JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **190**, 584–593 (1995)

Hypergeometric Reduction Formulas Involving Roots of Unity

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Submitted by H. M. Srivastava

Received June 14, 1993

A special $F_D^{(n-1)}$ with variables $(\omega, \omega^2, \ldots, \omega^{n-1})$, where $\omega = \exp(2\pi i/n)$, is expressed in terms of gamma functions. This formula is used in the proof of a reduction formula for a certain n-dimensional power series with variables $(x, \omega x, \omega^2 x, \ldots, \omega^{n-1}x)$. Further generalizations of the results are established. Similar results for $F_C^{(n-1)}$ are discussed for $n \leq 4$; the general case is an open problem.

1. INTRODUCTION

Reduction formulas for hypergeometric functions have been given considerable attention in the literature. Classical results involving single and double hypergeometric functions have been generalized by establishing reduction formulas for generalized Kampé de Fériet functions. Many of these formulas are in turn particular cases of reduction formulas for certain multiple power series. Some recent listings of such results are found in [2; 8; 9, pp. 28–32]. The formulas typically involve variables that are equal or opposite. (Also, certain parameters may be required to be equal.)

In this paper, we shall establish reduction formulas with a different kind of condition: A set of variables appearing in an expression to be reduced is *either* all nth roots of unity except unity itself *or* all nth roots of unity multiplied by a common factor. For convenience, we introduce

$$\omega = \exp\frac{2\pi i}{n}. \tag{1}$$

0022-247X/95 $6.00

The variable sets just mentioned are thus $(\omega, \omega^2, \ldots, \omega^{n-1})$ and $(z, \omega z, \omega^2 z, \ldots, \omega^{n-1}z)$, respectively. The elementary identity

$$1 - z^n = (1 - z)(1 - \omega z)(1 - \omega^2 z) \cdots (1 - \omega^{n-1} z) \tag{2}$$

will play a significant role in the following.

2. The Generalized Kummer Formula

A well-known hypergeometric summation formula due to Kummer [5, Sect. 23, Eq. (1)] may be written

$${}_2F_1[2a, b; 1 + 2a - b; -1] = \frac{1}{2}\Gamma\begin{bmatrix} a, 1 + 2a - b \\ 2a, 1 + a - b \end{bmatrix}. \tag{3}$$

Here, and in what follows, we use (cf. [6, Sect. 2.1.1]) the contracted notation

$$\Gamma\begin{bmatrix} \alpha_1, \ldots, \alpha_u \\ \beta_1, \ldots, \beta_v \end{bmatrix} = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_u)}{\Gamma(\beta_1) \cdots \Gamma(\beta_v)}.$$

Kummer's formula (3) may be proved by a straightforward application of the Eulerian integral representation for ${}_2F_1$. The generalization of (3) involves a special Lauricella function $F_D^{(n-1)}$, and again we start from a single Eulerian integral representation, the Schläfli–Picard integral representation; see e.g. [9, Sect. 9.4, Eq. (34)]. Next, we perform the substitution $t^n = u$. Thus,

$$\begin{aligned} &F_D^{(n-1)}[na; b, b, \ldots, b; 1 + na - b; \omega, \omega^2, \ldots, \omega^{n-1}] \\ &\quad = \Gamma\begin{bmatrix} 1 + na - b \\ na, 1 - b \end{bmatrix} \int_0^1 t^{na-1}(1 - t)^{-b}(1 - \omega t)^{-b} \cdots (1 - \omega^{n-1}t)^{-b}\, dt \\ &\quad = \Gamma\begin{bmatrix} 1 + na - b \\ na, 1 - b \end{bmatrix} \int_0^1 t^{na-1}(1 - t^n)^{-b}\, dt \\ &\quad = \Gamma\begin{bmatrix} 1 + na - b \\ na, 1 - b \end{bmatrix} \int_0^1 \frac{1}{n} u^{a-1}(1 - u)^{-b}\, du \\ &\quad = \frac{1}{n}\Gamma\begin{bmatrix} 1 + na - b \\ na, 1 - b \end{bmatrix} \Gamma\begin{bmatrix} a, 1 - b \\ 1 + a - b \end{bmatrix}. \end{aligned}$$

It may be readily verified that the multiplication is correct with respect to principal values of power functions; thus it does not imply any conditions. On the other hand, the integral representation itself requires Re $a > 0$, and Re$(1 - b) > 0$. These conditions are relaxed by analytical continuation, and we thus obtain the *generalized Kummer formula*,

$$F_D^{(n-1)}[na; b, b, \ldots, b; 1 + na - b; \omega, \omega^2, \ldots, \omega^{n-1}] = \frac{1}{n}\Gamma\begin{bmatrix} a, 1 + na - b \\ na, 1 + a - b \end{bmatrix}. \tag{4}$$

provided that neither $1 + na - b$ nor a is zero or a negative integer. However, if a is zero or a negative integer, we may take the appropriate limit using elementary properties of the gamma function. The result is

$$\begin{aligned} &F_D^{(n-1)}[-k; b, b, \ldots, b; 1 - k - b; \omega, \omega^2, \ldots, \omega^{n-1}] \\ &= \begin{cases} \dfrac{(nj)!(b)_j}{j!(b)_{nj}}, & k = nj, j \in \mathbb{N}_0, \\ 0, & k \in \mathbb{N}\backslash\{n, 2n, \ldots\}. \end{cases} \end{aligned} \tag{5}$$

Clearly, we get zero when na is a negative integer while a is not. Formula (4) for $n = 3$ was given by the author [3, Eq. (4.11a)]. For $n = 2$ we obtain, of course, Kummer's formula (3).

3. Multiple Sum Reduction Formula

Let $\{\Lambda(\mu)\}_{\mu=0}^{\infty}$ be a sequence of complex numbers. Then, subject to conditions of absolute convergence, the *reduction formula*

$$\sum_{m_1, \ldots, m_n = 0}^{\infty} \Lambda(m_1 + \cdots + m_n) \prod_{r=1}^{n} \frac{(b)_{m_r}(x\omega^{r-1})^{m_r}}{m_r!} = \sum_{j=0}^{\infty} \frac{\Lambda(nj)(b)_j x^{nj}}{j!} \tag{6}$$

holds. To prove (6), we rewrite the left-hand member as

$$\sum_{k=0}^{\infty} \frac{\Lambda(k)(b)_k x^k}{k!} A_k,$$

where

$$A_k = \sum_{m_2,\ldots,m_n=0}^{\infty} \frac{k!(b)_{k-(m_2+\cdots+m_n)}}{(k-(m_2+\cdots+m_n))!(b)_k} \prod_{r=2}^{n} \frac{(b)_{m_r}(\omega^{r-1})^{m_r}}{m_r!}$$

$$= \sum_{m_2,\ldots,m_n=0}^{\infty} \frac{(-k)_{m_2+\cdots+m_n}}{(1-b-k)_{m_2+\cdots+m_n}} \prod_{r=2}^{n} \frac{(b)_{m_r}(\omega^{r-1})^{m_r}}{m_r!}$$

$$= F_D^{(n-1)}[-k; b, b, \ldots, b; 1-k-b; \omega, \omega^2, \ldots, \omega^{n-1}].$$

We may now use (5), and we arrive at (6) without much effort. The particular case $n = 2$, where we have $\omega = -1$, was given by Srivastava [7, Eq. (17)].

An interesting particular case of (6) is obtained if we set

$$\Lambda(k) = \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\gamma_1)_k \cdots (\gamma_q)_k} \qquad (p \leq q) \tag{7}$$

and apply the multiplication formula for the Pochhammer symbol. The result is a reduction formula for a special generalized Kampé de Fériet function with n variables, viz.,

$$F^{p:1;\ldots;1}_{q:0;\ldots;0}\left[\begin{matrix}\alpha_1, \ldots, \alpha_p : b; b; \ldots; b \\ \gamma_1, \ldots, \gamma_q : -; -; \ldots; -\end{matrix} \middle| x, \omega x, \ldots, \omega^{n-1}x\right]$$
$$= {}_{np+1}F_{nq}\left[\begin{matrix}\Delta(n;\alpha_1), \ldots, \Delta(n;\alpha_p), b \\ \Delta(n;\gamma_1), \ldots, \Delta(n;\gamma_q)\end{matrix} \middle| (n^{p-q}x)^n\right]. \tag{8}$$

As usual, $\Delta(n; \alpha)$ denotes the array $\{\alpha/n, (\alpha+1)/n, \ldots, (\alpha+n-1)/n\}$. For $n = 2$, Eq. (8) becomes [2, Eq. (3.3)], which is, incidentally, also recorded as [9, Sect. 1.3, Eq. (44)].

4. A FURTHER GENERALIZATION

The transformations leading to the generalized Kummer formula may be generalized. Thus, if we modify the F_D in Section 2 by insertion of new variables $(y, \omega y, \omega^2 y, \ldots, \omega^{n-1}y)$ and, associated with these, new numerator parameters all of which are equal to β, we shall have to insert the product

$$(1 - yt)^{-\beta}(1 - \omega yt)^{-\beta} \cdots (1 - \omega^{n-1}yt)^{-\beta}$$

into the Schläfli–Picard integral. The product equals $(1 - y^n t^n)^{-\beta}$, which in turn becomes $(1 - y^n u)^{-\beta}$; the integral thus represents a ${}_2F_1$ with variable y^n. Now, any number of such products could be inserted, and we finally arrive at a reduction formula expressing a special F_D in terms of an F_D with fewer variables, viz.,

$$\begin{aligned} &F_D^{(n-1+Nn)}[na; [b]_{n-1}, [b_1]_n, \ldots, [b_N]_n; 1 + na - b; \Omega^-, \Omega y_1, \ldots, \Omega y_N] \\ &\quad = \frac{1}{n}\Gamma\begin{bmatrix} a, 1 + na - b \\ na, 1 + a - b \end{bmatrix} F_D^{(N)}[a; b_1, \ldots, b_N; 1 + a - b; y_1^n, \ldots, y_N^n], \end{aligned} \tag{9}$$

where for brevity

$$\Omega^- = (\omega, \omega^2, \ldots, \omega^{n-1}),$$
$$\Omega y = (y, \omega y, \omega^2 y, \ldots, \omega^{n-1} y),$$

and

$$[\cdot]_\lambda \text{ denotes a set of } \lambda \text{ equal elements.}$$

The formula holds, by analytical continuation, whenever $1 + na - b$ is not a negative integer or zero and the y-variables are suitably restricted. For negative integral values of a we may proceed as in Section 2. The result is

$$\begin{aligned} &F_D^{(n-1+Nn)}[-k; [b]_{n-1}, [b_1]_n, \ldots, [b_N]_n; 1 - b - k; \Omega^-, \Omega y_1, \ldots, \Omega y_N] \\ &\quad = \begin{cases} \dfrac{(nj)!(b)_j}{j!(b)_{nj}} F_D^{(N)}[-j; b_1, \ldots, b_N; 1 - b - j; y_1^n, \ldots, y_N^n], & k = nj, j \in \mathbb{N}_0, \\ 0, & k \in \mathbb{N}\backslash\{n, 2n, \ldots\}. \end{cases} \end{aligned} \tag{10}$$

Two particular cases of (9), viz. $(N, n) = (1, 3)$ and $(N, n) = (1, 2)$, have been given previously [3, Eqs. (4.10) and (6.1)].

Having generalized (4) we naturally seek a generalization of (6), and indeed a special Nn-dimensional power series is found to be reducible to an N-dimensional power series:

$$\sum_{m_1,\ldots,m_{Nn}=0}^{\infty} \Lambda(m_1+\cdots+m_{Nn}) \left\{ \prod_{r=1}^{n} \frac{(b_1)_{m_r}(y_1\omega^{r-1})^{m_r}}{m_r!} \right\} \times \cdots$$

$$\times \left\{ \prod_{r=Nn-n+1}^{Nn} \frac{(b_N)_{m_r}(y_N\omega^{r-1})^{m_r}}{m_r!} \right\} \tag{11}$$

$$= \sum_{j_1,\ldots,j_N=0}^{\infty} \Lambda(nj_1+\cdots+nj_N) \prod_{s=1}^{N} \frac{(b_s)_{j_s} y_s^{nj_s}}{j_s!}.$$

Again, $\{\Lambda(\mu)\}_{\mu=0}^{\infty}$ is a sequence of complex numbers, subject to conditions that will ensure absolute convergence. Note, furthermore, that ω^{r-1} is unaltered when r is replaced by $n+r$.

The proof of (11) is similar to that of (6), and we merely indicate the steps. Let L denote the left-hand member of (11). We first obtain

$$L = \sum_{k=0}^{\infty} \frac{1}{k!} \Lambda(k)(b_1)_k y_1^k A_k,$$

where A_k is a sum over $m_2, \ldots, m_{Nn}$. After a few manipulations, we apply (10) to A_k; this leads to the expression

$$L = \sum_{j=0}^{\infty} \frac{\Lambda(nj)(b_1)_j y_1^{nj}}{j!} F_D^{(N-1)} \left[-j; b_2, \ldots, b_N; 1-b_1-j; \left(\frac{y_2}{y_1}\right)^n, \ldots, \left(\frac{y_N}{y_1}\right)^n \right].$$

This may be written as a sum over $j, j_2, \ldots, j_N$, where we must, clearly, have $j \geq j_2 + \cdots + j_N$. Next, we introduce the summation index $j_1 = j - (j_2 + \cdots + j_N)$ and obtain, finally, the right-hand member of (11).

In particular, we might use the expression for Λ given by (7). Then, formula (11) gives us a generalization of (8), namely,

$$F_{q:0;\ldots;0}^{p:1;\ldots;1} \left[\begin{matrix} \alpha_1, \ldots, \alpha_p : [b_1]_n; \ldots; [b_N]_n \\ \gamma_1, \ldots, \gamma_q : - \quad ; \ldots; - \end{matrix} \middle| \Omega y_1, \ldots, \Omega y_N \right]$$

$$= F_{nq;0;\ldots;0}^{np;1;\ldots;1} \left[\begin{matrix} \Delta(n;\alpha_1), \ldots, \Delta(n;\alpha_p): b_1; \ldots; b_N \\ \Delta(n;\gamma_1), \ldots, \Delta(n;\gamma_q): -; \ldots; - \end{matrix} \middle| (n^{p-q}y_1)^n, \ldots, (n^{p-q}y_N)^n \right]. \tag{12}$$

Note that on the left-hand side we have a function of Nn variables grouped in sets of n variables, and parameters associated with one variable are similarly grouped.

5. Another Multiple Sum

The question naturally arises of whether other multiple series reduction formulas exist that are similar to (6). For instance, we might move the b-parameters from the numerator to the denominator:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \Lambda(m_1+\cdots+m_n)\prod_{r=1}^{n}\frac{(x\omega^{r-1})^{m_r}}{(b)_{m_r} m_r!}=\sum_{k=0}^{\infty} C_{k,n}x^k. \tag{13}$$

The first steps are similar to those in Section 3, and we obtain

$$C_{k,n}=\frac{\Lambda(k)}{(b)_k k!}F_C^{(n-1)}[-k, 1-b-k; b, b, \ldots, b; \omega, \omega^2, \ldots, \omega^{n-1}]. \tag{14}$$

However, there is no companion to (5) for $F_C^{(n-1)}$. Only the first three results of this kind can be stated.

For $n = 2$, the formula in question is Kummer's formula for a terminating series,

$${}_2F_1\left[\begin{matrix}-k, 1-b-k\\ b\end{matrix}\,\middle|\, -1\right]=\begin{cases}\dfrac{(-1)^j(2j)!}{(b)_j j!}, & k=2j,\ j\in\mathbb{N}_0,\\ 0, & k\in\{1,3,5,\ldots\}.\end{cases} \tag{15}$$

The corresponding double series reduction formula was given by Srivastava [7, Eq. (5)].

The case $n = 3$ was considered by Srivastava and the author [4]; the result may be written

$$F_4\left[-k, 1-b-k; b, b; \exp\frac{2\pi i}{3}, \exp\frac{4\pi i}{3}\right] = \begin{cases}\dfrac{(3j)!(2b-1+3j)_j}{j!(b)_j(b)_{2j}}, & k=3j,\ j\in\mathbb{N}_0,\\ 0, & k\in\mathbb{N}\backslash\{3,6,9,\ldots\}.\end{cases} \tag{16}$$

Finally, it will be shown in Section 6 that for $n = 4$ the formula reads

$$F_C^{(3)}[-k, 1-b-k; b, b, b; \mathrm{i}, -1, -\mathrm{i}]$$

$$= \begin{cases} \dfrac{(-1)^j (4j)!}{j!(b)_j (b)_{2j}}\, {}_3F_2\left[\begin{matrix} -j, 1-b-2j, -\frac{1}{2}+b+2j \\ \frac{b}{2}, \frac{b+1}{2} \end{matrix} \middle| 1\right], & k = 4j, j \in \mathbb{N}_0, \\ 0, & k \in \mathbb{N}\backslash\{4, 8, 12, \ldots\}. \end{cases} \tag{17}$$

This ${}_3F_2$ does not seem to fit into any of the known summation theorems.

While it is clear from the outset that we must get zero unless we have $k = nj$, $j \in \mathbb{N}_0$, Eqs. (15), (16), and (17) do not otherwise give any clue to a general expression. Neither do we find any similarity when comparing the proofs in Section 6 and in [4]; and in both cases we use transformations that do not lend themselves to generalization. We are really left with an open problem, as far as $C_{k,n}$ is concerned.

6. Proof of Equation (17)

We prove below two reduction formulas which evidently imply (17), when β, c, d are properly chosen; namely

$$F_C^{(3)}[-k, 1-b-k; \beta, \beta, b; \xi, -\xi, -1]$$

$$= \begin{cases} \dfrac{(-1)^q (2q)!}{(b)_q q!}\, {}_4F_3\left[\begin{matrix} -q, 1-b-q, \frac{1-b}{2}-q, 1-q-\frac{b}{2} \\ \beta, \frac{\beta}{2}, \frac{\beta+1}{2} \end{matrix} \middle| \xi^2\right], & k = 2q, q \in \mathbb{N}_0, \\ 0, & k \in \{1, 3, 5, \ldots\}; \end{cases} \tag{18}$$

and

$$
{}_4F_3\left[\begin{matrix}-q,1-b-q,1-c-q,1-d-q\\ b,c,d\end{matrix}\middle| -1\right]
$$

$$
=\begin{cases}\dfrac{(-1)^j(2j)!}{(b)_j j!}\,{}_3F_2\left[\begin{matrix}-j,1-b-2j,c+d-1+2j\\ c,d\end{matrix}\middle| 1\right], & q=2j,\ j\in\mathbb{N}_0,\\ 0, & q\in\{1,3,5,\ldots\}.\end{cases} \tag{19}
$$

To prove (18) we use the triple series transformation formula [9, Section 9.4, Eq. (135)]

$$
\sum_{\mu,\nu,\rho=0}^{\infty}\frac{A(\mu+\nu,\rho)}{(\beta)_\mu(\beta)_\nu}\frac{\xi^\mu\eta^\nu\zeta^\rho}{\mu!\nu!\rho!}=\sum_{\mu,\nu,\rho=0}^{\infty}\frac{A(\mu+2\nu,\rho)}{(\beta)_{\mu+2\nu}(\beta)_\nu}\frac{(\xi+\eta)^\mu(\xi\eta)^\nu\zeta^\rho}{\mu!\nu!\rho!}
$$

with

$$
\eta=-\xi,\qquad \zeta=-1,\qquad A(\lambda,\rho)=\frac{(-k)_{\lambda+\rho}(1-b-k)_{\lambda+\rho}}{(b)_\rho}.
$$

The right-hand member now reduces to a double sum which may be so arranged that Kummer's theorem applies. Then (18) is arrived at after a few manipulations.

Next, we recall Whipple's transformation [10, Eq. (3.4)], which may be written

$$
{}_4F_3\left[\begin{matrix}a,1+a-b,1+a-c,1+a-d\\ b,c,d\end{matrix}\middle| -1\right]
$$

$$
=\Gamma\left[\begin{matrix}c,d\\ 1+a,c+d-a-1\end{matrix}\right]{}_3F_2\left[\begin{matrix}b-\frac{a}{2},1+a-c,1+a-d\\ 1+\frac{a}{2},b\end{matrix}\middle| 1\right].
$$

To the right-hand member we now apply Thomae's transformation

$$
{}_3F_2\left[\begin{matrix}\alpha,\beta,\gamma\\ \delta,\varepsilon\end{matrix}\middle| 1\right]=\Gamma\left[\begin{matrix}\delta,\varepsilon,\sigma\\ \alpha,\beta+\sigma,\gamma+\sigma\end{matrix}\right]{}_3F_2\left[\begin{matrix}\delta-\alpha,\varepsilon-\alpha,\sigma\\ \beta+\sigma,\gamma+\sigma\end{matrix}\middle| 1\right],
$$

$$
\sigma=\delta+\varepsilon-\alpha-\beta-\gamma;
$$

compare e.g. [6, Eq. (2.3.3.7)] or [1, Sect. 3.2, Eq. (1)]. Using also the duplication formula for the gamma function, we arrive at

$$
{}_4F_3\left[\begin{matrix} a, 1+a-b, 1+a-c, 1+a-d \\ b, c, d \end{matrix}\,\middle|\, -1\right]
$$

$$
= 2^{-a}\Gamma\left[\begin{matrix} \frac{1}{2}, b \\ \frac{1+a}{2}, b-\frac{a}{2} \end{matrix}\right] {}_3F_2\left[\begin{matrix} \frac{a}{2}, 1+a-b, c+d-a-1 \\ c, d \end{matrix}\,\middle|\, 1\right], \tag{20}
$$

subject to convergence conditions. Finally, (19) follows from (20) without much effort.

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