A Matrix Representation for Automorphisms of Compact Riemann Surfaces

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ABSTRACT

In this paper we prove a result which has as corollaries theorems of Hurwitz, Accola, Grothendieck, and Serre on automorphisms of Riemann surfaces.

1. INTRODUCTION

Throughout this paper $W$ will denote a compact Riemann surface of genus $g \geq 2$ and $h$ a conformal automorphism of $W$. Let $\pi_1(W)$ and $H_1(W,\mathbb{Z})$ be the fundamental group and the first homology group of $W$. Let $\approx$ denote homology. By abuse of language we will refer to a curve in $H_1(W,\mathbb{Z})$ when we mean the homology class of the curve. For each integer $n$, $C(n, h)$ will be the subgroup of $H_1(W,\mathbb{Z})$ consisting of curves $\gamma$ with $\gamma - h(\gamma) \approx n\delta$ for some curve $\delta$ in $H_1(W,\mathbb{Z})$.

Grothendieck and Serre [4] have shown that if $C(n, h) = H_1(W,\mathbb{Z})$ for some integer $n \geq 3$, then $h$ is the identity. More recently, Earle [2] has pointed out that this is an immediate consequence of Minkowski's theorem (see p. 212 of [9]) about integer matrices which are congruent to the identity matrix modulo a prime.

The object of this paper is to obtain a basis for $H_1(W,\mathbb{Z})$ with respect to which the matrix of the action of $h$ is in a particularly nice form. This enables us to compute the index of $C(n, h)$ in $H_1(W,\mathbb{Z})$ when $h$ is of prime order. As a corollary we see that $C(n, h)$ is always a proper subgroup of $H_1(W,\mathbb{Z})$ unless $h$ is the identity or $h$ is the hyperelliptic involution and

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n = 2. In addition we obtain an upper bound on the number of elements in \( C(n, h) \) in any homology basis and show that there is always a homology basis which does not contain any element of \( C(n, h) \) for any \( n > 3 \). The method also gives another proof of a result of Accola [1] about how \( h \) can act on homology. This is all done in Sec. 4.

The method of the proof is to use the Schreier-Reidemeister rewriting process [8] to obtain a presentation for \( \pi_1(W) \) on which the action of \( h \) is easily calculated. This is done in Sec. 3. The matrix of the action induced by \( h \) on homology is calculated in Sec. 4.

2. NOTATION

The following notation will be fixed throughout the paper. \( H \) will be the group generated by \( h \); \( W_0 \) will be the factor surface, \( W_0 = W/H \); \( g_0 \) will be the genus of \( W_0 \); and \( t \) will be the number of fixed points of \( h \). If \( t \neq 0 \) and if \( h \) is conformal, then \( t > 2 \). When \( h \) is of prime order, for example, this can be seen by applying Theorem 3 of [3] when \( T = 1 \) and \( n = 0 \). For two curves \( A \) and \( B \), \( A \times B \) will be their intersection number. Automorphism will always mean a conformal map.

3. A PRESENTATION FOR \( \pi_1(W) \)

**Theorem 1.** Let \( W \) be a compact Riemann surface of genus \( g \geq 2 \). Let \( h \) be an automorphism of \( W \) of prime order \( p \neq 1 \). Let \( g_0 \) be the genus of the factor surface and \( t \) the number of fixed points of \( h \). Then

(i) If \( t > 0 \), \( \pi_1(W) \) is generated by the set

\[
\begin{align*}
    h^j(A_w), h^j(B_w), & \quad w = 1, \ldots, g_0, \quad j = 0, \ldots, p-1, \\
    h^k(X_i), & \quad i = 3, \ldots, t, \quad k = 0, \ldots, p-2.
\end{align*}
\]

\( \pi_1(W) \) has a single defining relation \( R \) in which each generator and its inverse occur exactly once. Further, for each \( i = 3, \ldots, t \), we have

\[
h(h^{p-2}(X_i)) \equiv \sum_{j=0}^{p-2} - h^j(X_i).
\]
(ii) If \( t = 0 \), \( \pi_1(W) \) is generated by the set

\[
\begin{align*}
&h^i(A_w), h^j(B_w), \quad w = 2, \ldots, g_0, \quad j = 0, \ldots, p - 1, \\
&A_1, B_1.
\end{align*}
\]

\( \pi_1(W) \) has a single defining relation \( R \) in which each generator and its inverse occur exactly once, \( h(A_1) = A_1 \) and \( h(B_1) = B_1 \).

(iii) In either case, the generators for \( \pi_1(W) \) have the following properties:

\[
\begin{align*}
&h^i(A_w) \times h^k(B_v) = \delta_{wv} \delta_{jk}, \text{ where } \delta_{rs} \text{ is the Kronecker delta and } w, v, i, \text{ and } k \\
&\text{vary over all possibilities.} \\
&h^i(A_w) \times h^k(A_v) = 0 \text{ for all possible } w, v, i \text{ and } k. \\
&h^i(B_w) \times h^k(B_v) = 0 \text{ for all possible } w, v, i \text{ and } k. \\
\text{However, when case (ii) occurs, } h^i(A_1) \times h^k(B_1) = 1 \text{ for all } j \text{ and } k.
\end{align*}
\]

**Proof.** Since parts (i) and (ii) are a slight modification of the result of Nielsen (see p. 31 of [10]), we give only a brief outline of a proof which will be need for the proof of (iii).

Macbeath [7] has shown that there exist Fuchsian groups \( G \) and \( M \) with \( G/M \) isomorphic to \( H \) and \( U/M \) isomorphic to \( W \), where \( U \) is the upper half plane. \( M \) is torsion free and thus isomorphic to \( \pi_1(W) \). \( G \) has presentation:

\[
\left\langle a_1, \ldots, a_{g_0}, b_1, \ldots, b_{g_0}, x_1, \ldots, x_t; x_1^p = 1, x_1 \cdots x_t \prod_{i=1}^{g_0} [a_i, b_i] = 1 \right\rangle.
\] (1)

Here \([a, b]\) denotes the commutator of \( a \) and \( b \).

We let \( \phi \) be the homomorphism of \( G \) onto \( H \) with kernel \( M \). Let \( \phi(x_i) = h^i, t = 1, \ldots, t \). By results of Harvey [5], replacing \( h \) by a conjugate homeomorphism if necessary, we may assume that \( \phi(b_i) = \phi(b_i) = \phi(a_i) = 1 \) for \( i = 2, \ldots, g_0 \). By the results of [3], we may also assume that \( \phi(a_1) = 1 \) when \( t \neq 0 \).

We apply the Schreier-Reidemeister rewriting process to \( G \) and obtain a presentation for \( M \). We use the notation of Chapter 2 of [8] and apply Theorem 2.9, p. 94.

We choose the Schreier system of representatives \( 1, x_1, \ldots, x_t^{p-1} \) in case (i) and \( 1, a_1, \ldots, a_j^{p-1} \) in case (ii), and carry out the tedious eliminations of generators and relations to obtain the desired result. We note that \( h \) acts on elements of \( M \) by conjugation by \( x_1^q \) or \( a_1^q \), where \( q \) is the integer which satisfies \( \phi(x_i)^q = h \) or \( \phi(a_i)^q = h \).

To simplify the notation we make the following substitution at the end of the calculation for case (i). Let \( j_0 \) be \( qj \) reduced modulo \( p \). Set \( h^i(A_w) = \)}
$S_{(g_0^p, \alpha_\pi)}$ and $h^j(B_w) = S_{(g_0^p, \alpha_\pi)}$ for $w = 1, \ldots, g_0$ and $j = 0, \ldots, p - 1$. Also we set $h^i(X_i) = S_{(g_0^p, \alpha_\pi)}$ for $i = 3, \ldots, t$ and $j = 0, \ldots, p - 2$. A similar substitution is made in case (ii).

To prove part (iii), recall that $W_0 = W/H$. Let $\hat{W}_0$ be $W_0$ with the images of the branch points removed and let $\hat{W}$ be $W$ with the fixed points of $h$ removed. Then $\pi_1(\hat{W}_0)$ has presentation

$$\left\langle \alpha_1, \ldots, \alpha_{g_0^p}, \beta_1, \ldots, \beta_{g_0^p}, \delta_1, \ldots, \delta_{t-1}, \delta_t \prod_{i=0}^{g_0} [\alpha_i, \beta_i] = 1 \right\rangle. \quad (2)$$

$\pi_1(W)$ and $\pi_1(W_0)$ are both isomorphic to quotients of $\pi_1(\hat{W}_0)$. We may assume that the homology classes of the images of $\alpha_1, \ldots, \alpha_{g_0^p}, \beta_1, \ldots, \beta_{g_0^p}$ on $W_0$ form a canonical homology basis consisting of simple curves, so that

$$\alpha_i \times \beta_i = \delta_{ij}, \quad \alpha_i \times \alpha_j = 0, \quad \text{and} \quad \beta_i \times \beta_j = 0 \quad \text{for each } i \text{ and } j. \quad (3)$$

If $M$ is the defining subgroup of the covering $\hat{W} \to \hat{W}_0$, one can perform a Schreier-Reidemeister calculation for $M$ and $\pi_1(W)$. Let $\hat{\phi}$ be the corresponding homomorphism. We make the same assumptions on $\phi$ as we did on $\phi$. Fix a base point $p$ on $W$. Let $p_k$ be the end point of the lifting of $\delta^k_1$ with initial point $p$. Following the notation of [8], $S_{y,c} = yc^{-1}$, where $yc$ is the coset representative of $yc$. If $\hat{\phi}(\gamma) = 1$, the homology class of $S_{\delta^k_1}$ is just that of the lifting of $\gamma$ with initial point $p_k$. Note that for a smooth covering any two liftings of a simple curve either coincide or are disjoint. In case (ii), of course, we replace $\delta_1$ by $\alpha_1$ and we notice that $S_{\delta^k_1, \alpha_1}$ is just the lifting of $\alpha_1^k$. Using the intersection numbers from (3) and the fact that two homology classes have intersection number zero if they contain disjoint curves, one can compute intersection numbers for curves in $M$ of the form $S_{y,c}$ whenever $c = \alpha_1, \ldots, \alpha_{g_0^p}$ or $\beta_1, \ldots, \beta_{g_0^p}$ and $y$ is either $\delta^k_1$ or $\alpha_1^k$ for some integer $k$.

$M$ is the quotient of $\hat{M}$ under a homomorphism which sends $S_{y,\alpha_1}$ to $h^{\delta^k_1}(A_1)$ and $S_{y,\beta_1}$ to $h^{\delta^k_1}(B_1)$ except that when $t = 0$, the homomorphism sends $S_{\alpha_1^{-1}, \alpha_1}$ to $A_1$. Here in case (i), $\hat{\phi}(\delta_1) = h^s$, and in case (ii), $\hat{\phi}(\alpha_1) = h^s$. The intersection properties follow from this fact.

4. THE ACTION OF $h$ ON $H_1(W, \mathbb{Z})$

Definition 1. An integral homology basis for $W$ is said to be adapted to $h$ if for every curve $\gamma$ in the basis either

(i) $h^j(\gamma)$ is in the basis for all $j = 0, \ldots, p - 1$, or
(ii) $h^j(\gamma)$ is in the basis for all $j = 0, \ldots, p - 2$ and $h^{p-1}(\gamma) = \sum_{i=0}^{p-2} - h^s(\gamma)$,
or

\[ (iii) \, \gamma = h^k(\delta), \] where \( k \) is an integer \( 0 < k < p - 2 \) and \( \delta \) satisfies (ii).

Let \( n \) be any positive integer. We set

\[ nH = \{ \gamma \in H_1(W,Z) | \gamma = n\delta \ \text{for some} \ \delta \in H_1(W,Z) \} \]

and

\[ C(n,h) = \{ \gamma \in H_1(W,Z) | \gamma - h(\gamma) \ \text{is in} \ nH \}. \]

Note that both \( nH \) and \( C(n,h) \) are subgroups of \( H_1(W,Z) \).

Remark. Let \( trh \) denote the trace of the action of \( h \) on the first homology group. We see that a homology basis adapted to \( h \) will contain precisely \( -(p-1)(trh) \) curves of type (iii) and \( 2g + (p-1)(trh) \) curves of type (i).

**Theorem 2.** If \( h \) is an automorphism of a compact surface \( W \) of genus \( g > 2 \) and if the order of \( h \) is a prime \( p \neq 1 \), then there exists a homology basis adapted to \( h \).

Proof. The relation \( R \) of Theorem 1 in each case lies in the commutator subgroup of \( \pi_1(W) \). Thus the homology classes of the generators for \( \pi_1(W) \) which are given in Theorem 1 will form an integral homology basis. The basis is obviously adapted to \( h \).

For the rest of this paper when we refer to the homology basis adapted to \( h \), we will mean the one obtained from Theorem 1.

Remark. If \( t \neq 0 \), the matrix of the action induced by \( h \) on this basis can be written as \( 2g_0 + t - 2 \) blocks along the diagonal, \( 2g_0 \) of which are \( p \times p \) permutation matrices with ones along the superdiagonal and a one in the lower left hand corner, and \( t - 2 \) of which are \( (p-1) \times (p-1) \) matrices with ones along the superdiagonal, every entry in the last row \(-1\), and zeros elsewhere. A similar statement holds when \( t = 0 \).

**Corollary 1.** Let \( h \) be an automorphism of prime order \( p \neq 1 \) on a compact surface of genus \( g > 2 \). Then there is a homology basis containing no curve in \( C(n,h) \) for any \( n \geq 3 \).
Proof. If \( t \) is positive, this follows directly from Theorem 2. If \( t = 0 \), replace \( A_1 \) by \( A_1 + A_2 \) and \( B_1 \) by \( B_1 + B_2 \). The Riemann-Hurwitz relation assures that \( g_0 \geq 2 \) if \( g \geq 2 \) and \( t = 0 \).

We fix some additional notation.

For any curve \( A \), let \( \{ A \} = A \) if \( h(A) = A \), and let \( \{ A \} = A + h(A) + \cdots + h^{p-1}(A) \) otherwise. For any curve \( A \), set \( [A] = \sum_{i=0}^{p-2}(j+1)h^j(A) \).

**Lemma 1.** Let \( \gamma \) be any curve in \( H_1(W, \mathbb{Z}) \). Then \( h(\gamma) \equiv \gamma \) if and only if \( \gamma = \sum_{i=1}^{g_0} (n_i(A_i) + m_i(B_i)) \) for some integers \( n_i \) and \( m_i \).

**Proof.** Write \( \gamma \) as an integral combination of the basis adapted to \( h \), apply \( h \), and equate coefficients.

**Corollary 2 (Accola [1]).** Let \( h \) be an automorphism on a compact surface of genus \( g \geq 2 \). Assume there are four independent closed curves \( C_1, C_2, C_3, \) and \( C_4 \) with \( C_1 \times C_2 = 1, C_2 \times C_4 = 1, \) and \( C_4 \times C_1 = 0 \) if \( i+j \equiv 1 \pmod{2} \). Suppose that \( h(C_i) = C_i \) for each \( i \). Then \( h \) is the identity.

**Proof.** We let \( n \) be the order of \( h \). Assume \( n \neq 1 \). Then the main steps in the proof of the theorem are to show that for a compact surface (1) if \( t \neq 0 \) and \( n \) is prime, then \( h(A) \equiv A \) and \( h(B) \equiv B \) for any two curves \( A \) and \( B \) only if \( A \times B \equiv 0 \pmod{n} \), and (2) if \( t = 0 \) and \( n \) is prime, then there is at most one such pair of curves \( A \) and \( B \) with \( h(A) \equiv A \), \( h(B) \equiv B \), and \( A \times B = 1 \). These two facts follow from Lemma 1 and Theorem 1, part (iii). The proof of (1) is direct. To see (2), assume there are four such curves. Let \( C_i = Y_i + a_i A_i + b_i B_i, C_2 = Y_2 + c_i A_i + d_i B_i, C_3 = Y_3 + e_i A_i + f_i B_i; \) and \( C_4 = Y_4 + g_i A_i + k_i B_i \). Here the \( Y \)'s are linear combinations of the \( \{ A_i \} \) and the \( \{ B_i \} \), where \( i = 2, \ldots, g_0 \). Using the intersection numbers for the \( C_i \) curves with each other and the fact that \( Y_i \times Y_i \equiv 0 \pmod{n} \), we obtain the following congruences modulo \( n \): \( af - eb \equiv 1 \equiv ck - d \). Also \( ad - bc \equiv ak - gb \equiv cf - de \equiv ek - fg \equiv 0 \). Then \( k(ad - bc) \equiv d(ak - bg) \equiv 0. \) Thus \( bcd - kbc \equiv 0 \). Since \( gd - kc \equiv -1 \), \( b \equiv 0 \). Similarly we can conclude \( f \equiv 0 \). This contradicts \( af - eb \equiv 1 \). Thus \( n = 1 \).

**Lemma 2.** Let \( \gamma \) be any curve in \( H_1(W, \mathbb{Z}) \). (i) Assume that \( p \) and \( n \) are relatively prime or \( t = 0 \). Then \( \gamma \) is in \( C(n, h) \) if and only if

\[
\gamma = \sum_{i=0}^{g_0} (m_i(A_i) + n_i(B_i)) + n\delta,
\]

where \( m_i \) and \( n_i \) are integers and \( \delta \) is in \( H_1(W, \mathbb{Z}) \). (ii) Assume that \( p = n \)
and \( t > 0 \). Then \( \gamma \) is in \( C(n, h) \) if and only if

\[
\gamma = \sum_{i=0}^{g_0} \left( m_i (A_i) + n_i (B_i) \right) + \sum_{i=3}^{t} r_i [X_i] + p \delta,
\]

where \( m_i, n_i \) and \( r_i \) are integers and \( \delta \) is in \( H_1(W, Z) \).

**Proof.** Write \( \gamma \) as in integral linear combination of the basis adapted to \( h \), and equate coefficients modulo \( n \).

Since \( nH \) is a normal subgroup, we can form the factor groups \( H_1(W, Z)/nH \) and \( C(n, h)/nH \). The first factor group is isomorphic to \((Z_n)^{2g}\) and has order \( n^{2g} \).

**Theorem 3.** If \( h \) is an automorphism of prime order \( p \neq 1 \) on a compact surface of genus \( g \geq 2 \), then \( C(n, h)/nH \) is isomorphic to \((Z_n)^m\), where

(i) \( m = 2g_0 \) if \( p \) and \( n \) are relatively prime or \( t = 0 \), and

(ii) \( m = 2g_0 + t - 2 \) if \( p = n \) and \( t > 0 \).

Here, of course, \( t \) is the number of fixed points of \( h \) and \( g_0 \) is the genus of the factor surface.

**Proof.** This follows directly from Lemma 2.

Since \( nH \) is a subgroup of \( C(n, h) \), the index of \( C(n, h)/nH \) in \( H_1(W, Z)/nH \) is equal to the index of \( C(n, h) \) in \( H_1(W, Z) \). Theorem 3 can be reformulated as

**Theorem 4.** Let \( h \) be an automorphism of prime order \( p \neq 1 \) on a compact surface of genus \( g \geq 2 \). Then the index of \( C(n, h) \) in \( H_1(W, Z) \) is

(i) \( n^{2g - 2g_0} \) if \( n \) and \( p \) are relatively prime or \( t = 0 \), and

(ii) \( n^{2g - 2g_0 - 1 + t} \) if \( p = n \) and \( t > 0 \).

Here \( t \) is the number of fixed points of \( h \) and \( g_0 \) the genus of the factor surface.

An immediate consequence of Theorem 3 and 4 is

**Theorem 5.** Let \( h \) be an automorphism of prime order \( p \neq 1 \) of a compact surface of genus \( g \geq 2 \). Then a homology basis for the surface can contain at most \( m \) elements of \( C(n, h) \), where \( m = 2g_0 \) if \( p \) does not divide \( n \) or \( t = 0 \) and \( m = 2g_0 + t - 2 \) if \( p = n \) and \( t > 0 \). Here \( t \) is the number of fixed points of \( h \), and \( g_0 \) is the genus of the factor surface.
Proof. Curves in a basis are primitive. If some basis contains $q$ curves in $C(n,h)$, then the order of $C(n,h)/nH$ is at least $n^q$.

Corollary 3 (Grothendieck and Serre [4]; Earle [2]). If $h$ is an automorphism of a compact surface $W$ of genus $g > 2$ and $h$ induces the identity on $H_1(W,Z/nZ)$ for some integer $n > 3$, then $h$ is the identity.

Proof. We first assume that $n$ is prime. Also it suffices to prove the theorem for all powers of $h$ which are of prime order. The hypothesis of the theorem is that the index of $C(n,h)$ in $H_1(W,Z)$ is 1. Assume that $h$ has prime order $p 
eq 1$. Then set $2g = 2g_0$ or $2g = 2g_0 + t - 2$. Use the Riemann-Hurwitz relation along with the fact that $g > 2$ and $t < 2g + 2$ to show that either $p = 1$, so $h$ is the identity, or $p = n = 2$, $g_0 = 0$, and $t = 2g + 2$. We can conclude that for any $n$, either $h$ is the identity or $n$ is a power of two and some power of $h$ is the hyperelliptic involution. Since there is a homology basis with respect to which the matrix of the action of the hyperelliptic involution is minus the identity matrix, the latter case will never occur, because that power of $h$ would not induce the identity on $H_1(W,Z/nZ)$ for any $n > 2$.

Corollary 4.

(i) Let $h$ be an automorphism of a compact surface of genus $g > 2$. Assume that the order of $h$ divides $n$ and that $p$ is the smallest prime dividing $n$. Assume that a homology basis for the surface contains more than $2(g-1)/p + 2$ curves $A$ for which $h(A) = A$. Then $h$ is the identity.

(ii) Let $h$ be an automorphism of a compact surface of genus $g > 2$. If there is a homology basis for the surface containing more than $g + 1$ curves $A$ for which $h(A) = A$, then $h$ is the identity.

Proof.

(i) We may assume that $h$ is of order $n$. We will show that for each prime $q$ dividing $n$, $h^{n/q}$ is the identity. Let $g_q$ be the genus of the factor surface obtained when identifying under the action of $h^{n/q}$. The Riemann-Hurwitz relation shows that $2(g-1)/q + 2 > 2g_q$. Let $r$ be a prime not dividing $n$. By Theorem 5, a homology basis can contain at most $2g_q$ elements of $C(r,h^{n/q})$. By assumption, if $h^{n/q}$ fixes $s$ curves in a homology basis, then $s > 2(g-1)/p + 2$. But $2(g-1)/p + 2 > 2(g-1)/q + 2 > 2g_q$.

Thus $s > 2g_q$, so $h^{n/q}$ is the identity.

(ii) Apply (i) with the fact that $p > 2$. 


Remark. Using the methods of Takao Kato [6], Theorem 5 can be extended to open surfaces. The number \( m \) must be replaced by the number \( m \) for the compact case plus the number of boundary components.

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