, B.K. Sarma, The spectrum of the corona of two graphs, SIAM J. Discrete Math. 21 (1) (2007) 47–56.
[6] D. Bonchev, A.T. Balaban, X. Liu, D.J. Klein, Molecular cyclicity and centricity of polycyclic graphs. I. Cyclicity based on resistance distances or reciprocal distances, Int. J. Quantum Chem. 50 (1994) 1–20.
[7] R.C. Entringer, D.E. Jackson, D.A. Snyder, Distance in graphs, Czechoslovak Math. J. 26 (1976) 283–296.

[8] P.W. Fowler, Resistance distances in fullerene graphs, Croat. Chem. Acta 75 (2002) 401–408.
[9] I. Gutman, B. Mohar, The Quasi-Wiener and the Kirchhoff indices coincide, J. Chem. Inf. Comput. Sci. 36 (1996) 982–985.
[10] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem. 72 (1847) 497–508.
[11] D.J. Klein, Resistance-distance sum rules, Croat. Chem. Acta 75 (2002) 633–649.
[12] D.J. Klein, Graph geometry, graph metrics and Wiener, MATCH Commun. Math. Comput. Chem. 35 (1997) 7–27.
[13] D.J. Klein, I. Lukovits, I. Gutman, On the definition of the hyper-wiener index for cycle-containing structures, J. Chem. Inf. Comput. Sci. 35 (1995) 50–52.
[14] D.J. Klein, J.L. Palacios, M. Randić, N. Trinajstić, Random walks and chemical graph theory, J. Chem. Inf. Comput. Sci. 44 (2004) 1521–1525.
[15] D.J. Klein, M. Randić, Resistance distance, J. Math. Chem. 12 (1993) 81–95.
[16] I. Lukovits, S. Nikolić, N. Trinajstić, Resistance distance in regular graphs, Int. J. Quantum Chem. 71 (1999) 217–225.
[17] I. Lukovits, S. Nikolić, N. Trinajstić, Note on the resistance distances in the dodecahedron, Croat. Chem. Acta 73 (2000) 957–967.
[18] R. Merris, Laplacian matrix of graphs: A survey, Linear Algebra Appl. 197–198 (1994) 143–176.
[19] J.L. Palacios, Resistance distance in graphs and random walks, Int. J. Quantum Chem. 81 (2001) 29–33.
[20] J.L. Palacios, Closed-form formulas for Kirchhoff index, Int. J. Quantum Chem. 81 (2001) 135–140.
[21] J.L. Palacios, Foster's formulas via probability and the Kirchhoff index, Methodol. Comput. Appl. Probab. 6 (2004) 381–387.
[22] B.Y. Sagan, Y.N. Yeh, P. Zhang, The wiener polynomial of a graph, Int. J. Quantum Chem. 60 (1996) 959–969.
[23] D. Stevanović, Hosoya polynomial of composite graphs, Discrete Math. 235 (1) (2001) 237–244.
[24] H. Wiener, Structural determination of paraffin boiling points, J. Amer. Chem. Soc. 69 (1947) 17–20.
[25] W. Xiao, I. Gutman, Resistance distance and Laplacian spectrum, Theor. Chem. Acc. 110 (2003) 284–289.
[26] H. Xu, The Laplacian spectrum and Kirchhoff Index of product and lexicographic product of graphs, J. Xiamen Univ. (Nat. Sci.) 42 (2003) 552–554 (in Chinese).
[27] H. Xu, The Laplacian spectrum of coronas, J. Xiamen Univ. (Nat. Sci.) 44 (2005) 745–748 (in Chinese).
[28] Y. Yang, X. Jiang, Unicyclic graphs with extremal Kirchhoff index, MATCH Commun. Math. Comput. Chem. 60 (1) (2008) 107–120.
[29] Y. Yang, H. Zhang, Kirchhoff index of linear hexagonal chains, Int. J. Quantum Chem. 108 (2008) 503–512.
[30] Y.N. Yeh, I. Gutman, On the sum of all distances in composite graphs, Discrete Math. 135 (1994) 359–365.
[31] H. Zhang, Y. Yang, Resistance distance and Kirchhoff index in circulant graphs, Int. J. Quantum Chem. 107 (2007) 330–339.
[32] H.-Y. Zhu, D.J. Klein, I. Lukovits, Extensions of the Wiener number, J. Chem. Inf. Comput. Sci. 36 (1996) 420–428.