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Unique Continuation for Solutions to the Induced Cauchy–Riemann Equations

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Let M be a real infinitely differentiable closed hypersurface in X, a complex manifold of dimension $n \geqslant 2$, and let $\bar{\partial}_M$ denote the induced Cauchy–Riemann operator on M. The problem considered in this paper is unique continuation for distribution solutions to the equation $\bar{\partial}_M u = 0$ (these solutions are called CR distributions). In a local version of the problem it is shown that a CR distribution u in an open set $U \subset M$ which vanishes on one side of a C^1 hypersurface $S \subset U$ which is noncharacteristic at a point $p \in S$ necessarily vanishes in a neighborhood of p. If the CR distribution u is a continuous function on U, then it is only necessary to assume that u vanishes on S in order to prove that u vanishes in a neighborhood of p in M. It is also proved that if u is a CR distribution on M, then the boundary of the support of u is foliated by complex hypersurfaces. Thus a global unique continuation theorem is obtained by assuming that such a set is not contained in M.

1. INTRODUCTION

Suppose M is a real infinitely differentiable closed hypersurface in X, a complex manifold of (complex) dimension $n \geqslant 2$. Then the Cauchy–Riemann operator $\bar{\partial}$ on X induces a differential operator on M which is called the induced Cauchy–Riemann operator, and which is denoted by $\bar{\partial}_M$. The problem con-

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sidered here is unique continuation for distribution solutions to the equation $\bar{\partial}_M u = 0$.

The problem of unique continuation exists in two versions. For the local version we show in Theorem 3.1 that if u is a CR distribution (a distribution solution of $\bar{\partial}_M u = 0$) in an open set $U \subset M$ which vanishes on one side of a C^2 hypersurface $S \subset U$ which is noncharacteristic at a point $p \in S$, then u necessarily vanishes in a neighborhood of p (we later show that we need only assume that S is a C^1 surface). The main tool here is the result from [9], which shows that locally every CR distribution on M is the generalized boundary values of a holomorphic function in the complement of M.

The global version of the problem of unique continuation involves putting global conditions on M which will ensure that any CR distribution on M which vanishes in an open set vanishes identically. In Theorem 4.2 we show that if u is a CR distribution on M, then the boundary of the support of u is foliated by complex hypersurfaces of M. This enables us to state a global unique continuation theorem by simply assuming that such a set is not contained in M. The foliation result is a special case of a more general result (Theorem 4.9) which may have independent interest.

For CR functions (CR distributions which are continuous functions) the unique continuation results can be improved. In this case it is only necessary to assume that the CR function u vanishes on a C^1 hypersurface $S \subset M$ in a neighborhood of a point $p \in S$ at which S is noncharacteristic. These results are easy corollaries of the general results and a removable singularity theorem from [4].

Several papers have recently been written concerning unique continuation of solutions to homogeneous linear partial differential equations of first order. The paper of Strauss and Treves [10] studies this problem for operators which satisfy the solvability conditions of Nirenberg and Treves [8]. In [5] the problem is examined for the induced Cauchy–Riemann equations for manifolds of arbitrary codimensions but with a restriction on the Levi algebra of the manifold.

2. Preliminaries

Let X be a complex manifold of complex dimension n and let $M \subset X$ be an infinitely differentiable closed submanifold of real dimension $2n - 1$. The induced Cauchy–Riemann complex on M is described in several places (e.g., [3, 6, 9]), so we will give only a brief description. Let $T(M)$ denote the tangent bundle of M, and let $T_p(M)$ denote the tangent space at the point p. The *holomorphic tangent space* at p is defined by $H_p(M) = T_p(M) \cap JT_p(M)$ where J is the almost complex structure on X induced by the complex structure. The bundle with fiber $H_p(M)$ at each point is denoted by $H(M)$ and is called the *holomorphic tangent bundle* of M.

Assuming the existence of a Hermitian metric on X, and the induced metric on M, we can choose a complement of $H(M)$ in $T(M)$, which we will denote by $Y(M)$. Corresponding to this decomposition we get a direct sum decomposition of the cotangent bundle, $T^*(M) = H^*(M) \oplus Y^*(M)$. If we let $T_c{}^*(M) = T^*(M) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(T(M), \mathbb{C})$ denote the complexified cotangent bundle, then we have the direct sum decomposition

$$T_c{}^*(M) = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M) \oplus Y_c{}^*(M),$$

where $Y_*{}^*(M)$ is the complexification of $Y^*(M)$, and $\Lambda_p^{1,0}(M)$ and $\Lambda_p^{0,1}(M)$ denote, respectively, the complex linear and the conjugate complex linear maps from $H_p(M)$ to $\mathbb{C}$. Then for a function ϕ defined on M we define $\bar\partial_M \phi = \pi d\phi$, where π: $T_c{}^*(M) \to \Lambda^{0,1}(M)$ is the orthogonal projection.

We can now identify the characteristics of $\bar\partial_M$. If we let σ denote the symbol of $\bar\partial_M$, then for $\xi \in T_p{}^*(M)$, $\sigma(\xi) = \pi\xi \in \Lambda_p^{0,1}(M)$. Clearly $\sigma(\xi) = 0$ if and only if $\xi \in Y_p{}^*(M)$, or equivalently, if and only if ξ annihilates the holomorphic tangent space $H_p(M)$. In particular if $p \in S$, where $S \subset M$ is a hypersurface defined by the equation $\phi(x) = 0$ with $d\phi(p) \neq 0$, then S is characteristic at p if and only if $d\phi(p) \in Y_p{}^*(M)$, or equivalently if and only if $T_p(S) = H_p(M)$. If S is a C^1 characteristic surface (i.e., S is characteristic at each of its points), then $T(S) = H(M)|_S$. Thus $T_p(S)$ is a complex subspace of $T_p(X)$ for each $p \in S$. We will refer to a real hypersurface of M which is also a complex manifold as a *complex hypersurface* of M.

We will let $\mathscr{D}'$ denote the sheaf of germs of distributions on X. More generally we let $\mathscr{D}'^{p,q}$ denote the sheaf of germs of (p, q) forms on X with distribution coefficients (i.e., $\mathscr{D}' = \mathscr{D}'^{0,0}$). In addition we let $\mathscr{D}'^{p,q}_M$ denote the subsheaf of $\mathscr{D}'^{p,q}$ consisting of these forms which have their supports contained in M. Finally let $'\mathscr{D}'$ denote the sheaf of germs of distributions on M. There is a natural injection μ_*: $'\mathscr{D}' \to \mathscr{D}'^{0,1}_M$ defined by $(\mu_* h, \phi) = (h, j^*\phi)$ for ϕ a compactly supported smooth form of type $(n, n-1)$, where j: $M \to X$ is the inclusion and j^* is the induced map on forms.

2.1. PROPOSITION [9]. *$\bar\partial_M h = 0$ if and only if $\bar\partial \mu_* h = 0$.*

Proposition 2.1 presents a useful extrinsic way of characterizing solutions to the induced Cauchy–Riemann equations.

Let $\bar{\mathscr{D}}'^{p,q}$ denote the quotient sheaf $\mathscr{D}'^{p,q}/\mathscr{D}'^{p,q}_M$. Since $\bar\partial$: $\mathscr{D}'^{p,q}_M \to \mathscr{D}'^{p,q+1}_M$, we get an induced map $\bar\partial$: $\bar{\mathscr{D}}'^{p,q} \to \bar{\mathscr{D}}'^{p,q+1}$. For $p = q = 0$, let $\bar{\mathcal{O}}$ denote the solution sheaf. Clearly for an open set $U \subset X$, $\bar{\mathcal{O}}(U) = \mathcal{O}(U - M) \cap \mathscr{D}'(U)/\mathscr{D}_M'(U)$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X. Thus $\bar{\mathcal{O}}(U)$ is the space of holomorphic functions on $U - M$ which have distribution extensions to U.

2.2. Lemma. *Given $\phi \in \mathcal{O}(U - M) \cap \mathcal{D}'(U)$ there exist unique distributions $w \in \mathcal{D}_M'(U)$ and $h \in {}'\mathcal{D}'(U \cap M)$ such that $\bar\partial\phi = \bar\partial w + \mu_* h$.*

Lemma 2.2 is an easy consequence of [9, Lemma 5.2]. The proof will not be presented. [9, Lemma 5.2] is stated in terms of hyperfunctions, but the same result is true for distributions. The proof is identical except for the substitution of the dual of the formal Cauchy–Kowalewski Theorem [9, Theorem 6.11] for the dual of the Cauchy–Kowalewski Theorem.

As a consequence of Lemma 2.2 we can make the following definition.

2.3. Definition. We define the *boundary value map* $b\colon \bar{\mathcal{O}} \to {}'\mathcal{D}'$ by $b\phi = h$, where h is the unique distribution determined by ϕ according to Lemma 2.2.

For $\phi \in \mathcal{O}(U - M) \cap \mathcal{D}'(U)$, $b\phi$ is defined as the boundary values of the image of ϕ in $\bar{\mathcal{O}}(U)$. If ϕ happens to be continuous up to M separately from each side of M, then $b\phi$ is the difference of the two boundary values (see [9]). In general we will call $b\phi$ the *(generalized) boundary values* of ϕ. Notice that b is a sheaf map and consequently is local in nature. In particular suppose $\phi \in \mathcal{O}(U - M) \cap \mathcal{D}'(U)$. Then $b\phi = 0$ in ω where $\omega \subset U \cap M$ is an open set, if and only if there is a $\phi_1 \in \mathcal{D}'(U)$ such that $\phi_1 = \phi$ in $U - M$ and $\bar\partial\phi_1 = 0$ in $(U - M) \cup \omega$.

Let ${}'\mathcal{O}$ denote the sheaf of germs of distribution solutions to the equation $\bar\partial_M h = 0$. A section of ${}'\mathcal{O}$ will be called a *CR distribution*. A section of ${}'\mathcal{O}$ which is a continuous function will be called a *CR function*. The following is a special case of the results in [9].

2.4. Theorem. *Suppose U is an open set in X. Then the sequence*

$$0 \to \mathcal{O}(U) \to \bar{\mathcal{O}}(U) \xrightarrow{b} {}'\mathcal{O}(U \cap M) \to H^1(U, \mathcal{O})$$

is exact.

Theorem 2.4 is especially useful in case $H^1(U, \mathcal{O}) = 0$ (which is true for example if U is a domain of holomorphy). In this case the boundary value map is surjective, so every CR distribution in $U \cap M$ is the boundary values of a holomorphic function in U. Since every point of M has a fundamental system of neighborhoods in X which are domains of holomorphy we see that locally at least every CR distribution is the boundary values of a holomorphic function. This result is a generalization and improvement of the extension theorem of Lewy [7] and of the results of Andreotti and Hill [1].

3. Local Unique Continuation

Suppose $S \subset M$ is a hypersurface which is a characteristic surface for $\bar\partial_M$. Then as pointed out in Section 2, S is actually a complex submanifold of M and of X. If $p \in S$ then S is defined near p by an equation $\phi(z) = 0$ where ϕ is a holomorphic function defined on a neighborhood of p in X and $d\phi(p) \neq 0$. Suppose $U \subset X$ is such a neighborhood with the additional property that if $\omega = U \cap M$, then $\omega - S = \omega^+ \cup \omega^-$ where ω^+ and ω^- are disjoint open subsets of M. For k a nonnegative integer define

$$f_k(z) = \begin{cases} \phi(z)^{k+1} & z \in \omega^+, \\ 0 & z \in \omega^- \cup S. \end{cases}$$

Clearly $f_k \in C^k(\omega)$ and $\bar{\partial}_M f_k = 0$ in $\omega - S$. Since each f_k is continuous in ω, a removable singularity result [4, Theorem 4.2] implies that $\bar{\partial}_M f_k = 0$ in ω for each $k \geqslant 0$. Since S is the intersection of the real hypersurfaces defined by $\operatorname{Re} \phi(z) = 0$ and $\operatorname{Im} \phi(z) = 0$, by multiplying ϕ be a complex constant of absolute value 1 and by shrinking U we can arrange $0 < |\operatorname{Im} \phi(z)| \leqslant c \operatorname{Re} \phi(z)$ for $z \in \omega^+$. Then if we define

$$f_\infty(z) = \begin{cases} \exp(-1/\phi(z)) & z \in \omega^+, \\ 0 & z \in \omega^- \cup S, \end{cases}$$

we see that f_∞ is a C^∞ CR function in ω. Each of these CR functions f_k, $0 \leqslant k \leqslant \infty$, vanishes in ω^- but does not vanish identically.

Finally notice that $1/\phi \in \mathcal{O}(U - M) \cap \mathcal{D}'(U)$. Hence $u = b(1/\phi)$ is a CR distribution in ω with supp $u = S \cap \omega$.

The previous two paragraphs amply demonstrate that there can be no local unique continuation for CR distributions across characteristic hypersurfaces. For noncharacteristic surfaces we have the result,

3.1. THEOREM. *Let $\Omega \subset M$ be open and let $p \in \Omega$. Let $r \in C^2(\Omega)$ satisfy $r(p) = 0$ and $dr(p) \neq 0$. Suppose that the level surface $S = \{x \in \Omega \mid r(x) = 0\}$ is noncharacteristic for $\bar{\partial}_M$ at p. Then there is a neighborhood ω of p such that if u is a distribution solution of the equation $\bar{\partial}_M u = 0$ in Ω and $u \equiv 0$ in $\Omega^+ = \{x : r(x) > 0\}$, then $u \equiv 0$ in ω.*

Remark. Thus we do have unique continuation across C^2 noncharacteristic hypersurfaces. As a consequence of the Section 4 results we will be able to prove this for C^1 hypersurfaces (Theorem 4.8).

Proof. Let U be a neighborhood of p in X which is a domain of holomorphy and satisfies $U \cap M \subset \Omega$. We may assume that the function r has been extended to U, belongs to $C^2(U)$, and that the manifold M is defined in U by the equation $\rho(z) = 0$ with $d\rho(z) \neq 0$ on $M \cap U$. As a submanifold of U, S is defined by the pair of equations $\rho(z) = 0$, $\tilde{r}(z) = r(z) + A\rho^2(z) = 0$ where A is any real constant. We will choose A so that the Levi form for the hypersurface $N = \{z \in U : \tilde{r}(z) = 0\}$ has at least one positive eigenvalue at p.

The Levi form for N at p is $L(w) = \langle \partial\bar{\partial}\tilde{r}(p), w \wedge \bar{w} \rangle$ restricted to $w \in H_p(N)$

(the pairing is that between $\Lambda^{1,1}T^*(X)$ and $\Lambda^{1,1}T(X)$). Since S is not characteristic at p we know that $H_p(N) \neq H_p(M)$. Choose $w \in H_p(N)$ such that $w \notin H_p(N)$ This means precisely that $w \in T_p(X)$, $\langle \partial \tilde{r}(p), w \rangle = 0$ and $\langle \partial \rho(p), w \rangle \neq 0$. Both of these conditions are independent of A since $\partial \tilde{r} = \partial r + 2A\rho\partial\rho$ and $\partial \tilde{r}(p) = \partial r(p)$. But $\partial\bar{\partial}\tilde{r} = \partial\bar{\partial}r + 2A\partial\rho \wedge \bar{\partial}\rho + 2A\rho\partial\bar{\partial}\rho$ so $\langle \partial\bar{\partial}\tilde{r}(p), w \wedge \bar{w} \rangle = \langle \partial\bar{\partial}r(p), w \wedge \bar{w} \rangle + 2A\,|\langle \partial\rho(p), w \rangle|^2$. Since $\langle \partial\rho(p), w \rangle \neq 0$, this will be positive if A is chosen to be sufficiently large.

Since the Levi form for N has a positive eigenvalue we can apply the classical Kontinuitätssatz to conclude that there is a neighborhood U' of p contained in U with the property that if $\phi \in \mathcal{O}(U_+')$, where $U_+' = \{z \in U' \mid \tilde{r}(z) > 0\}$, then there is a $\tilde{\phi} \in \mathcal{O}(U')$ with $\tilde{\phi} = \phi$ in U_+'.

Let $\omega = U' \cap M$. If $u \in {}'\mathcal{O}(\Omega)$ then $u \in {}'\mathcal{O}(U \cap M)$. Since U is a domain of holomorphy by Theorem 2.4 there is a $\phi \in \mathcal{O}(U - M) \cap \mathcal{D}'(U)$ such that $b\phi = u$. Since $u \equiv 0$ in Ω^+ by the remarks following Definition 2.3 we may assume that $\phi \in \mathcal{O}((U - M) \cup \Omega^+)$. In particular $\phi \in \mathcal{O}(U_+')$, so there is a $\tilde{\phi} \in \mathcal{O}(U')$ such that $\tilde{\phi} = \phi$ in U_+' and therefore in $(U' - M) \cup \Omega^+$. Hence $u = b\phi = b\tilde{\phi} = 0$ in $\omega = U' \cap M$. Q.E.D.

If we restrict our attention to CR functions we can improve the result of Theorem 3.1

3.2. Corollary. *Let $\Omega \subset M$ be open and let $p \in \Omega$. Let $r \in C^1(\Omega)$ satisfy $r(p) = 0$ and $dr(p) \neq 0$. Suppose that the level surface $S = \{x \in M \mid r(x) = 0\}$ is noncharacteristic for $\bar{\partial}_M$ at p. Then there is a neighborhood ω of p such that if u is a continuous solution of the equation $\bar{\partial}_M u = 0$ in Ω and if $u \equiv 0$ on S, then $u \equiv 0$ in ω.*

Proof. Choose ω_1 by Theorem 3.1 (as improved in Theorem 4.8). Given the CR function u in Ω, define

$$u_1(x) = \begin{cases} 0 & x \in \Omega^+, \\ u(x) & x \in \Omega^-. \end{cases}$$

Then $u_1 \in C(\Omega)$, and $\bar{\partial}_M u_1 = 0$ in $\Omega - S$. As a consequence of a removable singularity theorem [4, Theorem 4.2] we conclude that $\bar{\partial}_M u_1 = 0$ in Ω. By Theorem 3.1 $u_1 \equiv 0$ in ω_1, and so $u \equiv 0$ in $\omega_1 \cap \Omega^-$. Repeating the same argument with r replaced by $-r$ we get a neighborhood ω_2 with $u \equiv 0$ in $\omega_2 \cap \Omega^+$. Let $\omega = \omega_1 \cap \omega_2$ and the result follows. Q.E.D.

4. Global Unique Continuation

In this section we examine the following problem: What conditions on M will ensure that every CR distribution on M which vanishes on an open set is identically equal to zero? For convenience we make the following definition.

4.1. DEFINITION. The manifold M is said to have *global unique continuation* if every CR distribution on M which vanishes on an open set vanishes identically.

Obviously for M to have global unique continuation it is necessary that M be connected. Therefore, we will assume that M is connected in the remainder of this section.

Remark. For CR functions the property of global unique continuation takes on a different light as a result of Corollary 3.2. If M has global unique continuation then any CR function on M, which vanishes on a C^1 hypersurface of M that is noncharacteristic at a point, vanishes identically.

Let us approach the question from a different standpoint. Suppose M does not have global unique continuation. Then there is a nontrivial CR distribution u on M which vanishes on an open set. It is clear that Theorem 3.1 puts a strong restriction on the boundary of the open set on which u vanishes, or what is the same thing, on the boundary of the support of u. The implications of this restriction are the subject of the next theorem.

4.2. THEOREM. *Suppose u is a* CR *distribution on M. Let A denote the boundary of the support of u. If $A \neq \varnothing$ then for every point $p \in A$, there is a neighborhood U of p, a real valued function $f \in C^\infty(U)$ with $df(x) \neq 0$ for all $x \in U$, and a closed nowhere dense set $E \subset R$ such that*

(1) $A \cap U = \{x \in U \mid f(x) \in E\}$,

(2) *for each* $t \in E$, $S_t = \{x \in U \mid f(x) = t\}$ *is a complex manifold with* $T(S_t) = H(M)|_{S_t}$.

Remark. We will express the results of Theorem 4.2 by saying that the boundary of the support of u is foliated by complex hypersurfaces.

The proof of Theorem 4.2 will be given at the end of this section. First we wish to examine the consequences and point out some examples.

EXAMPLE 1. Let $M = \mathbb{C}^{n-1} \times \mathbb{R} \subset \mathbb{C}^n$. Any distribution on M which depends only on the real variable is a CR distribution. Consequently given any closed $E \subset \mathbb{R}$ there is a CR distribution with $\operatorname{supp} u = \mathbb{C}^{n-1} \times E$.

EXAMPLE 2. In the first example M itself is foliated by complex manifolds. In this example this is not the case. Let $M \subset \mathbb{C}^2$ be defined by the equation $\rho(z, w) = \operatorname{Im} w - h(\operatorname{Re} w) \operatorname{Re} z = 0$ where $h \in C^\infty(\mathbb{R})$. The Levi form for M must vanish identically on any complex submanifolds of M. A calculation shows that the Levi form vanishes if and only if $h(\operatorname{Re} w)(h(\operatorname{Re} w)\, h''(\operatorname{Re} w) -$

$2h'(\mathrm{Re}\, w)^2)\,\mathrm{Re}\, z = 0$. Now if $E \subset \mathbb{R}$ is any closed set then $\mathbb{R} - E = \bigcup_{k=1}^{\infty} I_k$, where $I_k = (a_k, b_k)$ are open intervals. Define

$$h(t) = \begin{cases} \exp -\left\{ \dfrac{b_k - a_k}{(t - a_k)(b_k - t)} \right\} & t \in I_k, \\ 0 & t \in E, \end{cases}$$

with suitable modifications if $a_k = -\infty$ or if $b_k = \infty$. Then $h \in C^\infty(\mathbb{R})$ and vanishes precisely on E. Furthermore an easy calculation shows that $hh'' - 2h'^2 > 0$ in $\mathbb{R} - E$. Consequently the only complex manifolds contained in M are those complex lines of the form $\mathbb{C} \times \{t\}$ with $t \in E \subset \mathbb{R} \subset \mathbb{C}$. The manifold M can be parametrized by $\mathbb{C} \times \mathbb{R}$ with $M = \{(z, t + ih(t)\,\mathrm{Re}\, z) : z \in \mathbb{C}, t \in \mathbb{R}\}$. With respect to this parametrization a distribution u on M is a CR distribution if and only if $2(1 + ih'(t)\,\mathrm{Re}\, z)(\partial u/\partial \bar{z}) - ih(t)(\partial u/\partial t) = 0$. Now if u is any distribution on $\mathbb{R}$ with $\mathrm{supp}\, u = E$ and if we consider u to be a distribution on $\mathbb{C} \times \mathbb{R}$ which is independent of z, then, since h vanishes to infinite order on E, u satisfies the above equation. Consequently u represents a CR distribution on M and $\mathrm{supp}\, u = \mathbb{C} \times E$.

EXAMPLE 3. Let $F \subset \mathbb{R}$ be closed and let $E = \partial F$. Let M be the manifold of Example 2. Then $\mathring{F} = \bigcup_{k=1}^{\infty} J_k$ where $J_k = (\alpha_k, \beta_k)$ are open intervals. Define

$$g(w) = \exp\{-1/(w - \alpha_k) - 1/(\beta_k - w)\} \qquad \text{if } \mathrm{Re}\, w \in J_k,$$

with suitable modification if $\alpha_k = -\infty$ or if $\beta_k = +\infty$. Then $g \in \mathcal{O}(\mathring{F} + i\mathbb{R})$. An easy calculation shows that if $\mathrm{Re}\, w \in J_k$ and $|\mathrm{Im}\, w| \leqslant B(\mathrm{Re}\, w - \alpha_k)$ and $|\mathrm{Im}\, w| \leqslant B(\beta_k - \mathrm{Re}\, w)$ then $|g(w)| \leqslant \exp(-[(1 + B^2)(\mathrm{Re}\, w - \alpha_k)]^{-1}) \cdot \exp(-[(1 + B^2)(\beta_k - \mathrm{Re}\, w)]^{-1})$. Define the function ϕ on $\mathbb{C}^2$ by

$$\phi(z, w) = \begin{cases} g(w) & \text{if } \mathrm{Re}\, w \in F, \\ 0 & \text{if } \mathrm{Re}\, w \notin F, \end{cases}$$

and let $f = \phi|_M$. Clearly $\phi \in \mathcal{O}(\mathbb{C}^2 - (\mathbb{C} \times E))$ and consequently $\bar{\partial}_M f \equiv 0$ in $M - (\mathbb{C} \times E)$. If $(z, w) \in M$, then $\mathrm{Im}\, w = h(\mathrm{Re}\, w)\,\mathrm{Re}\, z$. Clearly $h(\mathrm{Re}\, w) \leqslant d(\mathrm{Re}\, w, E)$. Thus if $|\mathrm{Re}\, z| \leqslant B$, then $|\mathrm{Im}\, w| \leqslant Bd(\mathrm{Re}\, w, E)$, and so

$$|f(z, w)| \leqslant \exp(-[(1 + B^2)\, d(\mathrm{Re}\, w, E)]^{-1}).$$

Consequently f vanishes on $\mathbb{C} \times E$ to infinite order and we conclude that $f \in C^\infty(M)$ and $\bar{\partial}_M f \equiv 0$. Thus f is a smooth CR function on M and $\partial(\mathrm{supp}\, f) = \mathbb{C} \times E$. Furthermore in this case it is only $\partial(\mathrm{supp}\, f)$ that is foliated and not $\mathrm{supp}\, f$.

The examples clearly illustrate that Theorem 4.2 cannot be improved upon even for smooth CR functions. We can now return to the original problem and solve it by legislating against the behavior indicated in Theorem 4.2.

4.3. THEOREM. *Suppose M is connected and contains no complex hypersurface the closure of which is foliated by complex hypersurfaces. Then M has global unique continuation.*

Proof. Suppose on the contrary that there is a nontrivial CR distribution u on M which vanishes on an open set. Let $p \in \partial(\operatorname{supp} u)$ and let S denote the component of $\partial(\operatorname{supp} u)$ which contains p. By Theorem 4.2 S is a complex hypersurface and since $\bar{S} \subset \partial(\operatorname{supp} u)$, it is easily seen that $\bar{S}$ is foliated by complex hypersurfaces. Q.E.D.

EXAMPLE 4. Let $h \in C^\infty(\mathbb{R})$ satisfy $h(t) \equiv 0$ for $t \leqslant 1$, $h(t) = t - \lambda$, λ the appropriate constant, for $t \geqslant 2$ with $h'(t) > 0$ for $t > 1$, and $h''(t) > 0$ for $1 < t < 2$ (choose $g \in C_0^\infty(\mathbb{R})$, nonnegative, $\operatorname{supp} g = [1, 2]$ and $\int g(t)\, dt = 1$. Solve $h'' = g$ with $h(0) = h'(0) = 0$). Let $M \subset \mathbb{C}^2$ be defined by $\operatorname{Im} w = h(|z|^2)$. Then the Levi form for M vanishes if and only if $|z| \leqslant 1$. Since the Levi form must vanish on any complex submanifold of M, the only such submanifolds are $S_t = \{(z, t) \mid |z| < 1\}$ for $t \in R$. Since $\bar{S}_t = \{(z, t) \mid |z| \leqslant 1\}$ is not foliated (if $|z| = 1$ then (z, t) is not contained in a complex submanifold) we conclude by Theorem 4.3 that M has global unique continuation. This is an example of a manifold which has global unique continuation but does not have local unique continuation.

The next result provides a partial converse of Theorem 4.3.

4.4. THEOREM. *Suppose $U \subset X$ is open and $\phi \in \mathcal{O}(U)$ has the property that $S = \{z : \phi(z) = 0\}$ is a closed subset of M. Then M does not have global unique continuation.*

Proof. By hypothesis $1/\phi \in \mathcal{O}(U - M)$ and since ϕ vanishes on S to finite order, $1/\phi$ has a distribution extension in $\mathcal{D}'(U)$. By Theorem 2.4 the generalized boundary value $b(1/\phi)$ is a CR distribution on $U \cap M$ and $b(1/\phi) = 0$ in $(U \cap M) - S$. Define

$$u = \begin{cases} b(1/\phi) & \text{in } U \cap M, \\ 0 & \text{in } M - S. \end{cases}$$

Then u is a CR distribution on M and $\operatorname{supp} u \subset S$, so M does not have global unique continuation. Q.E.D.

If we have some a priori knowledge about the size of $\partial(\operatorname{supp} u)$ we can improve Theorem 4.2. Let Λ^d denote Hausdorff measure in dimension d. Notice that the concepts $\Lambda^d(A) = 0$, $\Lambda^d(A) > 0$, $\Lambda^d(A) < \infty$ are invariant under coordinate changes, and consequently have meaning on a manifold.

4.5. COROLLARY. *Suppose u is a* CR *distribution on M and* $\partial(\operatorname{supp} u)$ *has locally finite Hausdorff measure in dimension* $2n - 2$. *Then* $\partial(\operatorname{supp} u)$ *is a complex hypersurface of M.*

Proof. The hypothesis implies that for each $p \in \partial(\operatorname{supp} u)$ the set E given by Theorem 4.2 is a finite set. The result now follows easily. Q.E.D.

It is well known that the Levi form for M must vanish on any complex hypersurface in M. Let F denote the set of points of M at which the Levi form vanishes. Then F is a closed set, and by Theorem 4.2 $\partial(\operatorname{supp} u) \subset F$ for any CR distribution on M.

4.6. THEOREM. (*a*) *If F has locally finite Hausdorff measure in dimension* $2n - 2$, *then* $\partial(\operatorname{supp} u)$ *is a complex hypersurface of M for any* CR *distribution u on M.* (*b*) *If* $\Lambda^{2n-2}(F) = 0$ *and M is connected then M has global unique continuation.*

Theorem 4.6 is an immediate consequence of the preceding discussion and Corollary 4.5. Theorem 4.6(b) improves the result of [5] where it was proved that M has global unique continuation if $F = \varnothing$.

4.7. COROLLARY. *Suppose M is connected and* $\Lambda^{2n-2}(F) = 0$. *If f is a* CR *function on M which vanishes on a hypersurface* $S \subset M$ *then* $f \equiv 0$.

Proof. Since $\Lambda^{2n-2}(S) > 0$, there is a point $p \in S$ at which S is not characteristic. The corollary now follows from Theorem 4.6(b) and the remark following Definition 4.1.

Finally we return to the local problem and state an improved version of Theorem 3.1.

4.8. THEOREM. *Let* $\Omega \subset M$ *be open and let* $\Omega^+ \subset \Omega$ *be open. Let* $p \in \partial\Omega^+ \cap \Omega$ *and suppose that* $\operatorname{Tan}(\partial\Omega^+, p)$, *the tangent cone of* $\partial\Omega^+$ *at p, is a hyperplane of* $T_p(M)$, *and that* $\operatorname{Tan}(\partial\Omega^+, p) \neq H_p(M)$. *Then there is a neighborhood* ω *of p such that if u is a* CR *distribution on* Ω *and* $u \equiv 0$ *in* Ω^+, *then* $u \equiv 0$ *in* ω.

Remark. The definition of tangent cone may be found in [2]. Notice that the theorem applies if $\Omega^+ = \{x: r(x) > 0\}$, where $r \in C^1(\Omega)$ satisfies $r(p) = 0$ and $dr(p)$ does not annihilate $H_p(M)$.

Proof. Suppose the theorem is not true. Then there is a CR distribution u on Ω with $u \equiv 0$ in Ω^+ and $p \in \operatorname{supp} u$. Since $p \in \partial(\operatorname{supp} u)$, by Theorem 4.2 there is a complex hypersurface $S \subset M$ with $p \in S \subset \partial(\operatorname{supp} u) \subset \Omega - \Omega^+$. Consequently $H_p(M) = T_p(S) = \operatorname{Tan}(S, p) \subset \operatorname{Tan}(\Omega - \Omega^+, p)$. However, since $\operatorname{Tan}(\partial\Omega^+, p)$ is a hyperplane of $T_p(M)$, $\operatorname{Tan}(\Omega - \Omega^+, p)$ is contained in one of the half spaces bounded by $\operatorname{Tan}(\partial\Omega^+, p)$. But the hyperplane $H_p(M)$ is contained in the half space if and only if $H_p(M) = \operatorname{Tan}(\partial\Omega^+, p)$, which is a contradiction. Q.E.D.

Proof of Theorem 4.2. Let $\Omega = M - \operatorname{supp} u$. Notice that as a result of Theorem 3.1 Ω has the following property: if $\omega \subset \Omega$ is an open set with a C^2 boundary, then for each $p \in \partial\omega \cap \partial\Omega$ we have $T_p(\partial\omega) = H_p(M)$. Consequently Theorem 4.2 is a special case of the following result.

4.9. THEOREM. *Let M be a C^∞ manifold of dimension n. Let H be a subbundle of the tangent bundle of M with fiber dimension $n - 1$. Suppose $\Omega \subset M$ is an open set with the property that if $\omega \subset \Omega$ is an open set with a C^2 boundary, then for each $p \in \partial\omega \cap \partial\Omega$ we have $T_p(\partial\omega) = H_p$. Then for each point $p \in \partial\Omega$, there is a neighborhood U of p, a real valued function $f \in C^\infty(U)$ with $df(x) \neq 0$ for all $x \in U$, and a closed nowhere dense set $R \subset \mathbb{R}$ such that*

(1) $\partial\Omega \cap U = \{x \in U \mid f(x) \in E\}$,

(2) *for each $t \in E$, $S_t = \{x \in U \mid f(x) = t\}$ is an integral manifold of H.*

We will need the following lemma, which contains the main elements of the analysis of this problem.

4.10. LEMMA. *Let X be a nonvanishing vector field on M defined near a point $p \in \partial\Omega$ which is a section of H. If γ is any integral curve of X, which intersects $\partial\Omega$, then $\gamma \subset \partial\Omega$.*

Proof. Suppose $p \in \gamma \cap \partial\Omega$ and choose local coordinates near p with p corresponding to 0 and $X = (\partial/\partial x_n)$. Then the curve γ is parametrized near 0 by $\gamma(t) = (0, t) \in \mathbb{R}^n$. Let $f(t) = d(\gamma(t), \sim\Omega)$ denote the Euclidean distance from $\gamma(t)$ to the complement of Ω. Let $S_t = \{x\colon |x - \gamma(t)| = f(t)\}$ denote the sphere with center $\gamma(t)$ and radius $f(t)$. For each t there is a point $x_t \in S_t \cap \partial\Omega$. If $f(t) \neq 0$ then S_t is a smooth hypersurface contained in $\bar{\Omega}$. By the hypothesis therefore $T_{x_t}(S_t) = H_{x_t}$. Consequently the tangent vector $(\partial/\partial x_n) \in T_{x_t}(S_t)$, which can happen if and only if $x_t = (x_t', t)$, where $x_t' \in \mathbb{R}^{n-1}$. This means in particular that if $s \neq t$ then $|x_t - \gamma(s)| > f(s)$. Rewriting this inequality we have $f^2(s) < f^2(t) + |s - t|^2$. Since the argument is symmetric in s and t we conclude that $|f^2(s) - f^2(t)| \leqslant |s - t|^2$ for all s and t. This can be true only if f is a constant function, and, since $f(0) = 0, f(t) \equiv 0$. Thus any integral curve of X which meets $\partial\Omega$ must stay in $\sim\Omega$.

Suppose there is a point $q \in \gamma \cap (\sim\bar{\Omega})$. Parametrize γ so that $\gamma(0) = q$ and suppose $\gamma(t_0) = p \in \gamma \cap \partial\Omega$. Let $\gamma(x; t)$ denote the integral curve of X with $\gamma(x; 0) = x$ and let $F_t(x) = \gamma(x; t)$ denote the corresponding flow. Then F_{t_0} is a diffeomorphism defined on a neighborhood of q and $F_{t_0}(q) = p$. Let U be a neighborhood of q with $q \in U \subset (\sim\bar{\Omega})$. Then $F_{t_0}(U)$ is a neighborhood of p.

Since $p \in \partial\Omega$, there is a point $y \in F_{t_0}(U) \cap \Omega$. Suppose $y = F_{t_0}(x)$, $x \in U$. Consider the curve $\gamma(x; t)$. $\gamma(x; 0) = x \in (\sim\bar{\Omega})$, and $\gamma(x; t_0) = y \in \Omega$, so there must be a t between 0 and t_0 with $\gamma(x; t) \in \partial\Omega$. However, by the previous paragraph we know that if an integral curve meets $\partial\Omega$ then that curve must stay in $(\sim\Omega)$ and this contradicts the fact that $\gamma(x; t_0) = y \in \Omega$. Q.E.D.

Proof of Theorem 4.9. Let $p \in \partial\Omega$ and let X be a nonvanishing section of H defined near p. Since the result is local we may assume that we are in $\mathbb{R}^n$ in a neighborhood $U = \{(x', x_n): |x'| < 1, |x_n| < 1\}$ of $p = 0$ and that $X = \partial/\partial x_n$. The integral curves of X are $\gamma(t) = (x', t)$, and by Lemma 4.10 these are completely contained in one of the three sets $\Omega, \partial\Omega, \sim\bar{\Omega}$. Thus if $\Omega_{n-1} = \{x' \mid (x', 0) \in \Omega \cap U_n\}$, we see that $\Omega \cap U_n = \Omega_{n-1} \times (-1, 1)$. Let H^{n-1} denote the bundle on the unit ball B_{n-1} in $\mathbb{R}^{n-1}$ defined by $H^{n-1} = H|_{B_{n-1}} \cap T(B_{n-1})$. Then Ω_{n-1}, B_{n-1}, and H^{n-1} satisfy the hypotheses of the theorem with the dimension decreased by one. Proceeding inductively we finally get new coordinates $(x_1, x_2, \ldots, x_n)$ such that if $U = \{x \mid |x_j| < 1, 1 \leqslant j \leqslant n\}$ there is an open set $\Omega_1 \subset \mathbb{R}^1$ such that $\Omega \cap U = \{x \in U: x_1 \in \Omega_1\}$. Set $f(x) = x_1$ and $E = \partial\Omega_1$ and we have proved part (1) of the theorem.

Clearly $T_p(S_0) = H_p$. It remains to show that $T(S_t) = H|_{S_t}$ for all $t \in E$. Choose any point $q \in S_t$ and repeat the argument with p replaced by q. We get a new function f' and level sets S_τ' and $T_q(S_0') = H_q$. Since the various level sets are disjoint and connected we must have $S_0' = S_t$ near q and consequently $T_q(S_t) = H_q$. Q.E.D.

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