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Tensor Products of AC_* Charges and AC Radon Measures Are Not Always AC_* Charges

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In this note we give an example of an AC_* charge, F, on $\mathbb{R}$ and an absolutely continuous Radon measure μ on $\mathbb{R}$ such that $F \otimes \mu$ is not an AC_* charge on $\mathbb{R}^2$.

Key Words: non-absolute integrals; variational measure; tensor product.

1. INTRODUCTION

In [1] the tensor problem was stated for the tensor product of AC_* charges and the Lebesgue measure. Later W. F. Pfeffer suggested looking at possible generalizations of this problem. In this note we show that a more general version of this problem is false. We also give a version of the tensor theorem which seems to be true and will be the subject of a forthcoming paper by this author.

The generalized version of the tensor problem is the following (for details of the definitions see Section 2 of this paper or [1]):

Let μ be an absolutely continuous Radon measure in $\mathbb{R}^n$, and let F be a charge in $\mathbb{R}^m$ where m and n are positive integers. (In this paper absolute continuity is always considered with respect to the Lebesgue measure.) Given a bounded BV set $B \subset \mathbb{R}^{m+n}$, let

$$B^y = \{x \in \mathbb{R}^m : (x, y) \in B\}$$

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and

$$(F \otimes \mu)(B) = \int_{\mathbb{R}^n} F(B^y)d\mu(y).$$

It is not difficult to see that $F \otimes \mu$ is a charge and the question is whether $F \otimes \mu$ is AC_* in $E \times \mathbb{R}^n$ whenever F is AC_* in a locally BV set $E \subset \mathbb{R}^m$.

This question is of interest because for nonabsolute integrals usually there is no Fubini's theorem and hence results concerning products are useful.

In this note we give a counterexample in the simplest possible setting. The charge F will be an AC_* charge on $\mathbb{R}$ (in fact, it will be supported on $[0, 1]$) and μ will be an absolutely continuous Radon measure supported on $[0, 1]$. Their tensor product $F \otimes \mu$ will not be an AC_* charge on $\mathbb{R}^2$. More sophisticated similar examples can be obtained in higher dimensions as well.

It is easy to see that if μ has an atomic part then the tensor conjecture is false. Our example shows that even for absolutely continuous measures one can obtain counterexamples. In our construction we give a μ such that its density function (its Radon–Nikodym derivative) ϕ is unbounded. It seems to us that the conjecture on tensor products is true if we assume that $\phi \in L^\infty(\mathbb{R}^n)$.

2. PRELIMINARIES

We denote by $d(A)$ and $|A|$ the diameter and the Lebesgue measure of the set $A \subset \mathbb{R}^m$, respectively.

The open ball centered at x and of radius r will be denoted by $B(x, r)$.

Closed intervals in $\mathbb{R}^m$ are Cartesian products of closed one-dimensional intervals. Figures in $\mathbb{R}^m$ are finite unions of closed m-dimensional intervals.

The Lebesgue density points (see for example [2, Corollary 6.2.6]) of a measurable set $A \subset \mathbb{R}^m$ form its essential interior, $\text{int}^* A$. The Lebesgue dispersion points (zero-density points) of A form its essential exterior $\text{ext}^* A$. The essential boundary of A equals $\partial^* A \stackrel{\text{def}}{=} \mathbb{R}^m \setminus (\text{int}^* A \cup \text{ext}^* A)$.

The measurable set $A \subset \mathbb{R}^m$ is a BV set if its perimeter $\|A\| \stackrel{\text{def}}{=} \mathscr{H}^{m-1}(\partial^* A)$ is finite (here $\mathscr{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure).

One-dimensional BV sets coincide (modulo sets of zero Lebesgue measure) with finite unions of closed intervals, that is, with one-dimensional figures. Recalling that the zero-dimensional Hausdorff measure is the counting measure, if the BV set $A \subset \mathbb{R}$ is equivalent to $\cup_{j=1}^k I_j$ (where the I_j's are disjoint closed intervals) then the essential boundary of A coincides with the endpoints of the intervals I_j and $\|A\| = 2k$ (the number of the endpoints of these intervals).

A function F defined on m-dimensional BV sets is an m-dimensional charge if it is

1. additive; that is, $F(A \cup B) = F(A) + F(B)$ holds for any two nonoverlapping BV sets A and B (recall that A and B are nonoverlapping if $\operatorname{int}^* A \cap \operatorname{int}^* B = \emptyset$);

2. continuous; that is, for every $\epsilon > 0$ there exists $\rho > 0$ such that $|F(A)| < \epsilon$ if $|A| < \rho$, $\|A\| < 1/\epsilon$, and $A \subset B(0, 1/\epsilon)$.

Assume $E \subset \mathbb{R}^m$, δ: $E \to [0, \infty)$ is a gauge on E if $\{x : \delta(x) = 0\}$ is thin; that is, it is of σ-finite $(m-1)$-dimensional Hausdorff measure.

A system $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^p$ is a partition if the A_i's are nonoverlapping BV sets and $x_i \in A_i$ for $i = 1, \dots, p$. The partition $\mathcal{P}$ is anchored in E if $x_i \in E$ for all $i = 1, \dots, p$. If a gauge δ is given on E then $\mathcal{P}$ is δ-fine if $A_i \subset B(x_i, \delta(x_i))$ holds for all $i = 1, \dots, p$.

The regularity, $r(A)$, of a BV set $A \subset \mathbb{R}^m$ is the number $|A|/d(A)\|A\|$ if $d(A)\|A\| \neq 0$; otherwise $r(A) \stackrel{\text{def}}{=} 0$. If $\eta > 0$ and $r(A) > \eta$ we say that A is η-regular. A partition $\mathcal{P}$ is η-regular if $r(A_i) > \eta$ for all $i = 1, \dots, p$.

If a one-dimensional BV set $A \subset \mathbb{R}$ is η-regular then

$$d(A) \geq |A| > \eta d(A)\|A\| \qquad \text{and hence} \qquad 1/\eta \geq \|A\|; \tag{1}$$

that is, A is equivalent to a finite union of less than $[1/2\eta]$ many closed intervals. Since $\|A\| \geq 2$ we also have

$$|A| > \eta d(A)\|A\| \geq 2\eta d(A). \tag{2}$$

Assume F is a charge defined on BV sets of $\mathbb{R}^m$, $\eta > 0, E \subset \mathbb{R}^m$ and δ: $E \to [0, \infty)$ is a gauge. Set

$$V_{\eta,\delta} F(E) = \sup_{\mathcal{P}} \sum_{i=1}^{p} |F(A_i)|,$$

where the supremum is taken for all η-regular, δ-fine partitions $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^p$. The F-variational outer measure of E is defined as

$$V_* F(E) = \sup_{0<\eta} \inf_{\delta} V_{\eta,\delta} F(E),$$

where the infimum is taken for all δ gauge functions defined on E. It is not difficult to show that $V_* F$ is a metric outer measure.

We say that F is an AC_* (m-dimensional) charge if $V_* F$ is absolutely continuous with respect to the Lebesgue measure (in $\mathbb{R}^m$).

3. THE MAIN RESULT

THEOREM 1. *There exists a charge F defined on BV sets of $\mathbb{R}$ and an absolutely continuous Radon measure μ in $\mathbb{R}$, such that $F \otimes \mu$ is not an AC_* charge in $\mathbb{R}^2$.*

Proof. Let C be the triadic Cantor set. We denote by C_k the union of the 2^k many closed intervals of length 3^{-k} which are used in the definition of C at step k. We denote these intervals by $G_{1,k}, \dots, G_{2^k,k}$. We denote by $G'_{1,k}, \dots, G'_{2^k,k}$ the intervals which are concentric with $G_{1,k}, \dots, G_{2^k,k}$, respectively, and which are of length

$$h_k = \frac{1}{k^2 2^{k+1} 3^{3k+2}}.$$

Observe that $C_{k+1} \cap \bigcup_{j=1}^{2^k} G'_{j,k} = \emptyset$.

On $G'_{j,k}$ we set for each $j = 1, \dots, 2^k$

$$\phi(x) = \frac{1}{h_k k^2 2^k}.$$

Observe that

$$\int_{\cup_j G'_{j,k}} \phi(x)dx = 2^k \frac{h_k}{h_k k^2 2^k} = \frac{1}{k^2}.$$

Set $\phi(x) = 0$ for $x \notin \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^k} G'_{j,k}$.

Put $\mu(A) = \int_A \phi(x)dx$. Observe that μ is an absolutely continuous Radon measure with support in $[0,1]$ and

$$\mu(G'_{j,k}) = \frac{1}{k^2 2^k} \tag{3}$$

for each $k \in \mathbb{N}$ and $j = 1, \dots, 2^k$.

To define the charge F we need an auxiliary Cantor type set E. We define this set as the intersection of some auxiliary nested closed sets E_k and each E_k will be a union of nonoverlapping closed intervals, each of length 3^{-k^2}.

Set $E_0 = I_{1,0} = [0,1]$, $N_0 = 1$.

Assume $k \geq 0$, E_k is defined and equals the union of nonoverlapping closed intervals $I_{j,k}$, $j = 1, \dots, N_k$, each of length 3^{-k^2}.

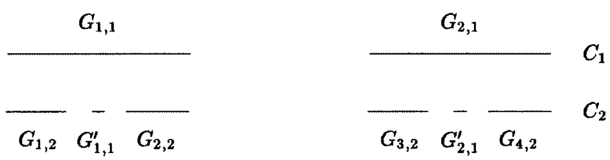

FIGURE 1

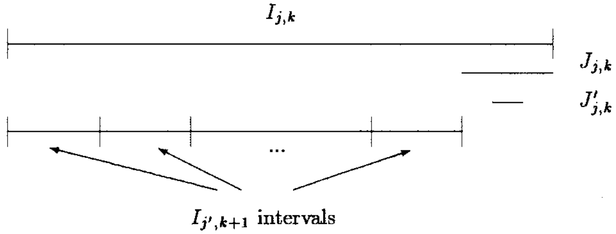

FIGURE 2

To define E_{k+1} divide each $I_{j,k}$ into $3^{-k^2}/3^{-(k+1)^2} = 3^{(k+1)^2-k^2}$ many nonoverlapping intervals, each of length $3^{-(k+1)^2}$, keep the first $3^{(k+1)^2-k^2} - 1$ of them, and delete the last one. Repeating this procedure in each $I_{j,k}$ we obtain the closed intervals $I_{j',k+1}$, $j' = 1, \dots, N_{k+1}$, where $N_{k+1} = (3^{(k+1)^2-k^2} - 1)N_k$. For $j = 1, \dots, N_k$ denote by $J_{j,k}$ the last deleted subinterval of $I_{j,k}$. Set $E_{k+1} = \bigcup_{j'=1}^{N_{k+1}} I_{j',k+1}$.

By induction we define E_k for all $k \in \mathbb{N}$ and set $E = \bigcap_{k=1}^{\infty} E_k$. Observe that for $k \geq 1$

$$|E_k| = \prod_{j=1}^{k} \left(1 - 3^{(j-1)^2-j^2}\right)$$

and hence $|E| > 0$.

Assume that k is fixed and we choose a $j \in \{1, \dots, N_k\}$. Then the interval $I_{j,k}$ contains a subinterval $J_{j,k}$ of length $3^{-(k+1)^2}$. Denote by $J'_{j,k}$ the subinterval of $J_{j,k}$ which is concentric with it and of length $(1/3)|J_{j,k}|$. Observe that $\operatorname{dist}(J'_{j,k}, E_{k+1}) = (1/3)|J_{j,k}| = 3^{-(k+1)^2-1}$. Set

$$\kappa_k = k^4 2^{k^2+1} 3^{(k+1)^2}.$$

Divide $J'_{j,k}$ into κ_k many equal subintervals, $J_{i,j,k}$, $i = 1, \dots, \kappa_k$. Then $|J_{i,j,k}| = 3^{-(k+1)^2-1}/\kappa_k$ for all i.

For $i = 1, \dots, \kappa_k/2$ on $\operatorname{int}(J_{2i,j,k})$ set $f(x) = |J_{j,k}|/|J_{2i,j,k}|$; on $\operatorname{int}(J_{2i-1,j,k})$ set $f(x) = -|J_{j,k}|/|J_{2i-1,j,k}|$. If $x \notin \bigcup_k \bigcup_j \bigcup_i \operatorname{int} J_{i,j,k}$ then set $f(x) = 0$.

Observe that if $[a, b]$ is an arbitrary interval and $a, b \notin J_{j,k}$ then

$$\int_{[a,b] \cap J'_{j,k}} f(x)dx = 0.$$

If A is one-dimensional BV set put

$$F(A) = \sum_{k=0}^{\infty} \sum_{j=1}^{N_k} \int_{A \cap J_{j,k}} f(x)dx.$$

By the above observation all but finitely many terms of the double sum in the definition of F equal zero; hence $F(A)$ is well defined.

Next we need to verify that F is an AC_* charge.

Assume that the BV set A is equivalent (modulo a set of measure zero) to the disjoint closed intervals $Z_1, \dots, Z_t$. We will denote this property later by $A \sim \bigcup_{s=1}^{t} Z_s$.

If no endpoint of an interval Z_s belongs to an interval of the form $J'_{j,k}$ then

$$F(Z_s) = \sum_{k=0}^{\infty} \sum_{j=1}^{N_k} \int_{Z_s \cap J_{j,k}} f = 0. \tag{4}$$

If at least one of the endpoints of Z_s belongs to an interval of the form J_{j_s, k_s} then, without limiting generality, we can assume that the other endpoint of Z_s either does not belong to an interval of the form $J'_{j,k} \subset J_{j,k}$, or if it belongs to such an interval then $k \geq k_s$. It is easy to see that in this case

$$|F(Z_s)| \leq 2|J_{j_s, k_s}|. \tag{5}$$

First we show that F is a charge. It is clearly additive. Assume $\epsilon > 0$ is given. Without limiting generality we assume $\epsilon < 1$. We need to choose $\rho > 0$ such that if the BV set $A \subset B(0, 1/\epsilon)$, $\|A\| < 1/\epsilon$, and $|A| < \rho$ then $|F(A)| < \epsilon$.

Assume $A \sim \bigcup_{s=1}^{t} Z_s$. Then $\|A\| < 1/\epsilon$ implies $t < 1/2\epsilon$. Choose k_0 such that

$$|J_{1, k_0}| = |J_{j, k_0}| < \epsilon^2/2, \qquad j = 1, \dots, N_{k_0}. \tag{6}$$

Since f is bounded on $H_{k_0} \overset{\text{def}}{=} \bigcup_{k=1}^{k_0-1} \bigcup_{j=1}^{N_k} J_{j,k}$ we can choose $\rho_1 > 0$ such that if the interval $Z_s \subset H_{k_0}$ and $|Z_s| < \rho_1$ then $|\int_{Z_s} f| = |F(Z_s)| < \epsilon^2$.

If $\int_{Z_s \cap H_{k_0}} f \neq 0$ then there exists $k \in \{1, \dots, k_0 - 1\}$ and $j \in \{1, \dots, N_k\}$ such that $Z_s \cap J'_{j,k} = \emptyset$. This implies that if we assume that $|Z_s| \leq |A| < \rho_2 = |J_{1, k_0}|/3$ then $Z_s \subset H_{k_0}$.

Therefore if $\int_{Z_s \cap H_{k_0}} f \neq 0$ then $|Z_s| \leq |A| \leq \rho \overset{\text{def}}{=} \min(\rho_1, \rho_2)$ implies that $|F(Z_s)| < \epsilon^2$.

If $\int_{Z_s \cap H_{k_0}} f = 0$ but $F(Z_s) \neq 0$ then an estimate of the form (5) holds with a $k_s \geq k_0$. Using (6) we obtain

$$|F(Z_s)| \leq 2|J_{j_s, k_s}| \leq 2|J_{1, k_0}| < \epsilon^2.$$

This implies

$$|F(A)| \leq \sum_{s=1}^{t} |F(Z_s)| \leq \epsilon^2/2\epsilon = \epsilon/2 < \epsilon.$$

Next we show that F is AC_*.

Since V_*F is a Borel measure, $V_*F(\mathbb{R}\setminus[0,1]) = 0$, and one can easily see that V_*F is absolutely continuous on each set $J_{j,k}$, $k = 0, 1, \dots, j = 1, \dots, N_k$ it is enough to show that it is absolutely continuous on E. Assume $S \subset E$, $|S| = 0$, $\eta > 0$ is fixed, and $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^{p}$ is an η-regular partition, anchored in $S \subset E$. Assume, furthermore, that $A_i \sim \bigcup_{s=1}^{t_i} Z_{s,i}$. Then (1) implies $t_i \leq 1/2\eta$ and (2) implies $|A_i| > 2\eta d(A_i)$.

Now, if $F(Z_{s,i}) \neq 0$, that is, (4) does not hold, then we can use (5) to obtain

$$|F(Z_{s,i})| \leq 2|J_{j_{s,i},k_{s,i}}|.$$

Since $x_i \in A_i$, $x_i \in S \subset E$, and $E \cap \operatorname{int}(J_{j_{s,i},k_{s,i}}) = \emptyset$ we also have $d(A_i) \geq |J_{j_{s,i},k_{s,i}}|/3$.

Hence, keeping in mind that $\|A_i\| \geq 2$, we obtain

$$|A_i| \geq \eta d(A_i)\|A_i\| \geq 2\eta|J_{j_{s,i},k_{s,i}}|/3 \geq \eta|F(Z_{s,i})|/3.$$

Since $t_i \leq 1/2\eta$ we obtain

$$|F(A_i)| \leq \sum_{s=1}^{t_i} |F(Z_{s,i})| \leq t_i \frac{3|A_i|}{\eta} \leq \frac{3|A_i|}{2\eta^2}.$$

Cover S by an open set G such that $|G| < \eta^2\epsilon/3$ and for $x \in H$ choose $\delta(x) > 0$ satisfying $B(x, \delta(x)) \subset G$. If $\mathcal{P}$ is δ-fine then $\bigcup_{i=1}^{p} A_i \subset G$ and hence

$$\sum_{i=1}^{p} |F(A_i)| \leq \sum_{i=1}^{p} \frac{3|A_i|}{2\eta^2} < 3|G|/\eta^2 < \epsilon.$$

This shows that using this δ we have $V_{\eta,\delta}F(S) < \epsilon$. Using that for each $\epsilon > 0$ one can choose a suitable δ and one can repeat the above argument for all $\eta > 0$ we obtain that $V_*F(S) = 0$. This completes the proof of the fact that F is AC_*.

Finally we show that $F \otimes \mu$ is not AC_*. Using Chapter 7 of [3] one can easily see that the Hausdorff dimension of $C \times E$ is $1 + \log 2/\log 3$, and hence $|C \times E| = 0$ but it is not of σ-finite one-dimensional Hausdorff measure; that is, it is not thin.

We show that $V_*F(C \times E) \neq 0$.

Assume η is sufficiently small (something like $\eta < 1/20$ is suitable). Since $C \times E$ is not thin if δ is an arbitrary two-dimensional gauge function then there exists an $x \in C \times E$ for which $\delta(x) > 0$. Choose and fix such an x.

Choose k such that $3^{-k^2} < \delta(x)/2$ and assume that G_x is the base interval of C_{k^2} which contains x and is of length 3^{-k^2}, and we denote by G'_x the corresponding middle third interval of G_x. Recall that G'_x is of length h_{k^2},

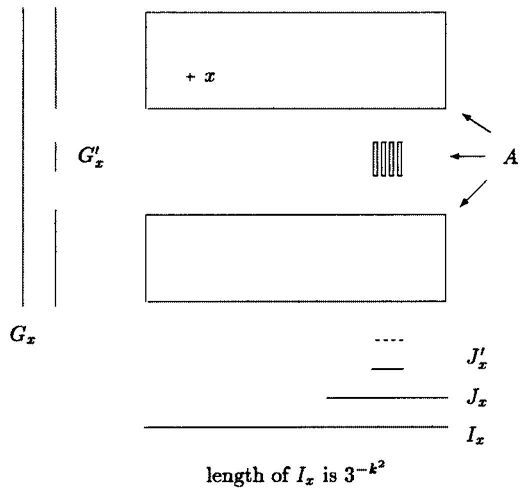

FIGURE 3

and it was used in the definition of ϕ and μ; furthermore by (3) we have $\mu(G'_x) = k^{-4}\, 2^{-k^2}$.

Similarly, assume that I_x is the base interval $I_{j,k}$ of length 3^{-k^2} which was used in the definition of E_k and J_x is the corresponding interval $J_{j,k}$ of length $3^{-(k+1)^2}$. Furthermore, the middle third of J_x, $J'_{j,k}$ will be denoted by J'_x and the intervals $J_{i,j,k}$ used in the definition of f will be denoted by $J_{x,i}(i = 1, \ldots, \kappa_k)$.

Observe that $C_{k^2+1} \cap G_x$ consists of two intervals, each of length $|G_x|/3$, and $G'_x \cap C_{k^2+1} = \emptyset$.

Set $A = (I_x \times (C_{k^2+1} \cap G_x)) \cup ((\bigcup_{i=1}^{\kappa_k/2} J_{x,2i}) \times G'_x)$.

Then $\|A\| \le 6 \cdot 3^{-k^2} + \kappa_k h_{k^2} < 7 \cdot 3^{-k^2}$, $d(A) < \sqrt{2} \cdot 3^{-k^2}$, and $|A| > (2/3)3^{-2k^2}$. This implies that A is η-regular. Since $A \subset G_x \times I_x \subset B(x, \delta(x))$, the one element partition $\mathcal{P} = \{(A, x)\}$ is η-regular and δ-fine.

On the other hand

$$(F \otimes \mu)(A) = \mu(C_{k^2+1} \cap G_x)F(I_x) + \mu(G'_x)\frac{\kappa_k}{2}\int_{J_{x,2}} f$$
$$= 0 + k^{-4}2^{-k^2}\frac{\kappa_k}{2}|J_{x,2}|\frac{|J_x|}{|J_{x,2}|} = \frac{\kappa_k}{k^4 2^{k^2+1} 3^{(k+1)^2}} = 1.$$

This implies that $V_{\eta,\delta}(F \otimes \mu)(C \times E) \ge 1$ holds for all gauge δ and hence $V_* F(C \times E) \ge 1$. Thus $V_*(F \otimes \mu)$ is not absolutely continuous.

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