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Simultaneous Approximation of a Set of Complex-Valued Functions in a Normed Vector Space

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We present a unifying characterisation theory for best simultaneous approximation of a set of complex-valued bounded functions on a compact topological space B in a normed vector space, by elements of a non-linear subset of $C(B)$. The linear problem in the uniform norm was first considered by Diaz and McLaughlin [*J. Approximation Theory* **2** (1969), 419–432] and was further developed by Blatt [*J. Approximation Theory* **8** (1973), 210–248] for non-linear subsets. We now generalise their approach to an arbitrary norm using the Hahn–Banach theory.

1. Introduction of Problem, Basic Definitions and Results

Let B be a compact topological space and $S(B)$ the linear space of complex-valued functions defined on B endowed with an arbitrary norm. For α a positive number, denote by $F \equiv F(\alpha)$ a nonempty subset of $S(B)$ such that if $F \in F$, then $\|f\| \leqslant \alpha$. Let $C(B)$ be the set of complex-valued continuous functions defined on B and $V \equiv V(B)$ a non-linear subset of $C(B)$. We wish to characterise a best simultaneous approximation, v_0, from V to F, if it exists, designated b.s.a. and given by $\sup_{f \in F} \|f - v_0\| = \inf_{v \in V} \sup_{f \in F} \|f - v\|$.

The case of the uniform norm has been treated by Blatt in [2]. In Section 2, we show that this problem is equivalent to finding a best one-sided approximation from V to a w^* upper semi-continuous function h^* (Definition 1.4) where h^* and V are defined now to be on a w^* compact subset of the dual space and h^* is set-valued. In Section 3, we obtain a sufficient condition that v_0 satisfies by generalizing the Kolmogoroff criterion. Furthermore, by im-

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posing on V that it is regular (Definition 1.7), the Kolmogoroff criterion is found to be a necessary condition for a global best approximation and we can further deduce a uniqueness result.

In Section 4, we develop the characterisation of a local best approximation for approximating families which depend on a parameter, with respect to which they have a Fréchet derivative. This includes the case when V is a set of generalized rational polynomials and, with the norm being L_1, we indicate in Section 5 how under appropriate conditions a local best approximation is (i) locally unique, (ii) locally strongly unique, and (iii) characterized by a generalized "alternation" theorem.

Notation. Let R, $\mathbb{C}$ be the fields of real, complex numbers, respectvely, endowed with the usual metric topologies given by $d(x, y) = |x - y|$. Let X and Y be topological spaces, X^* the dual of X, i.e., the set of complex-valued bounded linear functionals $X \to \mathbb{C}$.

Let $A(Y) := [E \subset Y \mid E \neq \varnothing]$ and $\mathscr{K}(Y) := [E \subset Y \mid E$ compact in the topology on Y and $E \neq \varnothing]$. E^c denotes the closure of E, $C(E)$ the complement of E and co$[E]$ the convex hull or cover of E. $W(L, \theta, \epsilon)$ is a w^* open neighbourhood (nbhd) of L, i.e.,

$$W(L, \theta, \epsilon) := \{l \in X^*\colon |(l - L)\, x| < \epsilon \text{ for all } x \in \theta; \\ \text{where } \theta \text{ is some finite subset of } X \text{ and } \epsilon > 0\}.$$

Where there is no loss of clarity we abbreviate $W(L, \theta, \epsilon)$ by $W(L)$ or W.

DEFINITION 1.1. $f\colon X^* \to A(Y)$ is w^* upper semi-continuous (u.s.c.) at $L \in X^*$ if to every open set G with $f(L) \subset G$ there exists a w^* open nbhd $W(L)$ such that $f(W(L)) \subset G$.

DEFINITION 1.2. $f\colon X^* \to R$ is w^* u.s.c. at $L \in X^*$ if to every real number $c > f(L)$ there exists a w^* open nbhd $W(L)$ with $f(l) < c$ for all $l \in W(L)$. The following theorems can be obtained by generalizations of standard topology arguments [9]:

THEOREM 1.1. *If $E \subset X^*$ is w^* compact and $f\colon E \to \mathscr{K}(Y)$ is w^* u.s.c. on E, then $f(E)$ is compact in Y.*

THEOREM 1.2. *If $E \subset X^*$ is w^* compact and $f\colon E \to R$ is w^* u.s.c. on E, then there exists an $L_0 \in E$ such that*

$$f(L_0) = \sup_{L \in E} f(L).$$

Henceforth, we shall further assume X to be a normed linear space. We recall that to each $x \in X$ we can associate the evaluation $\hat{x}\colon X^* \to \mathbb{C}$

given by $\hat{x}(L) \equiv Lx$. We remark that $\hat{x}$ is continuous. We shall omit the cap in the sequel when portraying x as a function on a subset of X^*.

DEFINITION 1.3. Let K be a subset of B^*, the unit norm ball of X^* satisfying

(i) K is w^* closed.

(ii) For every $f \in F$ and $v \in V$, there exists an $L \in K$ with $\operatorname{Re} L(f - v) = \| f - v \|$.

Remarks. (1) The existence of L in B^* in (ii) above guaranteed by the Hahn–Banach Theorem.

(2) We shall henceforth take all neighbourhoods of L to be in K.

(3) We understand by $l_n \to^{\theta} L$ that for this θ and any $\epsilon > 0$, there exists an $n_0 \equiv n_0(\theta, \epsilon)$ such that $l_n \in W(L, \theta, \epsilon)$ for all $n \geqslant n_0$. The following definitions are generalizations of corresponding ones in [5].

DEFINITION 1.4. For $L \in K$ let $h(L) := \{z \in \mathbb{C} \mid \text{there exists an } f \in F \text{ with } f(L) \equiv Lf = z\}$. Now define

$$h^*(L) := \bigcap_{\theta, \epsilon > 0} \Big(\bigcup_{l \in W(L, \theta, \epsilon)} h(l) \Big)^{\circ} \qquad \text{for } L \in K.$$

$h^*(L)$ is a set-valued mapping from K into $A(\mathbb{C})$ and is a closed set for each L.

THEOREM 1.3.

$$h^*(L) = \left\{ z \in C \;\middle|\; \begin{array}{l} \textit{for each } \theta \textit{ there exists at least} \\ \textit{one sequence } \{(l_n, z_n)\} \textit{ satisfying} \\ (1) \quad l_n \in K, \\ (2) \quad l_n \xrightarrow{\theta} L, \\ (3) \quad z_n \in h(l_n), \\ (4) \quad z_n \to z. \end{array} \right\}$$

Proof. Suppose first $z \in h^*(L)$. Then by Definition 1.4,

$$z \in \Big(\bigcup_{l \in W(L, \theta, \epsilon)} h(l) \Big)^{\circ} \qquad \text{for all } \epsilon > 0, \text{ and all } \theta.$$

For each θ then, we have $z \in (\bigcup_{l \in W(L, \theta, 1/n)} h(l))^{\circ}$ and so there exists a sequence $\{(l_n, z_n)\}$ depending possibly on θ with $| z - z_n | < 1/n$, $z_n \in h(l_n)$, $l_n \in K$ and $l_n \in W(L, \theta, 1/n)$. Conversely, if for each θ there exists a sequence $\{(l_n, z_n)\}$ satisfying the four conditions, then for any $\epsilon > 0$ there exists an n_0 such that for $n \geqslant n_0$, $l_n \in W(L, \theta, \epsilon)$ and by (3) $z_n \in h(l_n) \subset \bigcup_{l \in W(L, \theta, \epsilon)} h(l)$.

Now $z = \lim z_n$, therefore $z \in (\bigcup_{l \in W(L,\theta,\epsilon)} h(l))^\circ$. Since the arbitrary intersection of closed sets is again closed,

$$z \in \bigcap_{\epsilon > 0} \Big(\bigcup_{l \in W(L,\theta,\epsilon)} h(l) \Big)^\circ .$$

Finally, since this is true for each θ, the intersection may be taken over all such θ.

Remark. Suppose for each θ_i, $i = 1,\dots, m$, there exists sequences $\{(l_n, z_n)\}$ depending on θ_i satisfying (1)–(3) such that $\{z_n\}$ has limit points, but not in $h(L)$. Let $A(\theta_i)$ be the set of all such limit points. Then if these conditions are met by $\theta = \bigcup_{i=1}^{m} \theta_i$, we have $\varnothing \neq A(\theta) \subset A(\theta_i)$ for $i = 1,\dots, m$. Now $A(\theta_i)$ are closed subsets of the compact disc $\{z: |z| \leqslant \alpha\}$ and therefore the family $\{A(\theta): \theta \subset X\}$ satisfies the finite intersection property. Thus there exists a $z \in \bigcap_\theta A(\theta) \subset h^*(L)$ with $z \notin h(L)$. We employ a modified version of this argument, below.

COROLLARY. *If* $L = \lambda L_1 + (1 - \lambda) L_2$, *where* L, L_1, $L_2 \in K$ *and* $0 < \lambda < 1$, *then*

$$h^*(L) \subset \lambda h^*(L_1) + (1 - \lambda) h^*(L_2)$$

Proof. Since $h(L) \subset \lambda h(L_1) + (1 - \lambda) h(L_2)$, we need only consider $z \in h^*(L)$, $z \notin h(L)$. For any θ and $\epsilon > 0$, let $W^{(i)}(L_i, \theta, \epsilon)$ be a w^* open nbhd of L_i, $i = 1, 2$. Then $\lambda W^{(1)} + (1 - \lambda) W^{(2)}$ is a w^* open nbhd of $L = \lambda L_1 + (1 - \lambda) L_2$.

Setting

$$\bar{h}(L) = \bigcap_{\theta,\epsilon > 0} \Big(\bigcup_{l \in \lambda W^{(1)} + (1-\lambda) W^{(2)}} h(l) \Big)^\circ$$

it is obvious by their definitions that $h^*(L) \subset \bar{h}(L)$.

Now $\bar{h}(L) = \{z \in \mathbb{C} \mid$ for each θ, there exists sequence(s) $\{(l_n, z_n)\}$ satisfying (1)–(4) where $l_n = \lambda p_n + (1 - \lambda) q_n$ with $p_n \in W^{(1)}$, $q_n \in W^{(2)}\}$. Thus $\bar{h}(L) \subset h^*(L)$. Furthermore $p_n \to^\theta L_1$, $q_n \to^\theta L_2$, and since there exists an $f_n \in F$ with $l_n f_n = z_n$, we have $\{(v_n, w_n)\}$, $v_n \in h(p_n)$, $w_n \in h(q_n)$ and $z_n = \lambda v_n + (1 - \lambda) w_n$. Let $A(\theta)$ be the set of all limit points of all such sequence pairs $\{v_n, w_n\}$. Since $z_n \to z \notin h(L)$, $A(\theta) \cap h(L_1) \times h(L_2) = \varnothing$. Now $\{A(\theta): \theta \subset X\}$ satisfies the finite intersection property in the product topology.

Thus there exists a $(v, w) \in \bigcap_\theta A(\theta) \subset h^*(L_1) \times h^*(L_2)$ with $z = \lambda v + (1 - \lambda) w$.

DEFINITION 1.5. A non-void subset of $Y \subset X$ is an extremal subset of

X if a proper convex combination $\lambda x_1 + (1 - \lambda) x_2$, $0 < \lambda < 1$, of two points $x_1, x_2 \in X$, is in Y only if both x_1 and x_2 are in Y.

An extremal subset of X consisting of just one point is called an extremal point of X.

The collection of extremal points of X is denoted by ext (X).

LEMMA 1.1. *If C is a convex and compact subset in R^n then* $C =$ co[ext(C)] (*see, e.g.*, [13, *p.* 232]).

LEMMA 1.2. *Let ϕ be a continuous linear mapping of E_1 into E_2 (two Hausdorff locally convex topological spaces) and M be a compact subset of E_1. Then for every extremal point e_2 of $\phi(M)$ there exists at least one extremal point e_1 of M such that $\phi(e_1) = e_2$* (*see* [10, *p.* 333]).

Given $\phi_1, \ldots, \phi_N$ elements of X and M a w^* closed subset of B^*, let $\mathbf{\Phi}$ denote $(\phi_1, \ldots, \phi_N)^T$ and $\mathbf{0}$ the origin of N-space.

Let $[M, \mathbf{\Phi}]$ denote $[(L\phi_1, \ldots, L\phi_N)^T$ over all $L \in M]$. This is a compact set in Euclidean N-space as is its convex hull, co$[M, \mathbf{\Phi}]$ (see [4, p. 18]).

By $\| \mathbf{\Phi} \|$ we shall understand max $[\| \phi_i \|, i = 1, \ldots, N]$.

LEMMA 1.3. co$[M, \mathbf{\Phi}] =$ co[ext $(M), \mathbf{\Phi}]$.

Proof. By definition of extremal points and Lemma 1.2,

$$\text{ext}(\text{co}[M, \mathbf{\Phi}]) \subset \text{ext}[M, \mathbf{\Phi}] \subset [\text{ext}(M), \mathbf{\Phi}].$$

Furthermore, by applying Lemma 1.1,

$$\begin{aligned} \text{co}[M, \mathbf{\Phi}] &= \text{co}[\text{ext}(\text{co}[M, \mathbf{\Phi}])] \\ &\subset \text{co}[\text{ext}(M), \mathbf{\Phi}]. \end{aligned}$$

On the other hand, co$[\text{ext}(M), \mathbf{\Phi}] \subset$ co$[M, \mathbf{\Phi}]$. Hence the two are identical.

LEMMA 1.4. *If $\mathbf{0}$ is an interior point of* co$[M, \mathbf{\Phi}]$, *then there exists an $\epsilon > 0$ such that for all $\mathbf{\Phi}'$ satisfying $\| \mathbf{\Phi} - \mathbf{\Phi}' \| < \epsilon$, we have $\mathbf{0} \in$* co$[M, \mathbf{\Phi}']$.

Proof. Suppose to the contrary that for every $\epsilon > 0$, there exists a $\mathbf{\Phi}(\epsilon)$ with $\| \mathbf{\Phi} - \mathbf{\Phi}(\epsilon) \| < \epsilon$ and $\mathbf{0} \notin$ co$[M, \mathbf{\Phi}(\epsilon)]$. Then since co$[M, \mathbf{\Phi}(\epsilon)]$ is compact, there exists a separating hyperplane. That is, there exist constants $c_1(\epsilon), \ldots, c_N(\epsilon)$ not all zero, and a real number $\lambda(\epsilon)$ such that

$$\text{Re} \sum_{i=1}^{N} c_i(\epsilon) L\phi_i(\epsilon) \geqslant \lambda(\epsilon) > 0 \qquad \text{for all } L \in M.$$

Without loss of generality, we can normalise $c_i(\epsilon)$ so that $|c_i(\epsilon)| \leqslant 1$ for $i = 1,..., N$.

Let $\epsilon \to 0$. Then $L\phi_i(\epsilon) \to L\phi_i$ for each i, and we can also extract a subsequence from $c_i(\epsilon)$ such that $\lim_{\epsilon\to 0} c_i(\epsilon) = c_i$ for each i. Hence we can deduce that for all $L \in M$

$$\operatorname{Re} \sum_{i=1}^{N} c_i L\phi_i \geqslant 0.$$

It follows that co$[M, \mathbf{\Phi}]$ lies to one side of this hyperplane. Furthermore, $\mathbf{0}$ belongs either outside co$[M, \mathbf{\Phi}]$ or on a hyperplane supporting co$[M, \mathbf{\Phi}]$ at $\mathbf{0}$.

It could not, however, be in the interior of the convex hull for then there would be points of the convex hull to either side of this hyperplane.

Hence we have been led to a contradiction.

DEFINITION 1.6. A non-empty subset Σ of B^* is sign-extremal for $v_0 \in V \subset X$ if $\min_{L\in\Sigma} \operatorname{Re} L(v - v_0) \leqslant 0$ for all $v \in V$.

LEMMA 1.5. *If Σ is a w^* closed subset of B^* then Σ is sign-extremal for $v_0 \in V$ if and only if* ext(Σ) *is sign-extremal for v_0.*

The proof is given in [3, Lemma 2].

We define regular subsets of X in the sense of Brosowski.

DEFINITION 1.7. $V \subset X$ is regular at a point $v_0 \in V$ if for each real number $\lambda > 0$ and for each w^* closed subset A of B^* satisfying $\operatorname{Re} L(v - v_0) > 0$ for all $L \in A$, there exists a $v_\lambda \in V$ with

(R1) $\operatorname{Re} L(v_\lambda - v_0) > 0$ for all $L \in A$,

(R2) $\| v_\lambda - v_0 \| < \lambda$.

The subset V of X is regular if it is regular at every point of V. In [3, p. 155], Brosowski shows that if V is a linear space or a convex set then it is regular.

DEFINITION 1.8. An n-dimensional subspace V is an interpolating subspace on M, $M \subset$ ext(B^*) if for every set of n linearly independent functionals $L_1 ,..., L_n$ in M,

$$\det[L_i(\phi_j)] \neq 0,$$

where $V = \operatorname{span}[\phi_1 ,..., \phi_n]$.

In particular if M has finite cardinality $m > n$, then $\inf |\det[L_i(\phi_j)]| > 0$ when the inf is taken over all selections of n linearly independent functionals.

2. Conversion of Problem to Approximation of h^*

We first deduce a basic property of h^*.

LEMMA 2.1. *$h^*(L)$ is w^* u.s.c. on K and $h^*: K \to \mathscr{K}(\mathbb{C})$.*

Proof. Suppose at $L_0 \in K$ it is not w^* u.s.c. Then there exists an open neighbourhood G of $h^*(L_0)$ such that for every w^* open nbhd $W(L_0, \theta, \epsilon)$, there exists at least one $l \in W(L_0, \theta, \epsilon)$ with $h^*(l) \not\subset G$. If, however, $l \in W(L_0, \theta, \epsilon)$ then $W(l, \theta, \epsilon) \subset W(L_0, \theta, 2\epsilon)$. Thus for each θ and $\epsilon > 0$

$$\Big(\bigcup_{p \in W(l,\theta,\epsilon)} h(p)\Big)^{\circ} \subset \Big(\bigcup_{l \in W(L_0,\theta,2\epsilon)} h(l)\Big)^{\circ}$$

and $h^*(l) \subset h^*(L_0) \subset G$, leading to a contradiction.

Now $h^*(L_0)$ is a closed set and furthermore, $h^*(L_0)$ is bounded, since $\|f\| \leqslant \alpha$ for all $f \in F$ and the neighbourhoods of L_0 are subsets of K, a subset of B. Hence $h^*(L_0)$ is compact.

Remark. This proof does not depend on a countable base at L_0 as opposed to the proof given in [2, Lemma 2.1].

The following "distance" function is most suitable for our problem.

DEFINITION 2.1. $\hat{d}(A, b) = \sup_{a \in A} \operatorname{Re}(a - b)$.

We are now able to take the first step towards an equivalent formulation of our original problem.

LEMMA 2.2. $\sup_{f \in F} \|f - v\| = \sup_{L \in K} \hat{d}(h^*(L), v(L))$.

Proof. For any $f \in F$ and $L \in K$,

$$\operatorname{Re} L(f - v) \leqslant \sup_{z \in h(L)} \operatorname{Re}(z - Lv) \leqslant \sup_{z \in h^*(L)} \operatorname{Re}(z - Lv).$$

But there exists an $L \in K$ such that $\operatorname{Re} L(f - v) = \|f - v\|$. Therefore $\|f - v\| \leqslant \hat{d}(h^*(L), v(L)) \leqslant \sup_{L \in K} \hat{d}(h^*(L), v(L))$. The right-hand bound is independent of f. Therefore $\sup_{f \in F} \|f - v\| \leqslant \sup_{L \in K} \hat{d}(h^*(L), v(L))$. On the other hand, consider the sequence $\{L_n, z_n\}$ with $L_n \in K$, $z_n \in h^*(L_n)$ and

$$\lim_{n \to \infty} \operatorname{Re}(z_n - L_n v) = \sup_{L \in K} \sup_{z \in h^*(L)} \operatorname{Re}(z - Lv).$$

By Theorem 1.3., for $\theta = v$, there exists a sequence $\{q_k^{(n)}, \eta_k^{(n)}\}$ with (1) $q_k^{(n)} \in K$, (2) $q_k^{(n)} \to^{v} L_n$, (3) $\eta_k^{(n)} \in h(q_k^{(n)})$, (4) $\eta_k^{(n)} \to z_n$. Choose k_n so that (i) $|\eta_{k_n}^{(n)} - z_n| < 1/n$ and (ii) $|q_{k_n}^{(n)} v - L_n v| < 1/n$.

Then

$$\|f_{k_n} - v\| \geqslant \operatorname{Re} q_{k_n}^{(n)}(f_{k_n} - v) = \operatorname{Re}(\eta_{k_n}^{(n)} - q_{k_n}^{(n)} v)$$

$$\geqslant \operatorname{Re}(-L_n v + z_n + L_n v - q_{k_n}^{(n)} v + \eta_{k_n}^{(n)} - z_n)$$

$$\geqslant \operatorname{Re}(z_n - L_n v) - |\operatorname{Re}(L_n - q_{k_n}^{(n)})\, v| - |\operatorname{Re}(\eta_{k_n}^{(n)} - z_n)|$$

$$> \operatorname{Re}(z_n - L_n v) - 2/n.$$

Therefore

$$\sup_{f \in F} \|f - v\| \geqslant \lim_{n \to \infty} \|f_{k_n} - v\| \geqslant \lim_{n \to \infty} \operatorname{Re}(z_n - L_n v) = \sup_{L \in K} \hat{d}(h^*(L), v(L)).$$

A consequence of Lemma 2.2 is that we can reformulate our problem as that of finding the best approximation from V to h^* on K using the distance function $\hat{d}$ on $\mathbb{C}$ for approximating a set-valued function. It is desirable to investigate further the function on the right-hand side of Lemma 2.2.

LEMMA 2.3. *Set $g_v(L) := \hat{d}(h^*(L), v(L))$ for $L \in K$. Then g_v is a mapping of K into R and $g_v(L)$ is w^* u.s.c. on K for each v.*

Proof. Let $L_0 \in K$ and $\beta > g_v(L_0)$ with $\epsilon = (\beta - g_v(L_0))/2$ and

$$O \equiv \bigcup_{z \in h^*(L_0)} O_\epsilon(z) \qquad \text{where} \quad O_\epsilon(z) \equiv \{w \colon |w - z| < \epsilon\}.$$

O is w^* open and $h^*(L_0) \subset O$. By Lemma 2.1, h^* is w^* u.s.c. at L_0. Hence there exists a w^* open nbhd $W_1(L_0)$ such that for all $l \in W_1(L_0)$, $h^*(l) \subset O$. But for each $\eta \in h^*(l)$ where $l \in W_1(L_0)$, there exists a $z_\eta \in h^*(L_0)$ such that $|\eta - z_\eta| < \epsilon$ by definition of O. Therefore for $l \in W_1(L_0)$,

$$\hat{d}(h^*(l), v(L_0)) = \sup_{\eta \in h^*(l)} \operatorname{Re}(\eta - v(L_0))$$

$$\leqslant \sup_{\eta \in h^*(l)} \operatorname{Re}\{(z_\eta - v(L_0) - (z_\eta - \eta)\}$$

$$\leqslant \sup_{z \in h^*(L_0)} \operatorname{Re}(z - v(L_0)) + \epsilon$$

$$= g_v(L_0) + \epsilon.$$

Now

$$g_v(l) = \sup_{z \in h^*(l)} \operatorname{Re}(z - v(l)) \leqslant \sup_{z \in h^*(l)} \operatorname{Re}(z - v(L_0)) + |v(l) - v(L_0)|$$

$$= |v(l) - v(L_0)| + \hat{d}(h^*(l), v(L_0)).$$

Take a w^* open nbhd $W_2(L_0)$ such that $| lv - L_0 v | < \epsilon$ for all $l \in W_2(L_0)$. Then for all $l \in W_1(L_0) \cap W_2(L_0)$

$$g_v(l) < g_v(L_0) + 2\epsilon = \beta,$$

which completes the proof.

We remark that by Theorem 1.2, g_v attains its supremum on K.

LEMMA 2.4. *$g_v(L)$ is a convex functional on K in the following sense. Suppose $L = \lambda L_1 + (1 - \lambda) L_2$ where $L, L_1, L_2 \in K$ and $0 \leqslant \lambda \leqslant 1$. Then*

$$g_v(L) \leqslant \lambda g_v(L_1) + (1 - \lambda) g_v(L_2).$$

The proof follows from considering $\sup_{z \in h^*(L)} \operatorname{Re} z$ and applying the corollary to Theorem 1.3.

We now restate our problem as that of finding $\inf_{v \in V} \sup_{L \in K} g_v(L)$ and for convenience introduce the following non-negative functions:

$$\Delta(v) := \sup_{L \in K} g_v(L), \qquad \rho_V(h^*) := \inf_{v \in V} \Delta(v).$$

Furthermore, we set

$$M(v) := [L \in K \mid g_v(L) = \Delta(v)],$$

$$D[h^*, v] := [(L, z) \in K x \mathbb{C} \mid z \in h^*(L), \operatorname{Re}(z - Lv) = \Delta(v)],$$

$$\eta[h^*, v, L] := [z \in h^*(L) \mid \operatorname{Re}(z - Lv) = g_v(L)].$$

Since K and $h^*(L)$ are compact, $M(v)$, $D[h^*, v]$ and $\eta[h^*, v, L]$ are non-empty. We observe

$$\{(L, z) \mid L \in M(v), z \in \eta[h^*, v, L]\} = D[h^*, v].$$

LEMMA 2.5. *$M(v)$ is w^* compact in K.*

Proof. If $L \in C(M(v))$ then $g_v(L) < \Delta(v)$. Since, however, g_v is w^* u.s.c. on K, there exists a w^* open nbhd $U(L)$ such that

$$g_v(l) < \Delta(v) \qquad \text{for all } l \in U(L).$$

Hence $C(M(v))$ is w^* open and therefore $M(v)$ is w^* closed and the result follows.

LEMMA 2.6. $\operatorname{ext}(M(v)) \subset \operatorname{ext}(K)$

Proof. Suppose to the contrary, there exists an $L \in \text{ext}(M(v))$ and $L \notin \text{ext}\, K$. Then there exists $L_1, L_2 \in K$ and λ, $0 < \lambda < 1$ with $L = \lambda L_1 + (1-\lambda) L_2$. Hence $g_v(L) = \Delta(v) \leqslant \lambda g_v(L_1) + (1-\lambda) g_v(L_2)$ by Lemma 2.4. But $g_v(L) \leqslant \Delta(v)$ for all $L \in K$. Therefore $g_v(L_1) = g_v(L_2) = \Delta(v)$, i.e., $L_1, L_2 \in M(v)$, which contradicts $L \in \text{ext}(M(v))$.

We now consider relating two separate approaches to describing the envelope of F.

First we define $F^+(L) := \sup_{z \in h^*(L)} \operatorname{Re} z$. Since $g_v(L) = F^+(L) - \operatorname{Re} v(L)$ we have that $F^+(L)$ is w^* u.s.c. on K. Now define $U_F(L) := \sup_{f \in F} \operatorname{Re} Lf$. Let $\eta(L)$ denote the collection of all w^* open nbhds in K of L.

Let $U_F{}^+(L) := \inf_{W \in \eta(L)} \sup_{l \in W} U_F(l)$.

The characterization of the b.s.a. from a linear subspace has been obtained in [8] in terms of $U_F{}^+(L)$. It is now obtainable from the results in Section 3 by employing the following lemma.

LEMMA 2.7. *$U_F{}^+(L)$ is identical to $F^+(L)$ on K.*

Proof. Suppose to the contrary there exists an $L \in K$ with

$$F^+(L) = a \quad \text{and} \quad a > U_F{}^+(L).$$

Then there exists a $W \in \eta(L)$ with

$$\begin{aligned} a &> \sup_{l \in W} \sup_{f \in F} \operatorname{Re} lf \\ &= \sup_{l \in W} \sup_{z \in h(l)} \operatorname{Re} z \\ &= \sup \left\{ \operatorname{Re} z : z \in \Big(\bigcup_{l \in W} h(l) \Big)^{\circ} \right\}. \end{aligned}$$

On the other hand

$$\begin{aligned} a &\leqslant \sup \left\{ \operatorname{Re} z : z \in \bigcap_{W \in \eta(L)} \Big(\bigcup_{l \in W} h(l) \Big)^{\circ} \right\} \\ &\leqslant \sup \left(\operatorname{Re} z : z \in \Big(\bigcup_{l \in W} h(l) \Big)^{\circ} \right) \end{aligned}$$

leading to a contradiction.

Now suppose there exists an $L \in K$ with $F^+(L) = a < r = U_F{}^+(L)$. Since F^+ is w^* u.s.c. on K, there exists a $W \in \eta(L)$ such that for $a < r' < r$, $F^+(l) < r'$ for all $l \in W$. But $h(l) \subset h^*(l)$ for all $l \in W$. Therefore $\sup_{z \in h(l)} \operatorname{Re} z \leqslant \sup_{z \in h^*(l)} \operatorname{Re} z = F^+(l)$ for all $l \in W$ and $\sup \{\operatorname{Re} z : z \in (\bigcup_{l \in W(L)} h(l))^{\circ} \leqslant r'\}$. However, $\sup_{l \in W} \sup_{f \in F} \operatorname{Re} lf \geqslant r$, leading to a contradiction.

3. Characterisation of the Best Approximation to h^*

We first find circumstances under which $\rho_V(h^*)$ is bounded between two real numbers.

THEOREM 3.1. *Suppose $v_0 \in V$ and Ω a subset of K have the following properties*:

(i) $\operatorname{Re}(z - Lv_0) > 0$ *for all* $L \in \Omega$ *and* $z \in \eta[h^*, v_0, L]$.

(ii) *For no v in V do we have the inequality* $\operatorname{Re} L(v - v_0) > 0$ *satisfied for all* $L \in \Omega$.

Then $0 \leqslant \inf_{L\in\Omega} \hat{d}(h^*(L), v_0(L)) \leqslant \rho_V(h^*) \leqslant \Delta(v_0)$.

Proof. Suppose $0 \leqslant \rho_V(h^*) < \inf_{L\in\Omega} \hat{d}(h^*(L), v_0(L))$. Then there exists a $v \in V$ with $\rho_V(h^*) \leqslant \Delta(v) < \inf_{L\in\Omega} \hat{d}(h^*(L), v_0(L))$. Hence for every $L \in \Omega$, $\hat{d}(h^*(L), v(L)) < \hat{d}(h^*(L), v_0(L))$. Therefore for all $L \in \Omega$ and $z \in \eta[h^*, v_0, L]$

$$\operatorname{Re}(z - v(L)) \leqslant \sup_{z\in h^*(L)} \operatorname{Re}(z - v(L)) < \operatorname{Re}(z - v_0(L)).$$

Hence $0 < \operatorname{Re}[v(L) - v_0(L)]$, contradicting (ii).

We are now in a position to generalise the global Kolmogoroff criterion for a sufficient condition for the best approximation from V.

THEOREM 3.2. *$v_0 \in V$ is a best approximation to h^* if for all $v \in V$*

$$\min_{L\in M(v_0)} \operatorname{Re} L(v - v_0) \leqslant 0.$$

Proof. Take $\Omega = M(v_0)$ in Theorem 3.1. If there exists a $(L, z) \in D[h^*, v_0]$ such that $\operatorname{Re}(z - Lv_0) = \Delta v_0 = 0$, then obviously v_0 is a best approximation.

If for all $(L, z) \in D[h^*, v_0]$, $\operatorname{Re}(z - Lv_0) > 0$, then by Theorem 3.1

$$\Delta(v_0) = \inf_{L\in M(v_0)} \hat{d}(h^*(L), v_0(L)) \leqslant \rho_V(h^*) \leqslant \Delta(v_0)$$

and hence v_0 is a best approximation.

The condition of Theorem 3.2 is not always necessarily satisfied by a best approximation from V. However, if V is regular, we can prove the following.

THEOREM 3.3. *If $V \subset X$ is regular at v_0 then v_0 is a best approximation to h^* if and only if for all $v \in V$*, $\min_{L\in M(v_0)} \operatorname{Re} L(v - v_0) \leqslant 0$.

Proof. The sufficiency of the condition follows from Theorem 3.2. It remains to show the necessity.

Suppose there exists a $v \in V$ with $\min_{L\in M(v_0)} \operatorname{Re} L(v - v_0) = a > 0$.

Set $U := \{L \in K \mid \operatorname{Re} L(v - v_0) > a/2\}$. U is w^* open in K and contains $M(v_0)$. For all $L \in U^\circ$, $\operatorname{Re} L(v - v_0) \geqslant a/2$. By the regularity of V at v_0, for all real $\lambda > 0$, there exists a $v_\lambda \in V$ with $\operatorname{Re} L(v_\lambda - v_0) > 0$ for all $L \in U^\circ$ and $\| v_\lambda - v_0 \| < \lambda$. For $L \in U$ and $z \in h^*(L)$,

$$\begin{aligned}\operatorname{Re}(z - Lv_\lambda) &= \operatorname{Re}(z - Lv_0) + \operatorname{Re}(Lv_0 - Lv_\lambda)\\ &< \operatorname{Re}(z - Lv_0).\end{aligned}$$

Since $h^*(L)$ is compact for each $L \in U$, $\hat{d}(h^*(L), Lv_\lambda) < \hat{d}(h^*(L), Lv_0)$. On the other hand, $K \backslash U$ is weak* compact and is disjoint from $M(v_0)$. Therefore $\sup_{L \in K \backslash U} \hat{d}(h^*(L), v_0(L)) = E^* < \Delta(v_0)$. If we set $\lambda := \Delta(v_0) - E^*$ then for $z \in h^*(L)$ we have

$$\operatorname{Re}(z - Lv_\lambda) = \operatorname{Re}(z - Lv_0) + \operatorname{Re}(Lv_0 - Lv_\lambda) < \Delta(v_0).$$

Hence $\hat{d}(h^*(L), v_\lambda(L)) < \Delta(v_0)$ and $\Delta(v_\lambda) = \sup_{L \in K} \hat{d}(h^*(L), v_\lambda(L)) < \Delta(v_0)$.

We now formulate a uniqueness result for the best approximation, analogous to Theorem 3.13 in [2].

THEOREM 3.4. *If $V \subset X$ is regular and v_0 is a best approximation to h^* from V, then the best approximation is unique, in the case that* $\operatorname{Re} L(v - v_0) = 0$ *on a subset of $M(v_0)$ which is sign-extremal for v_0 implies $v = v_0$ on K.*

Proof. Suppose v_1 is another best approximation to h^*. For any $(L, z) \in D[h^*, v_0]$

$$\begin{aligned}\operatorname{Re}(z - Lv_1) &= \operatorname{Re}(z - Lv_0) + \operatorname{Re} L(v_0 - v_1)\\ &\leqslant \hat{d}(h^*(L), v_1(L))\\ &\leqslant \hat{d}(h^*(L), v_0(L))\\ &= \operatorname{Re}(z - Lv_0).\end{aligned}$$

Therefore $\operatorname{Re} L(v_1 - v_0) \geqslant 0$ for all $L \in M(v_0)$. But by Theorem 3.3 $\min_{L \in M(v_0)} \operatorname{Re} L(v_1 - v_0) \leqslant 0$. Hence

$$\Sigma' := \{L \in M(v_0) \mid \operatorname{Re} L(v_1 - v_0) = 0\} \neq \varnothing.$$

Assume $\Sigma' \neq M(v_0)$, otherwise the result follows trivially. It follows by Lemmas 8 and 9 in [3] that Σ' is sign-extremal and by the condition of our theorem $v_1 = v_0$ on K.

4. APPROXIMATING FUNCTIONS WITH A FRÉCHET DERIVATIVE

Let D be an open subset of a Banach space E with norm $\| \cdot \|_E$. Let V be the set of elements $v(\mathbf{a}) \in X$ which depend on the parameter $\mathbf{a} \in D$, i.e.,

$V: D \to X$ and $V = \{v(\mathbf{a}) \in X, \mathbf{a} \in D\}$. We shall henceforth assume that $v(\mathbf{a})$ has a Fréchet derivative with respect to $\mathbf{a}$ for each $\mathbf{a} \in D$, i.e., for any $\mathbf{b} \in E$ there exists a linear bounded mapping $v'_{\mathbf{a}} : E \to X$ which we denote by $v'[\mathbf{b}, \mathbf{a}]$ with

$$\| v(\mathbf{a} + \mathbf{b}) - v(\mathbf{a}) - v'[\mathbf{b}, \mathbf{a}]\| = o(\| \mathbf{b} \|_E) \qquad \text{as} \quad \| \mathbf{b} \|_E \to 0.$$

Let $\mathscr{L}[\mathbf{a}]$ denote the linear subspace of X consisting of all elements $v'[\mathbf{b}, \mathbf{a}]$, $\mathbf{b} \in E$. Let N be the dimension of $\mathscr{L}[\mathbf{a}]$ and $\Phi_1, \ldots, \Phi_N$ be a basis for $\mathscr{L}[\mathbf{a}]$ with $\Phi_i \equiv \Phi_i(a)$.

If $v(\mathbf{a})$ has a Fréchet derivative at $\mathbf{a}$, then

$$\| v(\mathbf{a} + t\mathbf{b}) - v(\mathbf{a})\| = O(t) \qquad \text{for any } \mathbf{b} \in E.$$

For

$$\begin{aligned} \| v(\mathbf{a} + t\mathbf{b}) - v(\mathbf{a})\| &\leqslant \| v'[t\mathbf{b}, \mathbf{a}]\| + \| v(\mathbf{a} + t\mathbf{b}) - v(\mathbf{a}) - v'[t\mathbf{b}, \mathbf{a}]\| \\ &= | t | \, \| v'[\mathbf{b}, \mathbf{a}]\| + o(t). \end{aligned}$$

Hence there exists a $t_0 > 0$ such that for all t, $0 < t < t_0$, $v(\mathbf{a} + t\mathbf{b})$ lies in the ϵ-locality of $v(\mathbf{a})$ defined by the norm sphere $S(v(\mathbf{a}), \epsilon)$ for some $\epsilon \equiv \epsilon(t_0, \mathbf{b})$ with $\epsilon > 0$. We define $v(\mathbf{a})$ to be a local best approximation to h^* when $\Delta(v(\mathbf{a})) \leqslant \Delta(v(\mathbf{c}))$ for all $v(\mathbf{c}) \in V$ and in an ϵ-locality of $v(\mathbf{a})$ for some $\epsilon > 0$.

THEOREM 4.1. $v(\mathbf{a})$ is a (local) best approximation to h^* implies that for all $\mathbf{b} \in E$

$$\min_{L \in M(v(\mathbf{a}))} \operatorname{Re} Lv'[\mathbf{b}, \mathbf{a}] \leqslant 0.$$

Proof. Suppose to the contrary, there exists a $\mathbf{b} \in E$ with

$$\min_{L \in M(v(\mathbf{a}))} \operatorname{Re} Lv'[\mathbf{b}, \mathbf{a}] > 0.$$

We show that there exists a better approximation to h^* than $v(\mathbf{a})$. Let U be the set of $L \in K$ for which

$$\operatorname{Re} Lv'[\mathbf{b}, \mathbf{a}] \geqslant 2\sigma > 0.$$

Since D is an open set in E, there exists a $t_0 > 0$ such that for all $t \in (0, t_0)$, $\mathbf{a} + t\mathbf{b} \in D$ and $v(\mathbf{a} + t\mathbf{b})$ lies in an ϵ-locality of $v(\mathbf{a})$. For $L \in U$

$$\begin{aligned} &\operatorname{Re} L[v(\mathbf{a} + t\mathbf{b}) - v(\mathbf{a})] \\ &\quad = \operatorname{Re} L[v'[t\mathbf{b}, \mathbf{a}]] + \operatorname{Re} L[v(\mathbf{a} + t\mathbf{b}) - v(\mathbf{a}) - v'[t\mathbf{b}, \mathbf{a}]] \\ &\quad \geqslant 2\sigma t - o(t). \end{aligned}$$

Hence there exists t_1 with $0 < t_1 \leqslant t_0$ such that for all t, $0 \leqslant t \leqslant t_1$ and $L \in U$

$$\operatorname{Re} L[v(\mathbf{a} + t\mathbf{b}) - v(\mathbf{a})] \geqslant \sigma t > 0$$

and therefore

$$\begin{aligned}\operatorname{Re}[z - Lv(\mathbf{a} + t\mathbf{b})] &= \operatorname{Re}[z - Lv(\mathbf{a})] + \operatorname{Re}[L(v(\mathbf{a}) - v(\mathbf{a} + t\mathbf{b}))] \\ &< \operatorname{Re}[z - Lv(\mathbf{a})].\end{aligned}$$

Therefore $\hat{d}(h^*(L), v(\mathbf{a} + t\mathbf{b})(L)) < \Delta(v(\mathbf{a}))$ for all $L \in U$. As shown,

$$\| v(\mathbf{a} + t\mathbf{b}) - v(\mathbf{a})\| \leqslant t \| v'[\mathbf{b}, \mathbf{a}]\| + o(t).$$

Hence there exists a t_2, $0 < t_2 \leqslant t_1$ such that for all t in $0 \leqslant t \leqslant t_2$

$$\| v(\mathbf{a} + t\mathbf{b}) - v(\mathbf{a})\| \leqslant 2t \| v'[\mathbf{b}, \mathbf{a}]\|.$$

We now consider the set $W = K \backslash U$. This is weak* compact, and does not contain any member of $M(v(\mathbf{a}))$. Therefore $\sup_{L \in W} \hat{d}(h^*(L), v(\mathbf{a})(L)) = E^* < \Delta(v(\mathbf{a}))$. Let τ be such that

$$0 < \tau < \min\left(t_2, \frac{\Delta(v(\mathbf{a})) - E^*}{2 \| v'[\mathbf{b}, \mathbf{a}]\|}\right).$$

For $L \in W$, $z \in h^*(L)$

$$\begin{aligned}\operatorname{Re}[z - Lv(\mathbf{a} + \tau\mathbf{b})] &\leqslant \operatorname{Re}[z - Lv(\mathbf{a})] + \operatorname{Re}[L(v(\mathbf{a}) - v(\mathbf{a} + \tau\mathbf{b}))] \\ &\leqslant \sup_{z \in h^*(L)} \operatorname{Re}[z - Lv(\mathbf{a})] + \| v(\mathbf{a}) - v(\mathbf{a} + \tau\mathbf{b})\| \\ &< E^* + 2\tau \| v'[\mathbf{b}, \mathbf{a}]\|.\end{aligned}$$

Therefore $\hat{d}(h^*(L), v(\mathbf{a} + \tau\mathbf{b})(L)) < \Delta(v(\mathbf{a}))$ for all $L \in W$. Hence $\Delta(v(\mathbf{a} + \tau\mathbf{b})) < \Delta(v(\mathbf{a}))$.

We remark that in this theorem, we can replace $M(v(\mathbf{a}))$ by its extremal points, denoted by $E_0(M)$, by applying Lemma 1.5.

COROLLARY 4.1. *If $v(\mathbf{a})$ is a (local) best approximation to h^* from V, then*

$$\mathbf{0} \in \operatorname{co}[(L\Phi_1, \ldots, L\Phi_N)^T \text{ over all } L \in M(v(\mathbf{a}))].$$

Proof. Suppose to the contrary that $\mathbf{0}$ does not belong to the convex hull. Since $[(L\Phi_1, \ldots, L\Phi_N)^T$ over all $L \in M(v(\mathbf{a}))]$ is a compact set in Euclidean N-space, there exists an N-dimensional vector $\mathbf{c} \in E$ so that

$$\operatorname{Re}\left(\sum_{i=1}^{N} c_i L\Phi_i\right) > 0 \qquad \text{for all } L \in M(v(\mathbf{a})).$$

But $\sum_{i=1}^{N} c_i\Phi_i \in \mathscr{L}[\mathbf{a}]$ and $\operatorname{Re} L(\sum_{i=1}^{N} c_i\Phi_i) > 0$ for all $L \in M(v(\mathbf{a}))$ would imply that $v(\mathbf{a})$ could not have been a (local) best approximation by the previous theorem. Hence the result. Here also we may replace $M(v(\mathbf{a}))$ by $E_0(M)$ as a consequence of Lemma 1.3.

We now wish to obtain a sufficient condition for $v(\mathbf{a})$ to be a local best approxmation. For any $\mathbf{b} \in E$, let $\mathbf{a} + t\mathbf{b}$ be represented by $\mathbf{a}(t)$ with $\mathbf{a}(0) = \mathbf{a}$. Suppose $v(\mathbf{a}(t))$ satisfies an additional condition (T), namely, that $(v(\mathbf{a}(t)) - v(\mathbf{a}))/t$ is in the linear span of $\{\Phi_i(\mathbf{a}(t))\}_{i=1}^{N}$, where

$$\|\Phi_i(\mathbf{a}(t)) - \Phi_i(\mathbf{a})\| = 0(t) \qquad \text{as} \quad t \to 0 \text{ for } i = 1,..., N.$$

THEOREM 4.2. *If $v(\mathbf{a}(t))$ satisfies (T) then a sufficient condition for $v(\mathbf{a})$ to be a local best approximation to h^* from V is that*

$$\mathbf{0} \in \textit{interior } \operatorname{co}[(L\Phi_1(\mathbf{a}),..., L\Phi_N(\mathbf{a}))^T \textit{ over all } L \in E_0(M)].$$

Proof. By the assumed condition and Lemma 1.4, for any $\mathbf{b} \in E$, there exists an $\epsilon_0 > 0$ with

$$\mathbf{0} \in \operatorname{co}[(L\Phi_1(\mathbf{a}(t)),..., L\Phi_N(\mathbf{a}(t)))^T \text{ over all } L \in E_0(M)]$$
$$\text{for all } t,\ 0 \leqslant t \leqslant \epsilon_0 .$$

Suppose to the contrary $v(\mathbf{a})$ is not a local best approximation to h^*. Then for all $\epsilon > 0$, there exists a t, $0 < t \leqslant \epsilon$ and $\mathbf{b} \in E$ such that $\mathbf{a}(t) \in D$ and $\rho_V(h^*) \leqslant \Delta(v(\mathbf{a}(t))) < \Delta(v(\mathbf{a}))$, i.e., for all $L \in K$

$$\hat{d}(h^*(L), v(\mathbf{a}(t))(L)) < \sup_{L\in K} \hat{d}(h^*(L), v(\mathbf{a})(L)).$$

Hence for all $L \in E_0(M)$ and $z \in h^*(L)$

$$\operatorname{Re}(z - v(\mathbf{a}(t))(L)) < \operatorname{Re}(z - v(\mathbf{a})(L)),$$

i.e., $\operatorname{Re}[L(v(\mathbf{a}(t)) - v(\mathbf{a}))] < 0$ for all $L \in E_0(M)$. Dividing through by t, we find

$$\mathbf{0} \notin \operatorname{co}[(L\Phi_1(\mathbf{a}(t)),..., L\Phi_N(\mathbf{a}(t)))^T \text{ over all } L \in E_0(M)].$$

Hence a contradiction follows by taking $\epsilon = \epsilon_0$.

5. APPROXIMATION OF REAL-VALUED FUNCTIONS BY GENERALISED RATIONALS IN INTERPOLATING SUBSPACES OF L_1

We may relate the results of Section 4 to the following setting. Suppose we are working in the space $S(B) := L_1(B, \Sigma, \mu)$ with the L_1 norm, abbreviated

$L_1(\mu)$, where B with an appropriate topology is a compact Hausdorff space, and μ is a σ-finite measure (see, e.g., [6, Chapter III]). If we further assume that B is the union of at most countably many atoms, say $B = \bigcup_{i \in I} A_i$, then it has been shown that $\text{ext}(B^*)$ is weak* closed and that each $L \in \text{ext}(B^*)$ has the representation

$$L(f) = \sum_{i \in I} f(A_i)\, \sigma(A_i)\, \mu(A_i) \qquad \text{for} \quad f \in L_1(\mu),$$

where $|\sigma(A_i)| = 1$ and $f(A_i)$ denotes the constant value of f a.e., on A_i. (See [1, p. 170, 175]; [12, Section 2]). The relevance of these points is immediate if we take K in Section 4, to be B^* or $\text{ext}(B^*)$ and recall Lemma 2.6 that $E_0(M) \subset \text{ext}(K)$, i.e., the above representation is valid for $E_0(M)$. Furthermore, the presence of atoms enables us to use the concepts of interpolating subspaces (see Definition 1.8). We remark that in computational work with the L_1 norm, we are obliged to discretise and hence our setting is a practical one.

Suppose we are given a set of real-valued functions $F \subset L_1(\mu)$ and we wish to characterise local best approximations from $V = R^+_{n,m}$. To recall, let $\{g_1, ..., g_n\}$ and $[h_1, ..., h_m]$ be fixed sets of linearly independent real-valued continuous functions on B.

Let

$$P := \text{span}\{g_1, ..., g_n\},$$
$$Q : \text{span}\{h_1, ..., h_m\},$$

and

$$Q^+ := \{q \in Q, q(x) > 0 \text{ on } B\}.$$

We define $R^+_{n,m} := \{p/q : p \in P, q \in Q^+\}$ and assume it is non-empty. Let $D := \{(\alpha_1, ..., \alpha_n; \beta_1, ..., \beta_m) \in E^{n+m}; \sum_{i=1}^m \beta_i h_i(x) > 0 \text{ on } B\}$.

For $(a_1, ..., a_n; b_1, ..., b_m) \in D$, $(c_1, ..., c_n; d_1, ..., d_m) \in E^{n+m}$ and real λ set

$$r_\lambda(x) := \frac{\sum_{i=1}^n (a_i + \lambda c_i)\, g_i(x)}{\sum_{i=1}^m (b_i - \lambda d_i)\, h_i(x)}$$

with the normalisation $\sum_{i=1}^m |b_i - \lambda d_i| = 1$. In particular

$$r_0(x) = \frac{\sum_{i=1}^n a_i g_i(x)}{\sum_{i=1}^m b_i h_i(x)} \in R^+_{n,m}.$$

For any $\mathbf{d} = (d_1, ..., d_m)$ we can always find a $\hat{\lambda} \equiv \hat{\lambda}(\mathbf{d}) > 0$ and a $\lambda^* \equiv \lambda^*(\mathbf{c}, \mathbf{d})$, $0 < \lambda^* \leqslant \hat{\lambda}$ such that

$$\left| \hat{\lambda} \sum_{i=1}^m d_i h_i(x) \right| < \sum_{i=1}^m b_i h_i(x) \qquad \text{on } B$$

and r_λ belongs to an ϵ-locality of r_0 in $R^+_{n,m}$ for $|\lambda| \leqslant \lambda^*$. We shall use the following abbreviations:

$$q_m(\lambda, \mathbf{d}, x) := \sum_{i=1}^{m} (b_i - \lambda d_i)\, h_i(x) \qquad \text{and} \qquad q_m(x) := \sum_{i=1}^{m} b_i h_i(x).$$

We have a simplification of our problem to that of approximating a single-valued w^* u.s.c. function $F^+: K \to R$ defined by $F^+(L) := \max_{z \in h^*(L)} z$ (see remarks after 2.6) with $\Delta(r_0) = \sup_{L \in K} g_{r_0}(L) = \sup_{L \in K} [F^+(L) - r_0(L)]$. We note that if $\phi_1, ..., \phi_N$ is a basis for

$$\mathscr{L}[r_0] := \frac{P}{q_m} + r_0 \frac{Q}{q_m},$$

and $0 < |\lambda| \leqslant \lambda^*$, then

$$\phi_i(\lambda, \mathbf{d}) := \frac{q_m}{q_m(\lambda, \mathbf{d})} \phi_i, \qquad i = 1, ..., N$$

is a basis for $P/q_m(\lambda, \mathbf{d}) + r_0(Q/q_m(\lambda, \mathbf{d}))$ and furthermore, condition (T) is satisfied.

THEOREM 5.1. *Let $p_n \in P$, $q_m \in Q^+$ and $r_0 = p_n/q_m$. If* (a) *r_0 is a locally best L_1 approximation to F^+ and* (b) *$\mathscr{L}[r_0] = P/q_m + r_0(Q/q_m)$ is an N-dimensional interpolating subspace of $L_1(\mu)$, with basis $\phi_1, ..., \phi_N$, then*

(i) *There exist $N + 1$ independent functionals $L_1, ..., L_{N+1}$ in* $\operatorname{ext}(M(r_0))$, *abbreviated $E_0(M)$ such that*

$$\mathbf{0} \in \textit{interior}\ \operatorname{co}[(L_i\phi_1, ..., L_i\phi_N)^T\ i = 1, ..., N + 1];$$

(ii) *0 is the only element ϕ of $P/q_m + r_0(Q/q_m)$ having the property $L_i\phi \geqslant 0$ for $i = 1, ..., N + 1$, L_i as in* (i);

(iii) *There exists a $\sigma \equiv \sigma(\mathbf{d})$,*

$$0 < \sigma \leqslant \lambda^* \textit{ such that for all } \lambda, |\lambda| \leqslant \sigma$$

$P/q_m(\lambda, \mathbf{d}) + r_0(Q/q_m(\lambda, \mathbf{d}))$ is an interpolating subspace on $\{L_i\}_{i=1}^{N+1}$ for which the result of (ii) *is valid by similar argument*;

(iv) *r_0 is a unique locally best approximation in the ϵ-locality of r_0 restricted to $|\lambda| \leqslant \sigma$ and denoted by $U(r_0, \sigma)$.*

The proofs are virtually identical to those given in [7, Theorem 4.2(i)–(iv).].

We now strengthen (iv) of Theorem 5.1 and show that under suitable conditions there is local strong unicity in the sense of Newman and Shapiro [11]. The proof corrects that given in [7, for Theorem 4.2(v)]. We will need the following lemma adapted from [4, p. 162].

LEMMA 5.1. *If* $r_0 := p_n/q_m \in R^+_{n,m}$ *such that*

$$\text{(c)}\quad \dim\left[\frac{P}{q_m} + r_0\frac{Q}{q_m}\right] = \dim\left[\frac{P}{q_m}\right] + \dim\left[\frac{Q}{q_m}\right] - 1$$

and if $p \in P$, $q \in Q$ *satisfy*

(i) $\| q \| = \| q_m \|$,
(ii) $p = r_0 q$,
(iii) $q(x) \geqslant 0$ *on* B,

then $p = p_n$, $q = q_m$.

THEOREM 5.2. *Under conditions* (a) *and* (b) *of Theorem* 5.1 *and* (c) *of Lemma* 5.1 *there exists a constant* $\gamma > 0$ *such that for all* $r_\lambda(x) \in U(r_0, \sigma)$

$$\Delta(r_\lambda) \geqslant \Delta(r_0) + \gamma \| r_\lambda - r_0 \|.$$

Proof. For $0 < |\lambda| \leqslant \sigma$, define for the set $U(r_0, \sigma)$

$$\gamma(r_\lambda) := \frac{\Delta(r_\lambda) - \Delta(r_0)}{\| r_\lambda - r_0 \|}$$

and suppose to the contrary, there exists a sequence $\{r_{\lambda_k}\} \in U(r_0, \sigma)$ $r_{\lambda_k} \neq r_0$ and $\gamma(r_{\lambda_k}) \to 0$. We may suppose $\gamma(r_{\lambda_k}) < \frac{1}{2}$ for $k \geqslant n_0$. Then we can show $0 < \| r_{\lambda_k} - r_0 \| < \infty$, $k \geqslant n_0$. For take any $f \in F$,

$$\begin{aligned}
\| r_{\lambda_k} - r_0 \| &\leqslant \| r_{\lambda_k} - f \| + \| r_0 - f \| \\
&\leqslant \sup_{f\in F} \| r_{\lambda_k} - f \| + \sup_{f\in F} \| r_0 - f \| \\
&\leqslant \Delta(r_{\lambda_k}) + \Delta(r_0) \\
&\leqslant 2\Delta(r_0) + \tfrac{1}{2} \| r_{\lambda_k} - r_0 \| \qquad \text{for} \quad k \geqslant n_0 \text{ by our supposition.}
\end{aligned}$$

Therefore $\| r_{\lambda_k} - r_0 \| \leqslant 4\Delta(r_0) \leqslant 4\alpha$, $k \geqslant n_0$. Now we show there exists a subsequence of r_{λ_k} relabelled the same, such that

$$\lim_{k\to\infty} r_{\lambda_k} = r_0 .$$

Since $0 < |\lambda_k| \leqslant \sigma$, either $\lim_{k\to\infty} \lambda_k = 0$ for every subsequence, in which case $\lim_{k\to\infty} r_{\lambda_k} = r_0$, or there exists a subsequence relabelled the same with $\lim_{k\to\infty} \lambda_k = \lambda_0$, where $0 < \lambda_0 \leqslant \sigma$. Assume the latter to be the case.

$$q_m(\lambda_k, \mathbf{d}_k) = \sum_{i=1}^{m} \beta_i^{(k)} h_i(x), \qquad \text{where} \quad \sum_{i=1}^{m} |\beta_i^{(k)}| = 1.$$

Hence for each i, $1 \leqslant i \leqslant m$, and for all k we have $|\beta_i^{(k)}| < 1$ and therefore $b_i - 1 \leqslant \lambda_k d_i^{(k)} \leqslant b_i + 1$. It follows that for each i, $\{d_i^{(k)}\}$ is a bounded sequence and we can extract a convergent subsequence such that $\lim_{k\to\infty} d_i^{(k)} = d_i^{(0)}$ and hence $\lim q_m(\lambda_k, \mathbf{d}_k) = q_m(\lambda_0, \mathbf{d}_0)$. By definition

$$\gamma(r_{\lambda_k}) \| r_{\lambda_k} - r_0 \| = \Delta(r_{\lambda_k}) - \Delta(r_0) \geqslant \max_{j=1,\ldots,N+1} L_j(r_0 - r_{\lambda_k}).$$

As $k \to \infty$, the left-hand side converges to zero. Now we apply the validity of (ii) to (iii) of Theorem 5.1 for

$$(\lim r_{\lambda_k} - r_0) \in \frac{P}{q_m(\lambda_0, \mathbf{d}_0)} + r_0 \frac{Q}{q_m(\lambda_0, \mathbf{d}_0)}$$

to obtain $\lim r_{\lambda_k} = r_0$.

By Lemma 5.1,

$$q_m(\lambda_0, \mathbf{d}_0) = q_m.$$

Consequently, as $k \to \infty$, we can say heuristically,

$$\frac{P}{q_m(\lambda_k, \mathbf{d}_k)} + r_0 \frac{Q}{q_m(\lambda_k, \mathbf{d}_k)} \to \frac{P}{q_m} + r_0 \frac{Q}{q_m}.$$

Next, we reason as follows. For $L_j \in E_0(M)$ and

$$\phi \in \frac{P}{q_m(\lambda_k, \mathbf{d}_k)} + r_0 \frac{Q}{q_m(\lambda_k, \mathbf{d}_k)}$$

we have by virtue of results (iii) and (ii) of Theorem 5.1 that for all k, including the limiting case,

$$c_k = \min_{\|\phi\|=1} \max_{j=1,\ldots,N+1} L_j\phi > 0.$$

But,

$$\gamma(r_{\lambda_k}) \| r_{\lambda_k} - r_0 \| \geqslant \max_{j=1,\ldots,N+1} L_j(r_0 - r_{\lambda_k})$$

and

$$\frac{r_0 - r_{\lambda_k}}{\| r_0 - r_{\lambda_k} \|} \in \frac{P}{q_m(\lambda_k, \mathbf{d}_k)} + r_0 \frac{Q}{q_m(\lambda_k, \mathbf{d}_k)}$$

and is of norm one. Therefore $\gamma(r_{\lambda_k}) \geqslant c_k > 0$.

Finally, if we let $c_0 = \min_{\|\phi\|=1} \max_{j=1,\ldots,N+1} L_j\phi$, $\phi \in \mathscr{L}[r_0]$ with $c_0 > 0$ as already deduced, we can show that for all ϵ, $0 < \epsilon < c_0$, we have that $c_k > c_0 - \epsilon$ for k sufficiently large. To prove this last conjecture, assume k

to be large enough that $q_m(\lambda_k, \mathbf{d}_k) \simeq q_m$ and hence $\phi_i(\lambda_k, \mathbf{d}_k) \simeq \phi_i$. Suppose now to the contrary there exists a convergent sequence (in ν)

$$\phi_\nu^{(k)} \in \frac{P}{q_m(\lambda_k, \mathbf{d}_k)} + r_0 \frac{Q}{q_m(\lambda_k, \mathbf{d}_k)} \qquad \text{with} \quad \| \phi_\nu^{(k)} \| = 1$$

and

$$\lim_{\nu \to \infty} \max_{j=1,\dots,N+1} L_j \phi_\nu^{(k)} = c_k \leqslant c_0 - \epsilon.$$

That is, there exists an $\tilde{N}(k)$ such that for $\nu \geqslant \tilde{N}(k)$

$$\max_{j=1,\dots,N+1} L_j \phi_\nu^{(k)} \leqslant c_0 - \tfrac{3}{4}\epsilon.$$

Assume $\nu \geqslant \tilde{N}(k)$. If we represent $\phi_\nu^{(k)}$ as $\sum_{i=1}^N a_i^{(\nu)} \phi_i(\lambda_k, \mathbf{d}_k)$ then $\{a_i^{(\nu)}\}_{i=1}^N$ are bounded by our assumption on $\phi_\nu^{(k)}$ and $\hat{\psi}_\nu := \sum_{i=1}^N a_i^{(\nu)} \phi_i$ satisfies $\| \hat{\psi}_\nu - \phi_\nu^{(k)} \| < \epsilon/4$ by our assumption on k. Hence $1 - \epsilon/4 < \| \hat{\psi}_\nu \| < 1 + \epsilon/4$. Now $\psi_\nu := \hat{\psi}_\nu / \| \hat{\psi}_\nu \|$ is of norm one, belongs to $\mathscr{L}[r_0]$ and

$$\begin{aligned} \| \psi_\nu - \phi_\nu^{(k)} \| &\leqslant \| \psi_\nu - \hat{\psi}_\nu \| + \| \hat{\psi}_\nu - \phi_\nu^{(k)} \| \\ &< (1 - \| \hat{\psi}_\nu \|) + \epsilon/4 \\ &< \epsilon/2. \end{aligned}$$

Consequently $-\epsilon/2 < \max L_j \psi_\nu - \max L_j \phi_\nu^{(k)} < \epsilon/2$, $j = 1,\dots, N+1$, and $\max_{j=1,\dots,N+1} L_j \psi_\nu < c_0 - \epsilon/4$, which is clearly impossible. Thus we have shown that for k sufficiently large $\gamma(r_{\lambda_k})$ is bounded away from zero and we have been led to a contradiction.

We can also reformulate Theorem 5.1 in terms of the more familiar "alternation" theorem. The result is identical to that given in [7, Theorem 4.4] with f replaced by F^+ and $E_0(S)$ by $E_0(M)$. For further applications, see [1, Theorem 4.3 et seq.].

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