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# LETTER TO THE EDITOR

## Representation of the Singularities of a Function

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*Abstract*—We show that a function $f$ having a little uniform smoothness can be locally represented at any point $x_0$ as

$$f(x) = |x - x_0|^h g\left(\frac{x - x_0}{|x - x_0|^{1+\beta}}\right) + r(x),$$

where $g$ is an indefinitely oscillating function; the value of $\beta$ of the regularity of $r$ are related by the 2-microlocal regularity of $f$. © 1999 Academic Press

### 1. INTRODUCTION

With the emergence of wavelet techniques and multifractal analysis, much effort has been made to try to sharpen the definition of the "regularity" of a function at a given point. The basic notion of Hölder exponent quickly proved to be inadequate in many cases, because it fully describes the behavior of the function only if one supposes that this function forms a "cusp" at the point of interest, like $x \mapsto \sqrt{|x|}$ does at 0. But a function can exhibit a very different behavior, for instance $x \mapsto \sqrt{|x|}\sin(1/x)$, and have the same Hölder exponent.

This difference is fairly important for the calculus of the singularity spectrum of a function, that is, the dimension $d(h)$ of the set of points where this function has Hölder exponent $h$. The multifractal formalism, which is at present the only practical way of calculating this spectrum, fails in some cases precisely because it only "sees" cusp points and misses the oscillating ones.

#### 1.1. 2-Microlocal Regularity

Let $\psi$ be a wavelet in $\mathcal{S}$ having an infinity of vanishing moments. We recall that $f$ belongs to the 2-microlocal space $C^{s,s'}(x_0)$ if and only if its wavelet coefficients verify

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$$|C_{ab}| \leq Ca^s \left(1 + \frac{|b - x_0|}{a}\right)^{-s'}$$

for a constant $C < \infty$ and $(a, b)$ in a neighborhood of $(0^+, x_0)$. A consequence of this definition is that for some neighborhood $V(x_0)$,

$$C^{s,0}(x_0) = C^s(V(x_0)). \tag{1.1}$$

The set $D = \{(s, s'), f \in C^{s,s'}(x_0)\}$ is called the 2-microlocal domain of $f$ at $x_0$. It is easy to see that $D$ is convex and that $(s, s') \in D$ implies $(t, t') \in D$ as soon as $t \leq s$ and $t' \leq s' + s - t$. The frontier $\partial D$ is thus a concave, decreasing curve (Fig. 1) that we parameterize from now on by $\sigma = s + s'$ and $\tau(\sigma) = -s$. The function $\tau$ is then convex, increasing (possibly equal to $+\infty$ from a certain $\sigma_0$), and its Legendre transform is

$$\tau^*(\beta) = \sup_\sigma \beta\sigma - \tau(\sigma).$$

2-microlocal regularity is a local notion, and it applies to distributions as well. However, if $\sigma_0 > 0$, that is, if $f \in C^{s,s'}(x_0)$ with $s + s' > 0$, then $f$ is indeed a function in a neighborhood of $x_0$, and a now classical theorem of Jaffard [2] states that

$$C^{s,s'}(x_0) \hookrightarrow C^s(x_0) \rightarrow C^{s,-s}(x_0). \tag{1.2}$$

In the following, we make the slightly stronger assumption that $f \in C^{\varepsilon,0}(x_0)$ for $\varepsilon > 0$ (that is, $f$ has a little uniform regularity around $x_0$). A consequence of (1.2) is then that the Hölder exponent of $f$ at $x_0$ is simply $H = -\tau(0)$.

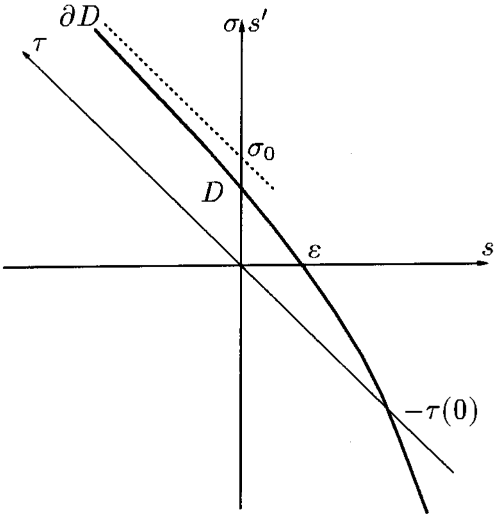


**FIG. 1.** 2-microlocal domain and frontier.

### *1.2. Oscillations*

A function $g \in L^\infty(\mathbb{R}^d)$ is called indefinitely oscillating if and only if $\exists h \in \mathcal{S}(\mathbb{R}^d)$ such that $\forall N \in \mathbb{N}$, it can be written

$$g = h + \sum_{|\alpha|=N+1} \partial^\alpha g_\alpha$$

with each $g_\alpha \in L^\infty(\mathbb{R}^d)$. This is a natural generalization (in particular to higher dimensions) of the oscillating behavior of the sine function.

If a function $f$ has oscillations at a point $x_0$, like $x \mapsto \sqrt{|x|}\sin(1/x)$ at 0, we would like to characterize it by a couple of exponents $(h, \beta)$, such that $f$ has a representation of the form

$$f(x) = |x - x_0|^h g\left(\frac{x - x_0}{|x - x_0|^{1+\beta}}\right) + r(x), \tag{1.3}$$

where $r$ is some rest smoother than $f$ at $x_0$ and $g$ is an indefinitely oscillating function. It is obvious that $h$ must be the Hölder exponent, but there is no unique possibility for $\beta$. We show that the largest possible $\beta$ is related to the smoothness of the rest $r$ via the 2-microlocal frontier $\tau$.

THEOREM 1. $\forall 0 \leq \beta \leq \partial^+\tau(\varepsilon)$, $\forall \alpha < \tau^*(\beta)$, $\forall 0 \leq h < -\tau(0)$, *there exists an indefinitely oscillating function* $g$ *such that* (1.3) *holds with* $r \in C^\alpha(x_0) \cap C^{\alpha/(1+\beta)}(V(x_0))$.

*If* $\tau^*(\beta) = \infty$ *then* $r \in C^\infty(V(x_0))$.

## 2. PROOF OF THE THEOREM

In the latter case ($\tau^*(\beta) = \infty$), since the 2-microlocal domain of $(-\Delta)^{-n/2}f$ is simply $D$ translated by $s \mapsto s + n$, we see that $(-\Delta)^{-n/2}f \in C^{h+n(1+\beta)}(x_0)$. This is precisely the definition of a *chirp* of type $(h, \beta)$; by a theorem of Meyer generalized by Xu [3] to dimensions higher than 1, $f$ can then be written as (1.3) with $r \in C^\infty$ in a neighborhood of $x_0$. We suppose now $\tau^*(\beta) < \infty$ and $x_0 = 0$.

The wavelet reconstruction formula (see [1]) yields

$$f(x) = C_\psi^{-1} \int_0^\infty \int_{\mathbb{R}^d} \frac{C_{ab}}{a^{d+1}} \psi\left(\frac{x - b}{a}\right) db\, da,$$

and we write $f = f_1 + f_2$ by splitting the integration domain into

- $D_1 = \{(a, b),\ 0 < a < |b|^{1+\beta}\}$
- $D_2 = \{(a, b),\ a > |b|^{1+\beta}\}$.

### 2.1. *Regularity of the Rest*

Let us bound the wavelet coefficients of $f_2$.

LEMMA 2.1. $\forall \alpha < \tau^*(\beta)$, $\exists C < \infty$, $\lambda > 0$ *such that* $a \geq |b|^{1+\beta}$ *implies*

$$|C_{ab}| \leq C \min(a^{\lambda}(a + |b|)^{\alpha}, a^{\alpha/(1+\beta)}).$$

*Proof.* If $\tau^*(\beta) < \infty$, $\exists \sigma \in \mathbb{R}$ such that $\tau^*(\beta) = \beta\sigma - \tau(\sigma)$. Since $\tau(\sigma)$ is the 2-microlocal frontier, $\forall \epsilon > 0$, $f \in C^{-\tau(\sigma)-\epsilon, \tau(\sigma)+\sigma}(0)$; that is, $\exists C < \infty$,

$$|C_{ab}| \leq C a^{\sigma-\epsilon}(a + |b|)^{-\tau(\sigma)-\sigma}, \tag{2.1}$$

and we suppose $a \geq |b|^{1+\beta}$, and hence

$$(a + |b|) \leq 2a^{1/(1+\beta)}.$$

Pointwise regularity: if $\tau^*(\beta) \leq -\tau(0)$, then $f \in C^{(\alpha-\tau(0))/2}(0)$ and the result comes trivially from (1.2). If $\tau^*(\beta) > -\tau(0)$, then $\tau(\sigma) + \sigma + \alpha \leq 0$ and (2.1) yields

$$\begin{aligned} |C_{ab}| &\leq C a^{\sigma-\epsilon}(a + |b|)^{-\tau(\sigma)-\sigma-\alpha}(a + |b|)^{\alpha} \\ &\leq C a^{\sigma-\epsilon-(\tau(\sigma)+\sigma+\alpha)/(1+\beta)}(a + |b|)^{\alpha}. \end{aligned}$$

Taking $\epsilon = (\tau^*(\beta) - \alpha)/(2(1 + \beta))$ gives the result with $\lambda = \epsilon$.

Global regularity: if $\beta \leq \partial^+\tau(\varepsilon)$, then $\tau(\sigma) + \sigma \leq 0$, and thus (2.1) yields

$$|C_{ab}| \leq C a^{\sigma-\epsilon-(\tau(\sigma)+\sigma)/(1+\beta)} \leq C a^{\tau^*(\beta)/(1+\beta)-\epsilon} \leq C a^{\alpha/(1+\beta)}. \quad \blacksquare$$

By (1.1) and (1.2), this lemma implies that $\tau = f_2 \in C^{\alpha}(0) \cap C^{\alpha/(1+\beta)}(V(0))$.

### 2.2. *The Oscillating Part*

Let $\psi_n = (-\Delta)^{-n/2}\psi$. By Fourier transform, $\widehat{\psi_n}(\xi) = |\xi|^{-n}\hat{\psi}$ so $\psi_n$ is still a wavelet in $\mathcal{S}$ with all its moments vanishing. Deriving under the integral, one verifies that

$$(-\Delta)^{-n/2} f_1(x) = C_{\psi}^{-1} \int\!\!\int_{D_1} \frac{C_{ab} a^n}{a^{d+1}} \psi_n\left(\frac{x - b}{a}\right) db\, da.$$

But since $(a, b) \in D_1$, we have $a^n \leq |b|^{n(1+\beta)} \leq (a + |b|)^{n(1+\beta)}$; moreover we know that $f \in C^{(h-\tau(0))/2}(0) \cap C^{\varepsilon}$, thus

$$|C_{ab}| \leq C a^{\varepsilon(-\tau(0)-h)/(h-\tau(0))}(a + |b|)^{h}$$

$$|C_{ab}| a^n \leq C a^{\varepsilon(-\tau(0)-h)/(h-\tau(0))}(a + |b|)^{h+n(1+\beta)}.$$

By (1.2), this proves that $(-\Delta)^{-n/2}f_1 \in C^{h+n(1+\beta)}(0)$. In other words, $f_1$ is a chirp of type $(h, \beta)$ and by the theorem of Meyer and Xu,

$$f_1(x) = |x|^h g\left(\frac{x}{|x|^{1+\beta}}\right),$$

with $g$ indefinitely oscillating.

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