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On Fundamental Solutions of Linear Parabolic Equations of Higher Order in Time and Associated Volterra Equations*

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1. INTRODUCTION

This paper is concerned with the initial-boundary value problem for the linear parabolic equation of higher order in t:

$$\sum_{k=0}^{l} A_{l-k}(x, t, D_x)\, D_t^k\, u(x, t) = f(x, t), \qquad \Omega \times (0, T], \tag{1.1}$$

$$B_j(x, t, D_x)\, u(x, t) = 0, \qquad j = 1, ..., m, \qquad \partial\Omega \times (0, T], \tag{1.2}$$

$$(D_t^k u)(x, 0) = u_k(x), \qquad k = 0, ..., l-1, \qquad \Omega, \tag{1.3}$$

and the same problem for the associated Volterra equation

$$\sum_{k=0}^{l} A_{l-k}(x, t, D_x)\, D_t^k u(x, t),$$
$$= \int_0^t B(x, t, s, D_x)\, u(x, s)\, ds + f(x, t), \qquad \Omega \times (0, T]. \tag{1.4}$$

Here $A_j(x, t, D_x)$, $j = 1, ..., l$, and $B(x, t, s, D_x)$ are linear differential operators in x with coefficients defined in $\bar{\Omega} \times [0, T]$ and $\bar{\Omega} \times \{(t, s)\colon 0 \leqslant s \leqslant t \leqslant T\}$, respectively, and $A_0(x, t, D_x) = 1$. $\{B_j(x, t, D_x)\}_{j=1}^m$ is a system of linear differential operators with coefficients defined on $\partial\Omega \times [0, T]$ which do not contain derivatives in t. The operator on the left side of (1.1) and (1.4) is assumed to be parabolic in the sense of Petrowsky.

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We plan to solve problems (1.1)–(1.3) and (1.4), (1.2), (1.3) in the space $L^p(\Omega)$, $1<p<\infty$, and formulate the problem as follows:

$$A(t, D_t)\, u(t) = f(t), \qquad 0 < t \leqslant T, \tag{1.5}$$

$$(D_t^k u)(0) = u_k, \qquad k = 0, ..., l-1, \tag{1.6}$$

and

$$A(t, D_t)\, u(t) = \int_0^t B(t, s)\, u(s)\, ds + f(t), \qquad 0 < t \leqslant T, \tag{1.7}$$

$$(D_t^k u)(0) = u_k, \qquad k = 0, ..., l-1, \tag{1.8}$$

respectively. Here $A(t, D_t)$ is the operator defined by

$$A(t, D_t) = \sum_{k=0}^{l} A_{l-k}(t)\, D_t^k, \tag{1.9}$$

where $A_0(t) = I$, $A_{l-k}(t) = A_{l-k}(x, t, D_x)$ for $k = 1, ..., l-1$, $A_l(t)$ is the realization of $A_l(x, t, D_x)$ in $L^p(\Omega)$ under the boundary conditions $B_j(x, t, D_x)\, u|_{\partial\Omega} = 0$, $j = 1, ..., m$, and $B(t, s) = B(x, t, s, D_x)$. Our main objective is to construct the fundamental solutions $U(t, s)$ and $W(t, s)$ for (1.5), (1.6) and (1.7), (1.8), respectively, which are bounded operator valued functions satisfying

$$A(t, D_t)\, U(t, s) = 0, \qquad 0 \leqslant s < t \leqslant T, \tag{1.10}$$

$$D_t^k U(t, s) = 0 \quad \text{at} \quad t = s \quad \text{for} \quad k = 0, ..., l-2, \tag{1.11}$$

$$D_t^{l-1} U(t, s) = I \quad \text{at} \quad t = s, \tag{1.12}$$

$$A(t, D_t)\, W(t, s) = \int_s^t B(t, \sigma)\, W(\sigma, s)\, d\sigma, \qquad 0 \leqslant s < t \leqslant T, \tag{1.13}$$

$$D_t^k W(t, s) = 0 \quad \text{at} \quad t = s \quad \text{for} \quad k = 0, ..., l-2, \tag{1.14}$$

$$D_t^{l-1} W(t, s) = I \quad \text{at} \quad t = s. \tag{1.15}$$

The operator $B(t, s)$ has the same order as $A_l(t)$, and the integral of the right of (1.8) should be understood as an improper integral:

$$\int_s^t B(t, \sigma)\, W(\sigma, s)\, d\sigma = \lim_{\varepsilon \to +0} \int_{s+\varepsilon}^t B(t, \sigma)\, W(\sigma, s)\, d\sigma.$$

With the aid of the fundamental solutions we give explicit representations of the unique solutions of (1.5), (1.6) and (1.7), (1.8) for initial values satisfying a compatibility condition and a Hölder continuous inhomogeneous term.

In the previous paper [10] we considered the case in which the boundary conditions (1.2) are independent of t. In this paper we investigate the case of time-dependent boundary conditions under more restrictive regularity assumptions on the coefficients of $A_j(x, t, D_x)$, $j = 1, ..., l$. As in the previous paper [10] we rely on the weighted elliptic estimates by S. Agmon and L. Nirenberg [1] in the construction of the fundamental solution of (1.5), (1.6) but following the method of [3] instead of [7]. Once this is accomplished, the fundamental solution of (1.7), (1.8) is constructed in just the same manner as [10] following the method of J. Prüss [6].

We show that $U(t, s)$ and $W(t, s)$ also satisfy the adjoint equation, i.e.,

$$\sum_{k=0}^{l} (-D_s)^k \,(U(t, s)\, A_{l-k}(s)) = 0, \tag{1.16}$$

$$\sum_{k=0}^{l} (-D_s)^k \,(W(t, s)\, A_{l-k}(s)) = \int_s^t W(t, \sigma)\, B(\sigma, s)\, d\sigma. \tag{1.17}$$

However, in the proof of (1.17) we need a strong smoothness hypothesis on the coefficients of $B(t, s)$ unlike the verification of (1.16) and in contrast with the case $l = 1$.

2. Assumptions and Theorems

Let Ω be a bounded domain in R^n, $n > 1$, with boundary $\partial\Omega$. We put $D_x = (D_1, ..., D_n)$, $D_i = \partial/\partial x_i$, $D_t = \partial/\partial t$, $D_x^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ for a multi-integer $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \geqslant 0$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. T denotes a positive number, and we put

$$\Delta = \{(t, s) : 0 \leqslant s < t \leqslant T\}, \qquad \bar{\Delta} = \{(t, s) : 0 \leqslant s \leqslant t \leqslant T\}.$$

We are interested in operators

$$\mathcal{A}(x, t, D_x, D_t) = \sum_{k=0}^{l} A_{l-k}(x, t, D_x)\, D_t^k \quad \text{and} \quad B(x, t, s, D_x), \tag{2.1}$$

where $A_j(x, t, D_x)$, $j = 1, ..., l$, are linear differential operators in x with coefficients defined in $\bar{\Omega} \times [0, T]$, $A_0(x, t, D_x) = 1$, and $B(x, t, s, D_x)$ is a linear differential operator in x with coefficients defined in $\bar{\Omega} \times \bar{\Delta}$.

Let s_j be the order of A_j. It is assumed that

$$s_l = 2m, \qquad s_j \leqslant 2mj/l, \qquad j = 0, ..., l-1,$$

for some integer $m \geqslant 1$ and that m and l are related by the condition $2m/l = d$, an even integer. The order of B is assumed to be $2m$.

In addition to (2.1) there are m linear differential boundary operators

$$B_j(x, t, D_x) = \sum_{|\beta| \leqslant m_j} b_{j,\beta}(x, t)\, D_x^\beta, \qquad j = 1, ..., m,$$

of respective order $m_j < 2m$ which do not contain D_t.

We denote by $A_j^\#(x, t, D_x)$ the sum of terms of $A_j(x, t, D_x)$ which are of order jd, and put

$$\mathcal{A}^\#(x, t, D_x, D_t) = \sum_{k=0}^{l} A_{l-k}^\#(x, t, D_x)\, D_t^k.$$

Similarly $B_j^\#(x, t, D_x)$ is the sum of terms of $B_j(x, t, D_x)$ which are of order m_j.

We state our basic assumptions.

(I.1) $\mathcal{A}(x, t, D_x, D_t)$ is parabolic in the sense of Petrowsky, i.e., for all real n-vectors $\xi \neq 0$, all $(x, t) \in \bar{\Omega} \times [0, T]$, and all complex numbers λ with $\operatorname{Re} \lambda \geqslant 0$, $\mathcal{A}^\#(x, t, i\xi, \lambda) \neq 0$.

(I.2) Ω is a bounded domain of class C^{2m}. At any point (x, t) of $\partial\Omega \times [0, T]$ let ν be the normal to $\partial\Omega$ at x and ξ be parallel to $\partial\Omega$ at x or $\xi = 0$. Let λ be any complex number with $\operatorname{Re} \lambda \geqslant 0$. Then, if $(\xi, \lambda) \neq 0$, the polynomials in s: $B_j^\#(x, t, \xi + s\nu)$, $j = 1, ..., m$, are linearly independent modulo the polynomial $\prod_{k=1}^{m} (s - s_k^+(\xi, \lambda))$ where $s_k^+(\xi, \lambda)$ are the roots of $\mathcal{A}^\#(x, t, i(\xi + s\nu), \lambda)$ with positive imaginary part.

(I.3) The coefficients of A_j, $j = 1, ..., l$, and their derivatives in t of order up to l are continuous in $\bar{\Omega} \times [0, T]$. These derivatives of order l are uniformly Hölder continuous in t in $\bar{\Omega} \times [0, T]$ with exponent ρ. As regards the coefficients of B_j

$$D_t^k D_x^\gamma b_{j,\beta}, \; k = 0, ..., l, \qquad |\gamma| \leqslant 2m - m_j, \qquad |\beta| \leqslant m_j, j = 1, ..., m,$$

are continuous on $\partial\Omega \times [0, T]$, and these derivatives with $k = l$ are uniformly Hölder continuous in t on $\partial\Omega \times [0, T]$ with exponent ρ.

(I.4) The coefficients of A_j, $j = 1, ..., l$, and their derivatives in t of order up to $2l$ are continuous on $\partial\Omega \times [0, T]$. As regards the coefficients of B_j

$$D_t^k D_x^\gamma b_{j,\beta}, \; k = 0, ..., 2l, \qquad |\gamma| \leqslant 2m - m_j, \qquad |\beta| \leqslant m_j, j = 1, ..., m,$$

are continuous on $\partial\Omega \times [0, T]$.

(I.5) The coefficients of B are continuous in $\bar{\Omega} \times \bar{\Delta}$ and uniformly Hölder continuous in (t, s) in $\bar{\Omega} \times \bar{\Delta}$ with exponent ρ.

Let $W^{j,p}(\Omega)$, $1<p<\infty$, be the usual Sobolev space with the norm

$$\|u\|_{j,p}=\left(\sum_{|\alpha|\leqslant j}\int_\Omega |D^\alpha u|^p\,dx\right)^{1/p}.$$

The norm of $L^p(\Omega)$ is simply denoted by $\|\ \|_p$. We use the notations $B(L^p, L^p)$, $B(L^p, W^{j,p})$ to denote the set of all bounded linear operators from $L^p(\Omega)$ to $L^p(\Omega)$, $W^{j,p}(\Omega)$, respectively. We denote the norm of $B(L^p, W^{j,p})$ by $\|\ \|_{B(L^p, W^{j,p})}$ and in case $j=0$ simply by $\|\ \|$ instead of $\|\ \|_{B(L^p, L^p)}$.

For a Banach space X and an interval I we denote by $C(I; X)$ the set of all functions which take values in X and are strongly continuous in I.

The operators $A_j(t)$, $j=0, ..., l$, are defined as follows: $A_0(t)=I$, $A_j(t)\,u=A_j(x, t, D_x)\,u$ for $u\in W^{jd,p}(\Omega)$ if $j=1, ..., l-1$, and $A_l(t)\,u=A_l(x, t, D_x)\,u$ for u in

$$D(A_l(t))=\{u\in W^{2m,p}(\Omega) \colon B_j(x, t, D_x)\,u|_{\partial\Omega}=0 \quad \text{for} \quad j=1, ..., m\}.$$

We put

$$A(t, D_t)=\sum_{k=0}^{l} A_{l-k}(t)\,D_t^k.$$

The operator $B(t, s)$ is defined by $B(t, s)\,u=B(x, t, s, D_x)\,u$ for $u\in W^{2m,p}(\Omega)$.

Throughout the paper we denote by C constants depending only upon the above basic assumptions.

THEOREM 1. *Under the assumptions* (I.1)–(I.3) *the fundamental solution* $U(t, s)$ *of* (1.5), (1.6) *exists. We have for* $j+kd\leqslant 2m$

$$\|D_t^k U(t, s)\|_{B(L^p, W^{j,p})}\leqslant C(t-s)^{l-1-k-j/d}, \tag{2.2}$$

$$\lim_{t-s\to 0} D_t^k U(t, s)=0 \qquad \textit{for} \quad k=0, ..., l-2 \tag{2.3}$$

in the strong operator topology of $B(L^p, W^{2m-kd-d,p})$, *and*

$$\lim_{t-s\to 0} D_t^{l-1} U(t, s)=I \tag{2.4}$$

in the strong operator topology of $B(L^p, L^p)$.

For given elements $u_0, ..., u_{l-1}$ of $L^p(\Omega)$ we assume that

(II) (Compatibility Condition) For $i=0, ..., l-2$, $u_i \in W^{2m,p}(\Omega)$, and

$$\sum_{k=0}^{i} \binom{i}{k} B_j^{(k)}(x, 0, D_x)\, u_{i-k} = 0, \qquad j=1, ..., m,$$

on the boundary $\partial\Omega$, where $B_j^{(k)}$ is the operator obtained by differentiating the corresponding coefficients of B_j k times in t.

LEMMA 2.1. *Under the assumption* (II) *the elements $v_0, ..., v_{l-2}$ defined successively by*

$$v_0 = u_0, \qquad v_i = u_i - \sum_{k=0}^{i-1} \binom{i}{k} D_t^{i-k} A_l(t)^{-1} A_l(0)\, v_k |_{t=0} \tag{2.5}$$

belong to $D(A_l(0))$.

Set

$$v_{l-1} = u_{l-1} - \sum_{k=0}^{l-2} \binom{l-1}{k} D_t^{l-1-k} A_l(t)^{-1} A_l(0)\, v_k |_{t=0}. \tag{2.6}$$

THEOREM 2. *Let the assumptions* (I.1)–(I.3) *be satisfied. Then, for any elements $u_0, ..., u_{l-1}$ of $L^p(\Omega)$ satisfying* (II) *and for any function $f(t)$ with values in $L^p(\Omega)$ which is Hölder continuous in $[0, T]$, the function $u(t)$ defined by*

$$u(t) = \sum_{i=0}^{l-1} u_i(t) + \int_0^t U(t, s) f(s)\, ds, \tag{2.7}$$

$$u_i(t) = \frac{t^i}{i!} A_l(t)^{-1} A_l(0)\, v_i - \int_0^t U(t, s)\, A(s, D_s) \left(\frac{s^i}{i!} A_l(s)^{-1} A_l(0)\, v_i \right) ds, \qquad i=0, ..., l-2, \tag{2.8}$$

$$u_{l-1}(t) = U(t, 0)\, v_{l-1} \tag{2.9}$$

is a solution of (1.5), (1.6), *where $v_0, ..., v_{l-1}$ are the elements defined by* (2.5), (2.6). *The initial conditions* (1.6) *are satisfied in the following sense: for $k=0, ..., l-1$*

$$\lim_{t \to 0} D_t^k u(t) = u_k \text{ strongly in } W^{2m-kd-d,p}(\Omega). \tag{2.10}$$

THEOREM 3. *Suppose that the assumptions* (I.1)–(I.4) *are satisfied. Then,* *the fundamental solution* $U(t, s)$ *of* (1.5), (1.6) *is* l *times continuously differentiable in* s, *and for* $j + kd \leqslant 2m$

$$\| D_s^k U(t, s) \|_{B(L^p, W^{j,p})} \leqslant C(t-s)^{l-1-k-j/d}, \tag{2.11}$$

$$\lim_{t-s \to 0} D_s^k U(t, s) = 0 \qquad \text{for} \quad k = 0, ..., l-2 \tag{2.12}$$

in the strong operator topology of $B(L^p, W^{2m-kd-d,p})$, *and*

$$\lim_{t-s \to 0} (-D_s)^{l-1} U(t, s) = I \tag{2.13}$$

in the strong operator topology of $B(L^p, L^p)$. *For* $(t, s) \in \Delta$ *and* $v \in D(A_l(s))$ *we have*

$$\sum_{k=0}^{l} (-D_s)^k (U(t, s) A_{l-k}(s)) v = 0. \tag{2.14}$$

The following theorem is concerned with the uniqueness of the solution.

THEOREM 4. *Suppose that the assumptions* (I.1)–(I.4) *are satisfied. Let* f *be a bounded function belonging to* $C((0, T]; L^p(\Omega))$. *If* u *is a function such that for* $k = 0, ..., l-1$

$$D_t^k u \in C([0, T]; W^{2m-kd-d,p}(\Omega)) \cap C((0, T]; W^{2m-kd,p}(\Omega)),$$

$D_t^l u \in C((0, T]; L^p(\Omega))$, $t \, \| D_t^k u(t) \|_{2m-kd,p}$ *is bounded in* $(0, T]$ *for* $k = 0, ..., l$, *and* (1.5) *holds, then for* $t \in (0, T]$

$$u(t) = \sum_{i=0}^{l-1} \sum_{k=i+1}^{l} (-D_s)^{k-1-i} (U(t, s) A_{l-k}(s)) \cdot D_s^i u(s)|_{s=0} + \int_0^t U(t, s) f(s) \, ds. \tag{2.15}$$

THEOREM 5. *Under the assumptions* (I.1)–(I.3), (I.5) *the fundamental solution* $W(t, s)$ *of* (1.7), (1.8) *exists. We have for* $j + kd \leqslant 2m$

$$\| D_t^k W(t, s) \|_{B(L^p, W^{j,p})} \leqslant C(t-s)^{l-1-k-j/d}, \tag{2.16}$$

$$\lim_{t-s \to 0} D_t^k W(t, s) = 0 \qquad \textit{for} \quad k = 0, ..., l-2 \tag{2.17}$$

in the strong operator toplogy of $B(L^p, W^{2m-kd-d,p})$, *and*

$$\lim_{t-s \to 0} D_t^{l-1} W(t, s) = I \tag{2.18}$$

in the strong operator topology of $B(L^p, L^p)$.

THEOREM 6. *Let the assumptions* (I.1)–(I.3), (I.5) *be satisfied. Then, for any elements* $u_0, ..., u_{l-1}$ *of* $L^p(\Omega)$ *satisfying* (II) *and for any Hölder continuous function* $f(t)$ *in* $[0, T]$ *with values in* $L^p(\Omega)$, *the function* $u(t)$ *defined by*

$$u(t) = \sum_{i=0}^{l-1} u_i(t) + \int_0^t W(t, s) f(s)\, ds, \tag{2.19}$$

$$\begin{aligned} u_i(t) = {} & \frac{t^i}{i!} A_l(t)^{-1} A_l(0)\, v_i \\ & + \int_0^t W(t, \tau) \left\{ \int_0^\tau B(\tau, s) \frac{s^i}{i!} A_l(s)^{-1} A_l(0)\, v_i\, ds \right. \\ & \left. - A(\tau, D_\tau) \left(\frac{\tau^i}{i!} A_l(\tau)^{-1} A_l(0)\, v_i \right) \right\} d\tau \qquad \text{for} \quad i = 0, ..., l-2, \end{aligned} \tag{2.20}$$

$$u_{l-1}(t) = W(t, 0)\, v_{l-1}, \tag{2.21}$$

where $v_0, ..., v_{l-1}$ *are the elements defined by* (2.5), (2.6) *is the solution of* (1.7), (1.8). *The integral of the right side of* (1.7) *exists in the improper sense*:

$$\int_0^t B(t, s)\, u(s)\, ds = \lim_{\varepsilon \to +0} \int_\varepsilon^t B(t, s)\, u(s)\, ds.$$

The initial conditions are satisfied in the sense of (2.10).

THEOREM 7. *Suppose that the assumptions* (I.1)–(I.5) *are satisfied. Then, the solution of* (1.7), (1.8) *is unique in the class of functions such that for* $k = 0, ..., l-1$

$$D_t^k u \in C([0, T]; W^{2m-kd-d, p}(\Omega)) \cap C((0, T]; W^{2m-kd, p}(\Omega)),$$

$D_t^l u \in C((0, T];\ L^p(\Omega))$, $t \, \| D_t^k u(t) \|_{2m-kd, p}$ *is bounded in* $(0, T]$ *for* $k = 0, ..., l$, and $\int_\varepsilon^t B(t, s)\, u(s)\, ds$ *is uniformly bounded for* $0 < \varepsilon < t \leqslant T$ *and converges as* $\varepsilon \to 0$ *in the strong topology of* $L^p(\Omega)$ *uniformly in any closed subset of* $(0, T]$.

THEOREM 8. *Suppose that the assumptions* (I.1)–(I.5) *are satisfied. If in addition the coefficients of* $B(t, s)$ *are* l *times continuously differentiable in* (t, s) *in* $\bar{\Omega} \times \bar{\Delta}$, *then the fundamental solution* $W(t, s)$ *of* (1.7), (1.8) *is* l *times continuously differentiable in* s, *and for* $j + kd \leqslant 2m$

$$\| D_s^k W(t, s) \|_{B(L^p, W^{j,p})} \leqslant C(t-s)^{l-1-k-j/d}, \tag{2.22}$$

$$\lim_{t-s \to 0} D_s^k W(t, s) = 0 \qquad \text{for} \quad k = 0, ..., l-2 \tag{2.23}$$

in the strong operator topology of $B(L^p, W^{2m-kd-d,p})$, *and*

$$\lim_{t-s\to 0}(-D_s)^{l-1}W(t,s)=I \tag{2.24}$$

in the strong operator topology of $B(L^p, L^p)$. *For* $(t,s)\in\Delta$ *and* $v\in D(A_l(s))$ *we have*

$$\sum_{k=0}^{l}(-D_s)^k\,(W(t,s)\,A_{l-k}(s))\,v=\int_0^t W(t,\sigma)\,B(\sigma,s)\,v\,d\sigma. \tag{2.25}$$

Let f *be a bounded function belonging to* $C((0,T];L^p(\Omega))$. *If* u *is a function in the class of functions stated in Theorem* 7 *and satisfies* (1.7), (1.8), *then*

$$\begin{aligned}u(t)=&\sum_{i=0}^{l-1}\sum_{k=i+1}^{l}(-D_s)^{k-i-1}\,(W(t,s)\,A_{l-k}(s))\cdot D_s^i u(s)|_{s=0}\\&+\int_0^t W(t,s)f(s)\,ds.\end{aligned} \tag{2.26}$$

3. Some Lemmas

Throughout this section we assume that the hypothesis (I.1)–(I.3) are satisfied.

The following lemma can be shown slightly modifying the proof of Theorem 5.2 of [1].

Lemma 3.1. *There exist constants* C_p *and* λ_p *such that the following inequality holds for* λ *with* $\operatorname{Re}\lambda\geqslant 0$, $|\lambda|\geqslant\lambda_p$, $t\in[0,T]$, *and* $u\in W^{2m,p}(\Omega)$:

$$\begin{aligned}&\sum_{j=0}^{2m}|\lambda|^{(2m-j)/d}\,\|u\|_{j,p}\\&\quad\leqslant C_p\Big\{\|\mathcal{A}(x,t,D_x,\lambda)\,u\|_p\\&\qquad+\sum_{j=1}^{m}|\lambda|^{(2m-m_j)}\,\|g_j\|_p+\sum_{j=1}^{m}\|g_j\|_{2m-m_j,p}\Big\},\end{aligned} \tag{3.1}$$

where g_j *is an arbitrary function in* $W^{2m-m_j,p}(\Omega)$ *coinciding with* $B_j(x,t,D_x)\,u$ *on* $\partial\Omega$ *for each* $j=1,\ldots,m$.

Replacing the unknown function u by $e^{-ct}u$ for some positive constant c if necessary we may and will assume that there exists an angle $\theta_0\in(\pi/2,\pi]$

such that the conclusion of Lemma 3.1 holds for $\lambda \in \Sigma = \{\lambda: |\arg \lambda| \leqslant \theta_0\} \cup \{0\}$. Hence, for $\lambda \in \Sigma$ the operator $A(t, \lambda)$ defined by

$$D(A(t, \lambda)) = D(A_l(t)), \qquad A(t, \lambda) = \sum_{k=0}^{l} A_{l-k}(t)\, \lambda^k,$$

especially $A_l(t)$, has a bounded inverse, and for any $f \in L^p(\Omega)$

$$\sum_{j=0}^{2m} |\lambda|^{(2m-j)/d} \, \| A(t, \lambda)^{-1} f \|_{j,p} \leqslant C \, \| f \|_p. \tag{3.2}$$

Furthermore, arguing as in the proof of Lemma 3.6 of [9] we can show that $A(t, \lambda)^{-1} f$ is l times continuously differentiable in t and

$$\sum_{j=0}^{2m} |\lambda|^{(2m-j)/d} \, \| D_t^k A(t, \lambda)^{-1} f \|_{j,p} \leqslant C \, \| f \|_p, \qquad k = 1, ..., l, \tag{3.3}$$

$$\sum_{j=0}^{2m} |\lambda|^{(2m-j)/d} \, \| D_t^l A(t, \lambda)^{-1} f - D_s^l A(s, \lambda)^{-1} f \|_{j,p} \leqslant C \, |t-s|^\rho \, \| f \|_p. \tag{3.4}$$

If (I.4) is also satisfied, $A(t, \lambda)^{-1} f$ is $2l$ times continuously differentiable in t and (3.3) holds for $k = 1, ..., 2l$.

The fundamental solution $U(t, s)$ of (1.5), (1.6) can be constructed as follows,

$$U(t, s) = U_0(t, s) + Z(t, s), \tag{3.5}$$

$$U_0(t, s) = (2\pi \sqrt{-1})^{-1} \int_\Gamma e^{\lambda(t-s)} A(t, \lambda)^{-1} \, d\lambda, \tag{3.6}$$

where Γ is a smooth contour running in $\Sigma \backslash \{0\}$ from $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$,

$$Z(t, s) = \int_s^t U_0(t, \tau) \, R(\tau, s) \, d\tau, \tag{3.7}$$

$$R(t, s) - \int_s^t R_1(t, \tau) \, R(\tau, s) \, d\tau = R_1(t, s), \tag{3.8}$$

$$\begin{aligned} R_1(t, s) &= -A(t, D_t) \, U_0(t, s) \\ &= -(2\pi \sqrt{-1})^{-1} \int_\Gamma e^{\lambda(t-s)} \sum_{k=1}^{l} A_{l-k}(t) \\ &\quad \times \sum_{i=1}^{k} \binom{k}{i} \lambda^{k-i} D_t^i A(t, \lambda)^{-1} \, d\lambda. \end{aligned} \tag{3.9}$$

LEMMA 3.2. *For $j+kd\leqslant 2m$*

$$\| D_t^k U_0(t,s)\|_{B(L^p,\,W^{j,p})}\leqslant C(t-s)^{l-1-k-j/d}, \tag{3.10}$$

$$\| D_s^k U_0(t,s)\|_{B(L^p,\,W^{j,p})}\leqslant C(t-s)^{l-1-k-j/d}. \tag{3.11}$$

For $j+kd\leqslant 2m$, $k>0$

$$\|(D_t+D_s)\,D_t^{k-1}\,U_0(t,s)\|_{B(L^p,\,W^{j,p})}\leqslant C(t-s)^{l-k-j/d}. \tag{3.12}$$

For $0\leqslant s<\tau<t\leqslant T$

$$\| D_s^{l-1}\,U_0(t,s)-D_\tau^{l-1}\,U_0(t,\tau)\|\leqslant C\log((t-s)/(t-\tau)), \tag{3.13}$$

$$\lim_{t-s\to 0} D_t^k U_0(t,s)=\lim_{t-s\to 0} D_s^k U_0(t,s)=0 \quad \text{for} \quad k=0,\dots,l-2 \tag{3.14}$$

in the strong operator topology of $B(L^p, w^{2m-kd-d,p})$, and

$$\lim_{t-s\to 0} D_t^{l-1}\,U_0(t,s)=\lim_{t-s\to 0}(-D_s)^{l-1}\,U_0(t,s)=I \tag{3.15}$$

in the strong operator topology of $B(L^p, L^p)$.

Proof. (3.10)–(3.13) are simple consequences of (3.2), (3.3). The equalities (3.14), (3.15) are verified as Proposition 3.2k of [5] and (3.21) of [10].

Lemma 3.3. *$R(t,s)$ is uniformly bounded in Δ. For $0\leqslant s<\tau<t\leqslant T$*

$$\begin{aligned}\| R(t,s)-R(\tau,s)\| \leqslant C\{&(t-\tau)^\rho+\log((t-s)/(\tau-s))\\ &+(t-\tau)(1-\log(t-\tau))\\ &+(t-s)\log(t-s)-(\tau-s)\log(\tau-s)\}.\end{aligned} \tag{3.16}$$

Proof. It readily follows from (3.2), (3.3) that $R_1(t,s)$ is uniformly bounded, and hence so is $R(t,s)$ in view of (3.8). With the aid of (3.2)–(3.4) we can easily show for $0\leqslant s<\tau<t\leqslant T$

$$\| R_1(t,s)-R_1(\tau,s)\|\leqslant C\{(t-\tau)^\rho+\log((t-s)/(\tau-s))\}. \tag{3.17}$$

Inequality (3.16) is a simple consequence of (3.8), (3.17).

LEMMA 3.4. *$Z(t,s)$ is l times continuously differentiable in t, and for $j+kd\leqslant 2m$*

$$\| D_t^k Z(t,s)\|_{B(L^p,\,W^{j,p})}\leqslant C(t-s)^{l-k-j/d}. \tag{3.18}$$

Proof. If $j+kd<2m$, Ineq. (3.18) is a simple consequence of Lemmas 3.2 and 3.3. We have for $0<k\leqslant l$

$$\begin{aligned} D_t^k Z(t,s) &= \int_s^t D_t^k U_0(t,\tau)(R(\tau,s)-R(t,s))\,d\tau \\ &\quad + \int_s^t (D_t+D_\tau)\, D_t^{k-1} U_0(t,\tau)\,d\tau\, R(t,s) \\ &\quad + D_t^{k-1} U_0(t,s)\cdot R(t,s). \end{aligned} \tag{3.19}$$

Inequality (3.18) for $k>0$, $j+kd=2m$ follows from this equality and Lemmas 3.2, 3.3. It follows from

$$\sum_{k=0}^{l} A_{l-k}(t)(-D_\tau)^k U_0(t,\tau)=0 \tag{3.20}$$

that

$$\int_s^t A_l(t)\, U_0(t,\tau)\,d\tau = I - \sum_{k=1}^{l} A_{l-k}(t)(-D_s)^{k-1} U_0(t,s). \tag{3.21}$$

Hence, we get

$$\begin{aligned} A_l(t)\, Z(t,s) &= \int_s^t A_l(t)\, U_0(t,\tau)(R(\tau,s)-R(t,s))\,d\tau \\ &\quad + \left\{ I - \sum_{k=1}^{l} A_{l-k}(t)(-D_s)^{k-1} U_0(t,s)\right\} R(t,s), \end{aligned} \tag{3.22}$$

from which (3.18) for $k=0$, $j=2m$ follows.

For operator valued functions F and G defined in Δ, we write for brevity

$$(F * G)(t,s) = \int_s^t F(t,\tau)\, G(\tau,s)\,d\tau \tag{3.23}$$

whenever the right side is meaningful.

The following lemma is easily established by induction.

LEMMA 3.5. *If $F(t,s)$ and $G(t,s)$ are i times continuously differentiable* *functions in Δ such that $F_j=(D_t+D_s)^j F$ and $G_j=(D_t+D_s)^j G$ are bounded for $j=0, ..., i$, then*

$$(D_t+D_s)^i (F*G)=\sum_{j=0}^{i}\binom{i}{j} F_{i-j}*G_j.$$

LEMMA 3.6. *If* (I.4) *is also satisfied, then $R(t, s)$ is l times continuously differentiable in (t, s), and for $i=0, ..., l$*

$$\|(D_t+D_s)^i R(t, s)\|\leqslant C. \tag{3.24}$$

Proof. Since $R(t, s)$ is the solution of (3.8), we have

$$R=\sum_{\nu=1}^{\infty} R_\nu, \qquad R_\nu=R_1*R_{\nu-1}, \qquad \nu=2, 3, ...$$

With the aid of Lemma 3.5 we can show by induction that there exist constants K_0, K such that for $i=0, ..., l$, $\nu=2, 3, ...$,

$$\|(D_t+D_s)^i R_\nu(t, s)\|\leqslant K_0 K^{\nu-1}(t-s)^{\nu-1}/(\nu-1)!,$$

from which (3.24) follows.

LEMMA 3.7. *If* (I.4) *is also satisfied, then $Z(t, s)$ is l times continuously differentiable in s, and the following inequality holds for $j+kd\leqslant 2m$, $0<k\leqslant l$:*

$$\| D_s^k Z(t, s)\|_{B(L^p, W^{j,p})}\leqslant C(t-s)^{l-k-j/d}. \tag{3.25}$$

Proof. With the aid of Lemma 3.2 and integration by parts we get for $0<k<l$

$$D_s^k Z(t, s)=\sum_{i=0}^{k}\binom{k}{i}\int_s^t D_\tau^{k-i} U_0(t, \tau)\cdot(D_\tau+D_s)^i R(\tau, s)\, d\tau. \tag{3.26}$$

Rewriting the summand with $i=0$ we obtain

$$\begin{aligned} D_s^k Z(t, s)=&\int_s^t D_\tau^k U_0(t, \tau)(R(\tau, s)-R(t, s))\, d\tau \\ &-D_s^{k-1} U_0(t, s)\cdot R(t, s) \\ &+\sum_{i=1}^{k}\binom{k}{i}\int_s^t D_\tau^{k-i} U_0(t, \tau)\cdot(D_\tau+D_s)^i R(\tau, s)\, d\tau. \end{aligned} \tag{3.27}$$

Calculating formally we get

$$
\begin{aligned}
D_s \int_s^t & D_\tau^{l-1} U_0(t, \tau) \cdot R(\tau, s)\, d\tau \\
&= -D_s^{l-1} U_0(t, s) \cdot R(s \cdot s) \\
&\quad + \int_s^t D_\tau^{l-1} U_0(t, \tau) \cdot (D_\tau + D_s)\, R(\tau, s)\, d\tau \\
&\quad - \int_s^t D_\tau^{l-1} U_0(t, \tau) \cdot D_\tau R(\tau, s)\, d\tau \\
&= \int_s^t D_\tau^{l-1} U_0(t, \tau) \cdot (D_\tau + D_s)\, R(\tau, s)\, d\tau - (-1)^{l-1} R(t, s) \\
&\quad + \int_s^t D_\tau^{l} U_0(t, \tau) \cdot R(\tau, s)\, d\tau \\
&= \int_s^t D_\tau^{l-1} U_0(t, \tau) \cdot (D_\tau + D_s)\, R(\tau, s)\, d\tau \\
&\quad + \int_s^t D_\tau^{l} U_0(t, \tau)(R(\tau, s) - R(t, s))\, d\tau \\
&\quad - D_s^{l-1} U_0(t, s) \cdot R(t, s). \qquad (3.28)
\end{aligned}
$$

It is not difficult to justify the above formal calculation. With the aid of (3.26) for $k = l - 1$ and (3.28) we see that (3.27) holds also for $k = l$. The inequality (3.25) is an easy consequence of Lemma 3.2, 3.3, and 3.6 and Eq. (3.27).

4. Proof of Theorems

Proof of Theorem 1. The assertions (2.2)–(2.4) follow from Lemmas 3.2 and 3.4. That $U(t, s)$ satisfies (1.10) can be verified without difficulty with the aid of (3.5), (3.19), (3.22).

Proof of Lemma 2.1. It is clear that $v_0 \in D(A_l(0))$. Suppose that $v_k \in D(A_l(0))$ for $k = 0, ..., i-1$, $i < l-2$. Put

$$w_k(t) = A_l(t)^{-1} A_l(0)\, v_k, \qquad k = 0, ..., i-1.$$

Then

$$v_i = u_i - \sum_{k=0}^{i-1} \binom{i}{k} D_t^{i-k} w_k(0), \qquad w_k(0) = v_k.$$

Differentiating both sides of

$$B_j(x, t, D_x)\, w_k(x, t) = 0, \qquad x \in \partial\Omega,$$

$i-k$ times in t and then letting $t=0$ we get

$$\sum_{\kappa=0}^{i-k} \binom{i-k}{\kappa} B_j^{(\kappa)}(x, 0, D_x)\, D_t^{i-k-\kappa} w_k(x, 0) = 0.$$

Hence, noting

$$\binom{i}{k}\binom{i-k}{\kappa} = \binom{i}{\kappa}\binom{i-\kappa}{k}, \qquad u_{i-\kappa} = \sum_{k=0}^{i-\kappa} \binom{i-\kappa}{k} D_t^{i-\kappa-k} w_k(0)$$

we obtain for $x \in \partial\Omega$

$$\begin{aligned}
B_j(x, 0, D_x)\, v_i(x) &= B_j(x, 0, D_x)\, u_i(x) \\
&\quad - \sum_{k=0}^{i-1} \binom{i}{k} B_j(x, 0, D_x)\, D_t^{i-k} w_k(x, 0) \\
&= B_j(x, 0, D_x)\, u_i(x) \\
&\quad + \sum_{k=0}^{i-1} \binom{i}{k} \sum_{\kappa=1}^{i-k} \binom{i-k}{\kappa} B_j^{(\kappa)}(x, 0, D_x)\, D_t^{i-k-\kappa} w_k(x, 0) \\
&= B_j(x, 0, D_x)\, u_i(x) \\
&\quad + \sum_{\kappa=1}^{i} \binom{i}{\kappa} B_j^{(\kappa)}(x, 0, D_x) \sum_{k=0}^{i-\kappa} \binom{i-\kappa}{k} D_t^{i-\kappa-k} w_k(x, 0) \\
&= \sum_{\kappa=0}^{i} \binom{i}{\kappa} B_j^{(\kappa)}(x, 0, D_x)\, u_{i-\kappa}(x) = 0.
\end{aligned}$$

Proof of Theorem 2. That the last term of the right side of (2.7) is a solution of (1.5), (1.6) with $u_0 = \cdots = u_{l-1} = 0$ can be shown as Theorem 6.2 of [4]. The assertion of Theorem 2 can be easily shown with the aid of this fact and a direct calculation.

LEMMA 4.1. *For* $k = 0, \ldots, l$

$$\| D_t^k U(t, s)\, A_l(s)^{-1} \|_{B(L^p,\, W^{2m-kd,p})} \leqslant C.$$

Proof. In view of (3.5) and Lemma 3.4 it suffices to show

$$\| D_t^k U_0(t, s)\, A_l(s)^{-1} \|_{B(L^p,\, W^{2m-kd,p})} \leqslant C.$$

This is a consequence of (3.3) with $k=1$, $\lambda=0$, and

$$A(t,\lambda)^{-1}A_l(t)^{-1}=\lambda^{-l}A_l(t)^{-1}-\sum_{k=0}^{l-1}\lambda^{k-l}A(t,\lambda)^{-1}A_{l-k}(t)A_l(t)^{-1}.$$

Proof of Theorem 3. The assertions (2.11)–(2.13) are consequences of Lemmas 3.2 and 3.7. In case $p=2$ for any element ϕ of $L^2(\Omega)$

$$U(t,s)A_l(s)^{-1}\phi=\frac{(t-s)^{l-1}}{(l-1)!}A_l(t)^{-1}\phi-\int_s^t U(t,\sigma)A(\sigma,D_\sigma)\left(\frac{(\sigma-s)^{l-1}}{(l-1)!}A_l(\sigma)^{-1}\phi\right)d\sigma \tag{4.1}$$

since both sides of this equality are solutions of

$$A(t,D_t)u(t)=0,\qquad s<t\leqslant T,$$
$$u(s)=\cdots=u^{(l-2)}(s)=0,\qquad u^{(l-1)}=A_l(s)^{-1}\phi$$

such that $\|D_t^k u(t)\|_{2m-kd,p}$ is bounded for $k=0,...,l$ by virtue of Lemma 4.1 and the uniqueness in this class of functions is already known in case $p=2$ in view of [2] or [8]. Approximating $\phi\in L^p(\Omega)$ by a sequence of elements of $L^p(\Omega)\cap L^2(\Omega)$ in the strong topology of $L^p(\Omega)$ we see that (4.1) holds also in case $p\neq 2$. Integrating by parts we get

$$\begin{aligned}\int_0^t U(t,\sigma)A_{l-k}(\sigma)D_\sigma^k\left(\frac{(\sigma-s)^{l-1}}{(l-1)!}A_l(\sigma)^{-1}\phi\right)d\sigma\\ =-U(t,s)A_l(s)^{-1}\phi\delta_{kl}+\frac{(t-s)^{l-1}}{(l-1)!}A_l(t)^{-1}\phi\delta_{kl}\\ +\int_s^t(-D_\sigma)^k(U(t,\sigma)A_{l-k}(\sigma))\cdot\frac{(\sigma-s)^{l-1}}{(l-1)!}A_l(\sigma)^{-1}\phi d\sigma\end{aligned} \tag{4.2}$$

for $k=1,...,l$. Combining (4.1) and (4.2) yields

$$\int_s^t\sum_{k=0}^{l}(-D_\sigma)^k(U(t,\sigma)A_{l-k}(\sigma))\cdot\frac{(\sigma-s)^{l-1}}{(l-1)!}A_l(\sigma)^{-1}\phi\,d\sigma=0. \tag{4.3}$$

Differentiating both sides of this equality l times in s we obtain

$$\sum_{k=0}^{l}(-D_s)^k(U(t,s)A_{l-k}(s))\cdot A_l(s)^{-1}\phi=0$$

from which (2.14) follows.

Proof of Theorem 4. Let u be a function in the statement of the theorem. Then, in view of (2.14) we have

$$\int_\varepsilon^t \sum_{k=0}^{l} (-D_s)^k (U(t,s) A_{l-k}(s)) \cdot u(s)\, ds = 0$$

for $0 < \varepsilon < t \leqslant T$. Integrating by parts in the left hand side and letting $\varepsilon \to 0$ we conclude (2.15).

Proof of Theorem 5. The theorem is proved in just the same manner as Theorem 1 of [10]; namely, the fundamental solution $W(t, s)$ of (1.7), (1.8) is constructed as follows:

$$W(t,s) = U(t,s) + A_l(t)^{-1} V(t,s), \tag{4.4}$$

$$V(t,s) = V_0(t,s) + \int_s^t P(t,\sigma) V(\sigma,s)\, d\sigma, \tag{4.5}$$

$$\begin{aligned} V_0(t,s) &= \int_s^t P(t,\sigma) A_l(\sigma) U(\sigma,s)\, d\sigma \\ &= \int_s^t (P(t,\sigma) - P(t,s)) A_l(\sigma) U(\sigma,s)\, d\sigma \\ &\quad + P(t,s) \int_s^t A_l(\sigma) U(\sigma,s)\, d\sigma, \end{aligned} \tag{4.6}$$

$$\begin{aligned} P(t,s) &= A_l(t) \int_s^t U(t,\tau) K(\tau,s)\, d\tau \\ &= \int_s^t A_l(t) U_0(t,\tau)(K(\tau,s) - K(t,s))\, d\tau \\ &\quad + \left\{ I - \sum_{k=1}^{l} A_{l-k}(t)(-D_s)^{k-1} U_0(t,s) \right\} K(t,s) \\ &\quad + \int_s^t A_l(t) Z(t,\tau) K(\tau,s)\, d\tau, \end{aligned} \tag{4.7}$$

where $K(t, s)$ is the operator valued function defined by

$$K(t,s) = B(t,s) A_l(s)^{-1}.$$

By the assumption (I.5) and the inequality (3.3) $K(t, s)$ is uniformly Hölder continuous in $\bar{\Delta}$:

$$\| K(t',s') - K(t,s) \| \leqslant C(|t'-t|^\rho + |s'-s|^\rho). \tag{4.8}$$

It follows from this inequality that for $s \leqslant \sigma \leqslant \tau \leqslant t$

$$\| K(\tau, \sigma) - K(t, \sigma) - K(\tau, s) + K(t, s) \| \leqslant C(t-\tau)^{\rho/2} (\sigma - s)^{\rho/2}. \quad (4.9)$$

With the aid of Lemmas 3.2, 3.4 and Ineqs. (4.8), (4.9) we can show that $P(t, s)$ is strongly continuous in $\bar{\Delta}$ and

$$\begin{aligned} &\| P(t, s) - P(t, \sigma) \| \\ &\quad \leqslant C\{(\sigma - s)^{\rho} + (t-\sigma)^{\rho/2} (\sigma - s)^{\rho/2} + \log((t-s)/(t-\sigma))\} \quad (4.10) \end{aligned}$$

for $0 \leqslant s < \sigma < t \leqslant T$. Using (2.2), (4.10) and noting

$$\begin{aligned} \int_s^t A_l(\sigma)\, U(\sigma, s)\, d\sigma = &- \sum_{k=1}^{l} A_{l-k}(t)\, D_t^{k-1} U(t, s) + I \\ &+ \sum_{k=1}^{l} \int_s^t A_{l-k}(\sigma)\, D_\sigma^{k-1} U(\sigma, s)\, d\sigma, \end{aligned}$$

we see that $V_0(t, s)$ is strongly continuous in $\bar{\Delta}$. Hence, the integral equation (4.5) can be solved by successive approximation and the solution $V(t, s)$ is strongly continuous in $\bar{\Delta}$. Arguing as in [10] it is not difficult to show that the operator valued function defined by (4.4) is the desired fundamental solution of (1.7), (1.8), the integral of the right side of (1.13) existing as an improper integral

$$\lim_{\varepsilon \to +0} \int_{s+\varepsilon}^t B(t, \sigma)\, W(\sigma, s)\, d\sigma.$$

Proof of Theorem 6. The theorem can be shown as Theorem 2 and Theorem 2 of [10].

Proof of Theorem 7. Let $u(t)$ be a solution of (1.7), (1.8) with $u_0 = \cdots = u_{l-1} = 0$ in the class of functions stated in the theorem. In view of Theorem 4 we have

$$u(t) = \int_0^t U(t, \tau) \int_0^\tau B(\tau, s)\, u(s)\, ds\, d\tau.$$

With the aid of (4.10) and the boundedness of $s \, \| A_l(s)\, u(s) \|_p$ we get

$$A_l(t)\, u(t) = \int_0^t P(t, s)\, A_l(s)\, u(s)\, ds,$$

which implies that $\| A_l(t)\, u(t) \|_p$ is bounded. Hence, with the aid of Gronwall's inequality we obtain $A_l(t)\, u(t) = 0$, from which we get $u(t) = 0$.

Proof of Theorem 8. Since $W(t, s)$ is the solution of the integral equation

$$W(t, s) = U(t, s) + \int_s^t U(t, \tau) \int_s^\tau B(\tau, \sigma)\, W(\sigma, s)\, d\sigma\, d\tau$$

$$= U(t, s) + \int_s^t W(t, \tau) \int_s^\tau B(\tau, \sigma)\, U(\sigma, s)\, d\sigma\, d\tau,$$

it is expressed as

$$W = \sum_{\nu=0}^{\infty} W_\nu, \qquad W_0 = U, \qquad W_\nu = W_{\nu-1} * Q, \qquad \nu = 1, 2, \ldots,$$

where

$$Q(t, s) = (B * U)(t, s) = -\sum_{k=1}^{l} K(t, t)\, A_{l-k}(t)\, D_t^{k-1} U(t, s)$$

$$+ K(t, s) + \sum_{k=1}^{l} \int_s^t D_\tau(K(t, \tau)\, A_{l-k}(\tau)) \cdot D_\tau^{k-1} U(\tau, s)\, d\tau \tag{4.11}$$

and the convolution $F * G$ of F and G is defined by (3.23).

LEMMA 4.2. *For* $0 \leqslant i \leqslant l$, $0 < k \leqslant l$, $j + kd \leqslant 2m$

$$\|(D_t + D_s)^i\, D_t^{k-1} U(t, s)\|_{B(L^p,\, W^{j,p})} \leqslant C(t-s)^{l-k-j/d}. \tag{4.12}$$

Proof. As is easily seen for $0 \leqslant i \leqslant l$, $0 < k \leqslant l$, $j + kd \leqslant 2m$

$$\|(D_t + D_s)^i\, D_t^{k-1} U_0(t, s)\|_{B(L^p,\, W^{j,p})} \leqslant C(t-s)^{l-k-j/d}. \tag{4.13}$$

An application of Lemma 3.5 yields

$$(D_t + D_s)^i\, D_t^{k-1} Z(t, s)$$

$$= \sum_{j=0}^{i} \int_s^t \binom{i}{j} (D_t + D_\tau)^{i-j}\, D_t^{k-1} U_0(t, \tau) \cdot (D_\tau + D_s)^j\, R(\tau, s)\, d\tau. \tag{4.14}$$

With the aid of (4.13), (4.14), and Lemma 3.6 we get

$$\|(D_t + D_s)^i\, D_t^{k-1} Z(t, s)\|_{B(L^p,\, W^{j,p})} \leqslant C(t-s)^{l-k-j/d+1}. \tag{4.15}$$

Combining (4.13) and (4.15) yields (4.12).

LEMMA 4.3. *For* $0 \leqslant i \leqslant l$

$$\|(D_t + D_s)^i\, Q(t, s)\| \leqslant C. \tag{4.16}$$

For $0 \leqslant s < \tau < t \leqslant T$

$$\| Q(t, s) - Q(\tau, s) \| \leqslant C \log((t-s)/(\tau - s)). \tag{4.17}$$

Proof. Inequality (4.16) follows from (4.11) and Lemma 4.2. It is also easily seen that $\| D_t Q(t, s) \| \leqslant C/(t-s)$, which implies (4.17).

LEMMA 4.4. *For* $j + kd \leqslant 2m$

$$\| D_s^k W_1(t, s) \|_{B(L^p, W^{j,p})} \leqslant C(t-s)^{l-k-j/d}. \tag{4.18}$$

Proof. Just as (3.27) was established we get

$$\begin{aligned} D_s^k W_1(t, s) = & \int_s^t D_\tau^k U(t, \tau)(Q(\tau, s) - Q(t, s))\, d\tau \\ & - D_s^{k-1} U(t, s) \cdot Q(t, s) \\ & + \int_s^t \sum_{i=1}^k \binom{k}{i} D_\tau^{k-i} U(t, \tau) \cdot (D_\tau + D_s)^i Q(\tau, s)\, d\tau \end{aligned} \tag{4.19}$$

for $0 < k \leqslant l$. Inequality (4.18) in this case is a simple consequence of (2.11) and Lemma 4.3. The proof of (4.18) for $k = 0$ is the same as that of (3.18) for $k = 0$.

By virtue of Lemma 4.4 one can show by induction that for $\nu \geqslant 2$

$$D_s^k W_\nu(t, s) = \int_s^t \sum_{i=0}^k \binom{k}{i} D_\tau^i W_{\nu-1}(t, \tau) \cdot (D_\tau + D_s)^{k-i} Q(\tau, s)\, d\tau$$

and there exist constants M_0, M such that for $j + kd \leqslant 2m$

$$\| D_s^k W_\nu(t, s) \|_{B(L^p, W^{j,p})} \leqslant M_0 M^{\nu-1} (t-s)^{l-k-j/d+\nu-1}/(\nu-1)!. \tag{4.20}$$

It follows from (2.11)–(2.13), (4.20) that (2.22)–(2.24) hold.

Let ϕ be any element of $L^p(\Omega)$. Then,

$$\begin{aligned} W(t, s) A_l(s)^{-1} \phi = & \frac{(t-s)^{l-1}}{(l-1)!} A_l(t)^{-1} \phi \\ & + \int_s^t W(t, \tau) \left\{ \int_s^\tau B(\tau, \sigma) \left(\frac{(\sigma - s)^{l-1}}{(l-1)!} A_l(\sigma)^{-1} \phi \right) d\sigma \right. \\ & \left. - A(\tau, D_\tau) \left(\frac{(\tau - s)^{l-1}}{(l-1)!} A_l(\tau)^{-1} \phi \right) \right\} d\tau \end{aligned}$$

since both sides are the solutions of

$$A(t, D_t)\, u(t) = \int_s^t B(t, \sigma)\, u(\sigma)\, d\sigma, \qquad s < t \leqslant T,$$

$$u(s) = \cdots = u^{(l-2)}(s) = 0, \qquad u^{(l-1)}(s) = A_l(s)^{-1}\, \phi,$$

and in view of Theorem 7 the solution of this problem is unique. With the aid of the argument by means of which we derived (4.30) from (4.2) we obtain

$$\int_s^t \left\{ \int_\sigma^t W(t, \tau)\, B(\tau, \sigma)\, d\tau - \sum_{k=0}^{l} (-D_\sigma)^k\, (W(t, \sigma)\, A_{l-k}(\sigma)) \right\}$$
$$\times \frac{(\sigma - s)^{l-1}}{(l-1)!}\, A_l(\sigma)^{-1}\, \phi\, d\sigma = 0.$$

Differentiating both sides of this equality l times in s and putting $v = A_l(s)^{-1}\, \phi$ we get (2.25). The verification of (2.26) is the same as that of (2.15).

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