Inequalities characterizing standard Sturmian and episturmian words

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Abstract

Considering the smallest and the greatest factors with respect to the lexicographic order we associate to each infinite word \( r \) two other infinite words \( \min(r) \) and \( \max(r) \). In this paper we prove that the inequalities \( as \leq \min(s) \leq \max(s) \leq bs \) characterize standard Sturmian words (proper ones and periodic ones) and that the condition “for any \( x \in A \) and lexicographic order \( < \) satisfying \( x = \min(A) \) one has \( xs \leq \min(s) \)” characterizes standard episturmian words.

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1. Introduction

We use notions and terminology of theoretical computer science (see \cite{11,12}).

Sturmian words have been extensively studied for a long time (see, for example \cite{2,14,15}). They are infinite words over a two letter alphabet (\( \{a, b\} \) in this paper) and have several characterizations \cite{12}. A recent one (the set of the factors of an infinite word contains...
exactly one palindrome of even length and two palindromes of odd length if and only if it is Sturmian) is in [5]. In [4] we began to study the episturmian words, an apparently interesting generalization to k letter alphabet A, with k ≥ 2.

In this paper a standard Sturmian word is presented as a cutting sequence of a half-line y = ax. We consider the two possible cases: a irrational (proper standard Sturmian words) and a rational (periodic standard Sturmian words). So our terminology is different from the most usual one where standard Sturmian word refers only to the proper standard Sturmian case (a irrational).

Now, let A be a finite alphabet and let < be a lexicographic order on the free monoid generated by A. Given an infinite word r over the alphabet A we denote by min(r) (resp. max(r)) the infinite word having for each integer k ≥ 0 its prefix of length k equal to the smallest (resp. greatest) factor of r of length k with respect to the lexicographic order <. With this notation and with the order a < b, we have the following property: an infinite word r satisfies min(r) = ar and max(r) = br if and only if it is proper standard Sturmian (see [20] and, in a different form, [6]). This result has been generalized to episturmian words in [9]: an infinite word s over a finite alphabet A is a standard Arnoux-Rauzy sequence if and only if for any x ∈ A and for any lexicographic order < for which x is the smallest letter of A we have xs = min(s).

Proposition 3.2 hereafter (an infinite word s over a finite alphabet A is standard episturmian if and only if for any x ∈ A and lexicographic order < satisfying x = min(A) we have xs ≤ min(s)) is an extension of the just recalled result of [9]. Proposition 3.1, which has been announced without proof in [18] and which is a corollary of Proposition 3.2, has an interest in itself (preliminary lemmas are descriptions of min(r) and max(r)). It may be remarked that our Propositions 3.1 and 3.2, which seem slight variations of [20,9] have fairly more complicated proofs than these two ones.

This paper is organized as follows.

In Section “2 Preliminaries” we recall the definitions of (proper and periodic) Sturmian words, we make some comments on the Fibonacci word (a very particular case of Sturmian word), we recall the definition of episturmian words, we recall the notion of extremal words and, finally, we prove some preliminary lemmas which will be very useful in the next section.

In Section “3 Results” we prove the characterization of standard Sturmian words, Proposition 3.1, and the characterization of standard episturmian words, Proposition 3.2.

We seize the occasion to mention an error in [8]. Theorem 5.1 there (not used elsewhere) is false and should be ignored. This has been corrected in [10] where complete results are given.

2. Preliminaries

2.1. Words

We refer to [11,12] for the notions of semigroup and monoid, for the notions of word, factor, prefix, suffix, free semigroup and free monoid over a finite alphabet and for the notion of morphism.
When $A$ is a finite alphabet, we denote by $A^+$ the free monoid over $A$ and by $A^*$ the free semigroup over $A$. We denote by $\varepsilon$ the empty word, which has length 0, and we have $A^+ = A^* \setminus \{\varepsilon\}$. In order to make clear the sequence of letters which gives a word $u$ we write $u = u(1)u(2)\ldots u(n)$, $n \geq 1$. By definition, $n$ is the length of $u$ and it is denoted by $|u|$. We denote by $A^n$ the set of words of length $n$ over the alphabet $A$. A right infinite word $w$ is a function from $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ into $A$. We write $w = w(1)w(2)\ldots w(i)\ldots$. We denote by $A^\omega$ the set of the right infinite words over $A$. If $x$ is a letter and $w$ is an infinite word then the infinite word $w' = xw$ is defined as follows: $w' = w'(1)w'(2)\ldots w'(i)\ldots$, where $w'(1) = x$ and, for $i > 1$, $w'(i) = w(i - 1)$.

The reversal of a finite word $u = u(1)u(2)\ldots u(m)$ is $u(m)u(m - 1)\ldots u(1)$ denoted by $\tilde{u}$ if $u$ is non-empty and is $\varepsilon$ itself if $u = \varepsilon$. A word $u$ is a palindrome (or a palindromic word) if and only if $u = \tilde{u}$.

If $u \in A^*$, the right palindromic closure of $u$ is the shortest palindrome $u^{(+)}$ having $u$ as a prefix. We have $u^{(+)} = uu\tilde{u}'$ where $v$ is the longest palindromic suffix of $u$ and $u = uv$.

For a finite or infinite word $m$, we denote by $m(i, j)$ the factor $v = m(i)m(i + 1)\ldots m(j)$ of $m$. The set of factors of $m$ is denoted by $F(m)$. The subset of the alphabet consisting of the letters which have at least one occurrence in $m$ is denoted by $Alph(m)$. A factor $u$ of an infinite word $m$ is right (resp. left) special in $m$ if there exist two letters $a$, $b$ such that $ua$ and $ub$ (resp. $au$ and $bu$) are also factors of $m$.

An infinite word $p$ is periodic (resp. ultimately periodic) if there exists $k \geq 1$ such that $p(j + k) = p(j)$ for each $j \geq 1$ (resp. for each $j \geq i$ for some $i \geq 1$). If $u$ is a finite word then $u^\omega$ is the periodic infinite word $uuu\ldots u\ldots$ having $|u|$ as a period.

Morphisms of $A^*$ extend to infinite words. In particular, if $b = x(1)\ldots x(i) \in A$, and $\varphi : A^* \rightarrow A^*$ is a morphism then $\varphi(b) = \varphi(x(1))\ldots$.

2.2. Fibonacci word

Before giving a general definition of Sturmian words we present the most famous of them, the Fibonacci word (see for instance [3,7,11–13]), which we describe now. Let $\varphi : \{a, b\}^* \rightarrow \{a, b\}^*$ be the morphism given by $\varphi(a) = ab$, $\varphi(b) = a$. Let $f_0 = b$ and, for $n \geq 0$,

$$f_{n+1} = \varphi(f_n).$$

Notice that, for $n \geq 0$, $|f_n|$ is the $n$th element of the sequence of Fibonacci numbers $F_n$.

For each $n \geq 2$, $f_n = f_{n-1}f_{n-2}$ implies that, for each $n \geq 1$, $f_n$ is a prefix of $f_{n+1}$.

Hence there exists a unique infinite word, called the Fibonacci word, denoted by $f$ such that, for each $n \geq 1$, $f_n$ is a prefix of $f$. The first letters of $f$ are

$$f = ababaababaababaababaababaababaababaababaababaababaababaababaababaababaababa... .$$

We are convinced that looking carefully at the properties of the Fibonacci word one can discover interesting properties of Sturmian (and episturmian!) words. Any case, the results of this paper were discovered in this way.
2.3. Sturmian words

There are many definitions of Sturmian words, see [12, Chap. 2]. In particular among them there is the following one.

Definition 2.1. An infinite word 

\[ s = s(1)s(2)s(3)\ldots, s(i) \in \{a, b\} \]

is Sturmian if there exist reals \( \alpha, \rho \in [0, 1] \), such that either for all \( i \)

\[ s(i) = a \text{ if } \lfloor \rho + (i + 1)\alpha \rfloor = \lfloor \rho + i\alpha \rfloor \]

or for all \( i \)

\[ s(i) = a \text{ if } \lceil \rho + (i + 1)\alpha \rceil = \lceil \rho + i\alpha \rceil \].

The infinite word \( s \) is proper Sturmian if \( \rho \) is irrational, it is periodic Sturmian if \( \rho \) is rational and it is standard Sturmian if \( \rho = 0 \).

Consider the grid \( g \) of the half-lines

\[ c_k = \{(x, y) | x = k, \ y > 0, \ k \text{ positive integer}\} \]

and

\[ l_h = \{(x, y) | y = h, \ x > 0, \ h \text{ positive integer}\} \]

and consider the half-line \( l \) which has the form

\[ \{(x, y) | y = \beta x, \ x > 0, \ \beta \in (0, \infty)\} \].

In this paper standard Sturmian words are obtained as the cutting sequence of a half-line \( l \) with respect to the grid \( g \), that is the sequences like those of Fig. 1, which we will describe more precisely hereafter.

Put in a sequence, by increasing distance from the origin, the intersection points of \( l \) with the half-lines of the grid. Let \( I(1), I(2), I(3), \ldots, I(n), I(n + 1), \ldots \) be this sequence.

If \( \beta \) is irrational, then for no \( I(j) \) its two coordinates are both integers. If \( \beta \) is rational, then there are infinitely many \( I(j) \) with the two coordinates integer.

In the first case, the cutting sequence of \( l \) is the infinite word \( s = s(1)s(2)\ldots s(j)s(j + 1)\ldots \), where \( s(j) = a \) if \( I(j) \) is on an horizontal half-line \( l_h \) of \( g \) and \( s(j) = b \) if \( I(j) \) is on a vertical half-line \( c_k \) of \( g \).

When \( \beta \) is rational, let \( I(n) \) be the intersection point which has integer coordinate and is at the shortest distance from the origin. Note that there exist suitable relatively prime integers \( p, q \) such that \( I(n) = (p, q) \) and \( \beta = q/p \). If \( I(n) \neq (1, 1) \) we proceed on \( I(1), I(2), \ldots, I(n - 1) \) in the same way used for the \( \beta \) irrational case and we obtain a word \( w = s(1)s(2)\ldots s(n - 1) \). If \( I(n) = (1, 1) \) we put \( w = \varepsilon, \varepsilon \) the empty word.

Note that \( w \), which is a prefix of our cutting sequence in construction, is restricted to the interior of the rectangle \( R \) of vertices \((0, 0), (p, 0), (p, q), (0, q)\) and, by geometrical
reasons is a palindrome (we point out again that the point (0, 0) is not considered at all and that \(w\) registers \(p + q - 2\) intersections with lines in the strict interior of \(R\)).

For the point \(I(n)\) we have two possible choices: \(ab\) or \(ba\) (the first choice means that we consider the horizontal cut before the vertical one, the second means the opposite). By geometrical reasons and coherence of choice the cutting sequence of \(I\) will be either the infinite word \((wab)\) or the infinite word \((wba)\).

For example the half-line of Fig. 1 (I) has equation \(y = \frac{3}{2}x\) and its cutting sequence is the periodic Sturmian word \(abaababaababaababaababaababaababaababaab\ldots = (wab)\), where \(w = aba\). The half-line of Fig. 1 (II) has equation \(y = \Phi x\) where

\[
\Phi = \frac{\sqrt{5} + 1}{2}
\]

and its cutting sequence is the Fibonacci word, \(f = abaaababaababaababaababaababaababaababa\ldots\), the most well-known (proper) Sturmian word. With a similar procedure, for any irrational \(\beta\), we have the cutting sequence of \(y = \beta x\).

**Remark 2.1.** Note that the cutting sequences \(b^0\) \((y = 0\) corresponding to the real value 0\) and \(a^0\) \((x = 0\) corresponding to \(\infty\)) will be considered in the sequel in the class of periodic Sturmian words. Note also that with our convention on \(a\) and \(b\) the usual order on real numbers corresponds to the opposite lexicographic order on the cutting sequences.

**Remark 2.2.** Sturmian words may play a role in the presentation of the notion of incommensurable segments in secondary school [19] where some easy geometric constructions concerning octagons, decagons and dodecagons and attributable to the Pythagorean School...
are presented: for example, see Fig. 2, \(BC\) and \(B_1C_1\) are incommensurable segments (hint: \(AD = 2BC + B_1C_1\) and \(BC = A_1D_1\)).

2.4. Episturmian words

We follow the notations and terminology of [4,9].

**Definition 2.2.** A right infinite word is *episturmian* if the set of its factors is closed under reversal and it has at most one right (or equivalently left) special factor of each length.

A right infinite word is *standard* episturmian if all its left special factors are prefixes of it. Standard words (also called characteristic in the Sturmian case) are characterized in [4] using the right palindromic closure. More precisely \(s\), right infinite, is standard episturmian if and only if there exists a right infinite word \(A = x(1)x(2)\ldots, x(i) \in A\), called its *directive word* such that the sequence \(u_1 = u, u_2, u_3, \ldots\) of palindromic prefixes of \(s\) (which are also right special factors of \(s\)) is given by \(u_{n+1} = (u_n x(n))^{(+)}\).

Consider for any \(a \in A\) the morphisms \(\psi_a\) given by \(\psi_a(a) = a, \psi_a(x) = ax\) and \(\overline{\psi}_a\) given by \(\overline{\psi}_a(a) = a, \overline{\psi}_a(x) = xa\) for \(x \in A, x \neq a\).

A right infinite word \(s\) is standard episturmian if there exist \(A\) as before and an infinite sequence of right infinite words \(s^{(0)} = s, s^{(1)}, \ldots\) given by \(s^{(i-1)} = \psi_{x(i)}(s^{(i)})\), moreover \(A\) is the directive word of \(s\).

Consider the prefix \(x(1)x(2)\ldots x(n)\) of the directive word \(A = x(1)x(2)\ldots, x(i) \in A\), we write \(\mu_n = \psi_{x(1)} \cdots \psi_{x(i)}(x(n + 1))\). The standard episturmian word directed by \(A\) has the \(h_n = \mu_n(x(n + 1))\) as prefixes and we have the important relation

\[ u_{n+1} = h_{n-1}u_n, \]
2.5. Lexicographic order

Let $<$ be a total order over the alphabet $A$. The lexicographic order on $A^+$ is defined as follows: for all $u, v \in A^+$

$$u < v \iff \begin{cases} 
\text{either } u \text{ is a proper prefix of } v \\
\text{or } u = ras, \ v = rbt \text{ for some } a, b \in A, \ a < b, \text{ and } r, s, t \in A^*. 
\end{cases}$$

The lexicographic order of infinite words arises from that of their prefixes. For example $f < (abaab)^\omega$, where $f$ is the Fibonacci word, because $abaabaabaa < abaababaabab$.

2.6. Extremal words

Let $r \in \{a, b\}^\omega$ and let $k$ be a positive integer. We denote by $\min(r \mid k)$ and by $\max(r \mid k)$, respectively, the smallest and the greatest factor of $r$ of length $k$ in the lexicographic order.

**Definition 2.3.** Let $r \in \{a, b\}^\omega$. We say that the pair

$$(\min(r \mid k), \max(r \mid k))$$

is $k$ extremal for $r$.

It is clear that $\min(r \mid k)$ (resp. $\max(r \mid k)$) is a prefix of $\min(r \mid k+1)$ (resp. $\max(r \mid k+1)$). So we can define, by taking the limits, the infinite words

$$\min(r) = \lim_{k \to \infty} \min(r \mid k) \text{ and } \max(r) = \lim_{k \to \infty} \max(r \mid k).$$

**Remark 2.3.** For $a^\omega$ and $b^\omega$, we have $a^\omega = \min(a^\omega) = \max(a^\omega)$ and $b^\omega = \min(b^\omega) = \max(b^\omega)$; for any other $r \in \{a, b\}^\omega$, we have

$$\min(r) < \max(r).$$

**Definition 2.4.** Let $r \in \{a, b\}^\omega$. We say that the pair

$$(\min(r), \max(r))$$

of infinite words is extremal for $r$.

**Remark 2.4.** Given an infinite word $r$, if there is a positive integer $i$ such that, for each positive integer $k$, $r(i, i + k - 1) = \min(r \mid k)$ then $i$ is called a minimal rank for $r$. This definition was given in [16] for obtaining a very short proof of Shirshov’s Theorem.
2.7. Some preliminary lemmas

It is useful to note the following

Fact. Let \( r \) be in \( \{a, b\}^\omega \) such that \( (\min(r), \max(r)) = (ar, br) \). If \( k \) is a positive integer and \( u \) is the prefix of \( r \) of length \( k - 1 \) then

\[ (\min(r|k), \max(r|k)) = (au, bu). \]

Example 2.1. The following pairs

\[(a, b), (aa, ba), (aab, bab), (aaba, baba), (aabaa, babaa)\]

are respectively 1, 2, 3, 4 and 5-extremal for \( f \), the Fibonacci word. Note that if we cancel the first letter in the first or in the second component of each pair the remaining suffix is always a prefix of \( f \).

This phenomenon is very well known and holds not only for Fibonacci word but also for any proper standard Sturmian word \( s \). In fact, for this class of words we always have

\[ (\min(s), \max(s)) = (as, bs). \]

This result is proved in [1] (where the terminology of Lyndon words and infinite Christoffel words is used) and it is also mentioned in [12, Exercise 2.2.13].

We proved (see [20]) the following.

Proposition 2.1. Let \( r \) be in \( \{a, b\}^\omega \). The following condition are equivalent:

(a) \( r \) satisfies \( (\min(r), \max(r)) = (ar, br) \);

(b) \( r \) is proper standard Sturmian.

Remark 2.5. The proof of Proposition 2.1 is actually in this paper and it is “contained in” that of Proposition 3.1. Indeed, for sake of completeness and as [20] is unpublished, we repeat in the appropriate parts of Proposition 3.1 the arguments of the proof of Proposition 2.1.

Now before proving Lemmas 2.2–2.6, needed for proving our hereafter Proposition 3.1, we give an example of infinite words satisfying the hypotheses of these lemmas.

Example 2.2. An infinite word satisfying the conditions of the following lemmas is \( s = (wab)^\omega \), for \( w = aba \), i.e. the word \( s = (abaab)^\omega \). We have \( \min(s) = \min(s') = (awb)^\omega \) and \( \max(s) = \max(s') = (bwa)^\omega \), where \( s' = (wba)^\omega \).

Definition 2.5. Let \( s \in \{a, b\}^\omega \) and \( w \in F(s) \). We say that the property \( P(s, w) \) holds if \( (awb, bwa) \) is \( (|w| + 2) \)-extremal for \( s \).

The previous example shows that \( P((wab)^\omega, aba) \) holds. Similarly \( P((aabaab)^\omega, aabaa) \) holds. The aim of the following lemmas is precisely to show that if \( P(s, w) \) holds then \( wab \) is a conjugate of \( wba \) and \( s = (wab)^\omega \) or \( s = (wba)^\omega \). In other words, if \( P(s, w) \) holds then \( s \) is periodic Sturmian having form \( s = (wab)^\omega \) or \( s = (wba)^\omega \).
Lemma 2.2. Let $s \in \{a, b\}^*$ and $w \in F(s)$ such that $P(s, w)$ holds. If $|w| = 0$ then $s$ has no left special factors, no right special factors and $s = (ab)^0$ or $s = (ba)^0$. If $|w| > 0$ then for each integer $i, 1 \leq i \leq |w|$, we have:

(a) the prefix $w(1, i)$ is the unique left special factor of $s$ of length $i$;
(b) the suffix $w(|w| - i + 1, |w|)$ is the unique right special factor of $s$ of length $i$;
(c) $w(1, i) = w(|w| - i + 1, |w|)$.

Proof. The case $|w| = 0$ is immediate since $aa$ and $bb$ are not factors of $s$.

Now suppose $|w| > 0$. We proceed by induction. Let us prove that (a)–(c) hold for $i = 1$.

Clearly, $w(1)$ is a left special factor of $s$ for length 1. Suppose that $y \in \{a, b\}, y \neq w(1)$, is also left special. Then $aa, bb \in F(s)$ and this contradicts that $(aw(1), bw(1))$ is 2-extremal for $s$. Then only the first letter of $w$ is a left special factor of $s$ of length 1 and (a) holds for $i = 1$.

Clearly, the last letter of $w$, $w(|w|)$, is a right special factor of $s$ for length 1. Suppose that $y \in \{a, b\}, y \neq w(|w|)$ is also right special. Then $aa, bb \in F(s)$ and this contradicts that $(aw(1), bw(1))$ is 2-extremal for $s$. Then only the last letter of $w$ is a right special factor of $s$ of length 1 and (b) holds for $i = 1$.

Moreover, the first and the last letter of $w$ must coincide (in the opposite case $aa, bb \in F(s)$ and this contradicts that $(aw(1), bw(1))$ is 2-extremal for $s$) and clearly $w(1) = w(|w|)$. So (c) holds for $i = 1$.

This completes the proof that (a)–(c) hold for $i = 1$.

Now, we can suppose that (a)–(c) hold for $i, 1 \leq i < |w|$, and we will prove that they hold also for $i + 1$.

Let $u = w(1, i)$ and $ux, x \in \{a, b\}$, be prefixes of $w, |u| < |w|$. By induction, $u$ is the unique left special factor of $s$ of length $i$. Also $ux$ is left special and we have to prove that it is the unique left special factor of $s$ of length $i + 1$. Suppose that $v$ of length $i + 1$ is also left special. The prefix of $v$ of length $i$ must be $u$ because it is the unique left special factor of $s$ of length $i$ and so the last letter of $v$ must be $y, y \neq x$. Then $aua, bab \in F(s)$ and this contradicts that $(aux, bux)$ is $(i + 2)$-extremal for $s$. Then only $ux$ is a left special factor of $s$ of length $i + 1$. So (a) holds for $i + 1$.

Now, suppose that $u = w(|w| - i + 1, |w|)$ and $xu, x \in \{a, b\}$, are suffixes of $w, |u| < |w|$. By induction, $u$ is the unique right special factor of $s$ of length $i$ and $\tilde{u}$ is a prefix of $w$. Clearly, $xu$ is right special and we have to prove that it is the unique right special factor of $s$ of length $i + 1$. Suppose that $v$ of length $i + 1$ is also right special. The suffix of $v$ of length $i$ must be $u$ because it is the unique right special factor of $s$ of length $i$ and so the first letter of $v$ must be $y, y \neq x$. So $xu, yu \in F(s)$ with $x \neq y$ and consequently $u$ is left special and $u = \tilde{u}$, a prefix of $w$. Then

$$aua = a\tilde{u}a \in F(s)$$
and

$$bub = b\tilde{u}b \in F(s)$$

and this contradicts that $(auw(i + 1), bw(i + 1))$ is $(i + 2)$-extremal for $s$. Then only $xu$ is a right special factor of $s$ of length $|u| + 1$. So (b) holds for $i + 1$. 
Finally, by way of contradiction, suppose that \(ux = w(1, i)x\) is a prefix of \(w\) and that \(yw(w| - i + 1, |w|) = y\bar{u}\) is a suffix of \(w\) with \(x \neq y\). In all the possible cases (\(u\) is a palindrome or \(u\) is not a palindrome) there exists a prefix \(v\) of \(u\) such that \(awv(a\bar{v}\bar{w}\bar{b})\) belong to \(F(s)\) and we are in contradiction with the extremality of \((avw(|v| + 1), bvw(|v| + 1))\). So \(w(i + 1) = w(|w| - i)\) and consequently \(w(1, i + 1) = w(|w| - i, |w|)\). So (c) holds for \(i + 1\). This completes the proof that (a)--(c) hold for \(i + 1\). □

**Example 2.2 (continued).** Let \(s = (abaab)^o\). Then (resp. \(ab\), resp. \(aba\)) is the unique left special factor of \(s\) of length 1 (resp. 2, resp. 3) and (resp. \(ba\), resp. \(aba\)) is the unique right special factor of \(s\) of length 1 (resp. 2, resp. 3).

**Lemma 2.3.** Let \(s \in \{a, b\}^o\) and \(w \in F(s)\) such that \(P(s, w)\) holds. Then \(w\) is a palindrome.

**Proof.** Clearly, \(w\) is a palindrome in case \(|w| = 0\) and, by previous lemma (point (c)), this also holds in the case \(|w| > 0\). □

**Lemma 2.4.** Let \(s \in \{a, b\}^o\) and \(w \in F(s)\) such that \(P(s, w)\) holds. Then, for each \(i \geq |w| + 1\),

(a) no factor of \(s\) of length \(i\) is left special;

(b) no factor of \(s\) of length \(i\) is right special.

**Proof.** (a) Suppose that the factor \(v\) of \(s\) of length \(|v| = |w| + 1\) is left special. Then \(v = wx\) for some letter \(x\). So \(xwv(x) \in F(s)\). If \(x = a\) then \(awa \in F(s)\). As \(awa < awb\), we in contradiction with \(awb = \min(r \mid |w| + 2)\). If \(x = b\) then \(bwb \in F(s)\). As \(bwa < bwb\), we are in contradiction with \(bwa = \max(r \mid |w| + 2)\). As in each of the two possible cases we are in contradiction, \(v\) is not left special and \(s\) has no left special factor of length \(|w| + 1\). Since a prefix of a left special factor must also be left special, \(s\) has no left special factor of length \(i > |w|\).

(b) Similar to part (a). □

**Example 2.2 (continued).** Let \(s = (abaab)^o\). No factor of \(s\) of length equal or greater than 4 is left special or right special.

**Lemma 2.5.** Let \(s \in \{a, b\}^o\) and \(w \in F(s)\) such that \(P(s, w)\) holds. Then \(s\) has exactly \(|w| + 2\) distinct factors of length \(|w| + 2\).

**Proof.** The argument is based on the previous lemmas. When \(|w| = 0\), by Lemma 2.2, \(s\) has exactly two factors of length 2: \(ab, ba\). Let \(|w| > 0\). By Lemma 2.2, \(s\) has exactly one right special factor of length \(i\), \(1 \leq i \leq |w|\). It follows that \(|F(s) \cap A_{i+1}^+| - |F(s) \cap A_{i}^+| = 1\) for \(1 \leq i \leq |w|\). As \(s\) has clearly two factors of length 1, namely \(a, b\), it follows that \(s\) has \(|w| + 1\) factors of length \(|w|\). As \(w\) is right special, \(s\) has \(|w| + 2\) factors of length \(|w| + 1\). As \(s\) has no right special factors of length \(|w| + 1\), the number of factors of \(s\) of length \(|w| + 2\) remains \(|w| + 2\). □
Example 2.2 (continued). Let \( s = (abaab)^\omega \). The factors of \( s \) of length 5 are \( abaab, baaba, aabab, ababa, babaa \). They are the conjugates of \( abaab \).

The next lemma is an “ad hoc” variation of a classical well-known result: an infinite word over a finite alphabet whose right special factors are of bounded length is ultimately periodic.

Lemma 2.6. Let \( s \in \{a, b\}^\omega \) and \( w \in F(s) \) such that \( P(s, w) \) holds. Then \( w \) is a palindromic prefix of a proper standard Sturmian word and \( s \) is periodic Sturmian with least period equal to \( |w| + 2 \). Moreover, \( \min(s) = (awb)^\omega \) and \( \max(s) = (bwa)^\omega \).

Proof. Denote by \( u_i, 1 \leq i \leq |w| + 3 \), the factor of \( s \) beginning at rank \( i \) and having length \( |w| + 2 \).

By Pigeon Hole Principle and Lemma 2.5 there exist \( i, j \), \( i \neq j \), such that \( u_i = u_j \) and, for some prefixes \( v' \) and \( v'' \) of \( s \), we have:

\[
    s = v'u_is'
\]

and

\[
    s = v''u_js'',
\]

where \( s' \) and \( s'' \) are infinite words which are equal because in \( s \), by Lemma 2.4, there is no right special factor of length \( |w| + 2 \).

So \( s \) is ultimately periodic. Moreover, it is periodic (otherwise there will be factors of length greater than \( |w| \) which should be left special, in contradiction with Lemma 2.4). The period of \( s \) cannot be \( i \) with \( i \leq |w| \) (otherwise \( s \) could not have special factors of length \( i \), in contradiction with Lemma 2.2). The period of \( s \) cannot be \( |w| + 1 \) (otherwise \( awa, bwb \in F(s) \) which is impossible by extremality of \( (awb, bwa) \)).

So necessarily the period of \( s \) is at least \( |w| + 2 \). On the other hand, by the above construction, the period of \( s \) is at most \( |w| + 2 \). Consequently, the least period of \( s \) is exactly \( |w| + 2 \). Hence \( s \) can be presented under the form:

\[
    s = u'awbawb...awb...
\]

or the form

\[
    s = u''bwabwa...bwa...
\]

for some \( u' \) and \( u'' \), respectively, suffixes of \( awb \) and \( bwa \).

Since \( awb \) and \( bwa \) are factors of \( s \), we have that either \( awb \) is a factor of \( (bwa)^2 \) or \( bwa \) is a factor of \( (awb)^2 \). Consequently, \( awb \) and \( bwa \) are conjugate, i.e. there exist words \( w_1 \) and \( w_2 \) such that \( awb = w_1w_2 \) and \( bwa = w_2w_1 \).

We proved in [17] that if \( awb \) and \( bwa \) are conjugate then \( w \) is a palindromic prefix of a proper standard Sturmian word. Applying this here we get that \( s \) is a periodic Sturmian word.

Finally, being \( (awb, bwa) \) an extremal pair, we have \( \min(s) = (awb)^\omega \) and \( \max(s) = (bwa)^\omega \). \( \Box \)
Example 2.2 (continued). Let $s = (abaab)^\omega$. Note that $aba$ is a palindromic prefix of the Fibonacci word.

3. Results

Now we prove our

Proposition 3.1. Let $s$ be an infinite word on $\{a,b\}$. The following conditions are equivalent:

(i) $s$ is a proper standard Sturmian word or a periodic standard Sturmian word;

(ii) $as \leq \min(s) \leq \max(s) \leq bs$.

Proof. (i) $\Rightarrow$ (ii). We have to prove that a proper standard Sturmian word as well as a periodic standard Sturmian word verifies the inequalities in (ii). Let us distinguish the two possible cases.

Proper. Let $s = s(1)s(2) \ldots s(i) \ldots$ be a proper standard Sturmian word. The pair $(a,b)$ is 1-extremal for $s$.

Now, let $u = s(1)s(2) \ldots s(i)$ be the prefix of length $i$ of $s$. Put $c = s(i+1)$ and suppose by induction hypothesis, that

$$(au, bu)$$

is $(i+1)$-extremal for $s$. We will prove that

$$(auc, buc)$$

is $(i+2)$-extremal for $s$. By contradiction, assume that

$$(aud, bue)$$

is $(i+2)$-extremal for $s$. Suppose, for instance, $d \neq c$. As $aud < auc$ we have $d = a$ and $c = b$, thus $au$ is right special. As in a proper standard Sturmian word the mirror image of a right special factor is a prefix we have that $\tilde{au}$ is a prefix of $s$. Hence $a = c = b$. Contradiction.

Similarly, we reach a contradiction if we suppose that $e \neq c$.

So the minimal (resp. maximal) element in the lexicographic order of length $n$ is $av$ (resp. $bv$) with $v$ prefix of $s$ of length $n-1$. Hence $as = \min(s)$ and $bs = \max(s)$ and in conclusion (ii) is satisfied in the stronger form $as \leq \min(s) < \max(s) = bs$.

Periodic. Let us consider first the case $|Alph(s)| = 1$. If $s = a^{\omega}$ then $as = \min(s) = \max(s) < bs$. If $s = b^{\omega}$ then $as \leq \min(s) = \max(s) = bs$. In both cases (ii) is satisfied. So we can suppose that $s$ is a periodic standard Sturmian word with $|Alph(s)| = 2$, i.e., different from $a^{\omega}$ and from $b^{\omega}$.

As seen in Section 2.3, $s$ has form $(wab)^\omega$ or $(wba)^\omega$ where $w$ is a palindrome corresponding to the cutting points of the half-line $y = (q/p)x$ with the lines of the grid in the strict interior of the rectangle of vertices $(0,0), (p,0), (p,q), (0,q)$, for suitable relatively primes $p, q$. 


Suppose for instance \((wab)^\omega\) (the other case is similar).

Now, note that \(w\) is also a palindromic prefix of infinitely many proper standard Sturmian words. In particular when \(x\) is suitably near to \(q/p\), the cutting sequence of \(y = ax\) contains the factors of any large required length which are in the cutting sequence of \(y = (q/p)x\). More precisely, the cutting sequence, say \(s'\), of \(y = ax\) with \(x = q/p + \varepsilon\), for a suitable small positive \(\varepsilon\), begins with \((wab)^2\) and consequently \(s'\) contains as factors all the factors of \(s\) having length at most \(|w| + 2\) and in particular those having length at most \(|w| + 1\).

It is also well known that \(wab\) is a conjugate of \(wba\) (i.e. there exist words \(w_1\) and \(w_2\) such that \(awb = w_1 w_2\) and \(bwa = w_2 w_1\), see, for example, [11,12,17]) and consequently \((wab)^\omega = u'(wba)^\omega\), for some prefix \(u'\) of \(wab\), and so there exist prefixes \(u''\) and \(u'''\) of \(wab\) such that \(s = u''(awb)^\omega = u'''(bwa)^\omega\). Consequently \(awb, bwa \in F(s)\) and \(aw, bw \in F(s)\).

As \(aw, bw \in F(s)\) \(\cap \{a, b\}^{\lfloor |w| + 1}\) \(= F(s') \cap \{a, b\}^{\lfloor |w| + 1}\) and as the pair \((aw, bw)\) is \(((|w| + 1)\text{-extremal for } s')\) (see the proof given just above for the proper case), then \((aw, bw)\) is also \(((|w| + 1)\text{-extremal for } s)\).

Now, as in \(s\) two factors of length \(|w| + 2\) must contain the same number of occurrences of \(a\) and as \(awb, bwa \in F(s)\) we have \(awa, bwb \notin F(s)\) and consequently \((awb, bwa)\) is \(((|w| + 2)\text{-extremal for } s)\).

As \(awb\) is minimal and it is a factor of the periodic word \(s\) we have that \(\min(s) = (awb)^\omega\). Similarly considering \(bwa\) which is maximal, we have that \(\max(s) = (bwa)^\omega\).

As \(awa\) is a prefix of \(as\) and \(awb\) is a prefix of \(\min(s)\) we have \(as < \min(s)\). On the other hand, \(\max(s) = (bwa)^\omega = b(awb)^\omega = bs\). So (ii) is satisfied in the stronger form \(as < \min(s) < \max(s) = bs\).

The case \(s = (wba)^\omega\) leads to \(as = \min(s) < \max(s) < bs\).

This ends the proof of (i) \(\Rightarrow\) (ii).

(ii) \(\Rightarrow\) (i). Let us distinguish the following cases:

1. \(as = \min(s)\) and \(\max(s) = bs\);
2. \(as = \min(s)\) and \(\max(s) < bs\);
3. \(as < \min(s)\) and \(\max(s) = bs\);
4. \(as < \min(s)\) and \(\max(s) < bs\).

Case 1: \(as = \min(s)\) and \(\max(s) = bs\).

For any \(i \geq 1\), the prefix \(u\) of length \(i\) of \(s\) is clearly a left special factor of \(s\). We will prove that \(u\) is the unique left special factor of \(s\) of length \(i\).

By contradiction, suppose that \(v \in F(s)\), \(|v| = |u|, v \neq u\) and \(v\) is left special. Since \((au, bu)\) is \((i + 1)\text{-extremal for } s\), we have both \(au \leq av\) and \(bu \geq bv\). So \(u \leq v\) and \(u \geq v\). Contradiction.

So for each \(i\), \(s\) has a unique left special factor of length \(i\) which is its prefix of length \(i\).

So \(s\) is proper standard Sturmian.

Case 2: \(as = \min(s)\) and \(\max(s) < bs\).

First, note that from \(\max(s) < bs\) it follows that the letter \(a\) occurs in \(s\).

If \(s = a^\omega\) then it verifies Case 2 \((a \cdot d^\omega = a^\omega = \min(d^\omega) = \max(d^\omega) < b \cdot a^\omega)\) and it is clearly periodic standard Sturmian.

If \(s\) contains also an occurrence of \(b\) then \(\max(s) = bs\) for some \(s' \in \{a, b\}^\omega\).
Let $w$ be the longest common prefix of $s$ and $s'$. Then, for some $x, y \in \{a, b\}$, $y \neq x$, we have

- $awx$ is a prefix of $as = \min(s)$,
- $bwx$ is a prefix of $bs$,
- $bwy$ is a prefix of $bs' = \max(s)$.

As $\max(s) = bs' < bs$ we have $y < x$ and consequently $y = a$ and $x = b$. Then $(aw_{b}, bw_{a})$ is $(|w| + 2)$-extremal for $s$. By Lemma 2.6, $w$ is a palindromic prefix of a proper standard Sturmian word and $s$ is a periodic Sturmian word of period $|w| + 2$. Being $(aw_{b}, bw_{a})$ an extremal pair, we have also $\min(s) = (aw_{b})^{o}$ and $\max(s) = (bwa)^{o}$.

As $as = \min(s) = (aw_{b})^{o}$, we have that $wb$ is a prefix of $s$ and consequently $s$ is the periodic standard Sturmian word $(wba)^{o}$.

**Case 3:** $as < \min(s)$ and $\max(s) = bs$.

Similarly to Case 2 we first consider the case $s = b^{o}$ and then we prove that the other words satisfying this case are the periodic standard Sturmian words of the form $(wab)^{o}$.

**Case 4:** $as < \min(s)$ and $\max(s) < bs$.

We will prove that this case is impossible. The two inequalities imply $s \in \{a, b\}^{o} - \{a^{o}, b^{o}\}$. There exist two infinite words $s'$ and $s''$ satisfying $\min(s) = as'$ and $\max(s) = bs''$ and two finite words $w'$ and $w''$ which are respectively the longest common prefix of $s$ and $s'$ and longest common prefix of $s$ and $s''$. So:

- $aw'_{a}$ is a prefix of $as$,
- $aw''_{a}$ is a prefix of $as' = \min(s)$,
- $bw''_{a}$ is a prefix of $bs'' = \max(s)$,
- $bw''_{b}$ is a prefix of $bs$.

If $|w'| < |w''|$ then $w'a$ is a prefix of $w''$ and consequently $bw'a$ is prefix of $\max(s)$. As $aw'b$ is a prefix of $\min(s)$ then $P(s, w')$.

If $|w'| = |w''|$ then $w' = w''$. As $aw'_{b}$ is a prefix of $\min(s)$ and $bw'_{a}$ is a prefix of $\max(s)$ then $P(s, w')$. If $|w''| < |w'|$ then $w''b$ is a prefix of $w'$ and consequently $aw''_{b}$ is prefix of $\min(s)$. As $bw''_{a}$ is a prefix of $\max(s)$ then $P(s, w'')$.

In any case there exists a prefix $w$ of $s$ (indeed the shorter between $w'$ and $w''$) such that $P(s, w)$.

From Lemma 2.6, we deduce that $as < (aw_{b})^{o} < (bwa)^{o} < bs$. As $w$ is a prefix of $s$, the prefixes of $as$ and $(aw_{b})^{o}$ of length $|w| + 1$, as well as the prefixes of $bs$ and $(bwa)^{o}$ of the same length, coincide. Now, whatever happens for the value $a$ or $b$ of $s(|w| + 1)$, one of the two inequalities $as < (aw_{b})^{o}$ or $(bwa)^{o} < bs$ holds and the other fails. Contradiction. This ends the examination of Case 4.

Following the previous proposition, all the words over $\{a, b\}$ satisfying $as \leq \min(s) \leq \max(s) \leq bs$ are proper or periodic standard Sturmian words. More precisely we proved the following: let $s$ be an infinite word on $\{a, b\}$. Then: (i) $as = \min(s) < \max(s) = bs$ if and only if $s$ is a proper Standard Sturmian word; (ii) $as = \min(s) < \max(s) < bs$ if and only if $s$ is a periodic Sturmian word having the form $(wba)^{o}$ for some palindromic word $w$ such that $wab$ is a conjugate of $wba$; (iii) $as < \min(s) < \max(s) = bs$ if and only if $s$ is a periodic Sturmian word having the form $(wab)^{o}$ for some palindromic word $w$ such that $wab$ is a conjugate of $wba$; (iv) it is impossible that $as < \min(s) \leq \max(s) < bs$. 


Remark 3.1. Proposition 3.1 is a particular case of the next proposition as easily seen if we observe that on a two letter alphabet proper and periodic episturmian words are exactly the proper and periodic Sturmian words and that when A is a two letter alphabet there are only two lexicographic orders over \(A^*\).

**Proposition 3.2.** For an infinite word \(s\) over a finite alphabet \(A\) the following properties are equivalent:

(i) \(s\) is standard episturmian;

(ii) for any \(x \in A\) and lexicographic order \(<\) satisfying \(x = \min(A)\) we have \(xs \leq \min(s)\).

**Proof.** Denote by \(s_i\) the prefix of length \(i\) of the infinite word \(s\).

(i) \(\Rightarrow\) (ii). Here \(s\) is standard episturmian. With \(x = \min(A)\) suppose by contradiction \(xs > \min(s)\) and let \(n\) be minimal such that \(xs_{n-1} > \min(s \mid n)\). Then \(xs_{n-1} < \min(s \mid n-1)\). Set \(s_{n-1} = s_{n-2}y\), \(y \in A\), and \(w = \min(s \mid n-1)\). Then \(\min(s \mid n) = wz\) for some \(z \in A\). From \(xs_{n-2} \leq w\) and \(s_{n-2}y > wz\) it follows \(w = xs_{n-2}\) and \(y > z\).

As \(s_{n-2}y\) and \(s_{n-2}z\) are factors of \(s\), \(s_{n-2}\) is right special, hence is a palindromic prefix, \(u_1\) say, of \(s\). Here the notations for \(s\), its directive word, and so on, are as in the Preliminaries.

We can assume that among all standard episturmian words \(s\) and lexicographic orders \(<\) such that \(xs > \min(s)\) where \(x = \min(A)\) we have chosen one with minimal value of \(p\). Consider \(s^{(1)}\) given by \(\psi_{s_1}(s^{(1)}) = s\) and the sequence \(u_1^{(1)} = e, u_2^{(1)}, \ldots, u_i^{(1)}, \ldots\) of its palindromic prefixes. It easily follows from the definition of \(\psi_{s_1}\) that if \(dx_1e x_1\) is a prefix of \(s\) then \(uv\) is a prefix of \(s^{(1)}\) where \(d = \psi_{s_1}(u)\) and \(x_1e = \psi_{s_1}(v)\). By [4, Lemma 3] we know that \(u_p^1 = \psi_{s_1}(u_{p-1}^1)x_1\). It follows that \(u_p^1 < y\) is a prefix of \(s^{(1)}\) and that \(xu_{p-1}^1z \in F(s^{(1)})\). Indeed consider \(xu_{p-1}^1z\) for instance. If \(x \neq x_1\) then \(x_1xs_{n-2}z \in F(s)\). If \(z = x_1\) then \(x_1xu_p = \psi_{s_1}(v)\) for some factor \(v\) of \(s^{(1)}\) and \(v = xu_{p-1}^1x_1\). If \(z \neq x_1\) then \(x_1xu_p = \psi_{x_1}(v)\) for some factor \(v\) of \(s^{(1)}\), and \(v = xu_{p-1}^1z\). Case \(x_1 = x_1\) has similar proof.

Now, as \(xu_{p-1}^1y > xu_{p-1}^1z\) because \(y > z\), we get \(xu_{p-1}^1y > \min(s^{(1)} \mid (2 + |u_{p-1}^1|))\) and this contradicts the minimality of \(p\).

Thus \(xs \leq \min(s)\).

(ii) \(\Rightarrow\) (i). We will show first that \(s\) satisfying (ii) is episturmian. If, for some \(n \in \mathbb{N}_+\), \(s\) has at least two different left special factors \(u, v\) of length \(n - 1\), take \(n\) minimal with this property. Set \(u = u'y, v = v'z, y, z \in A\). Then \(u', v'\) are left special whence, by minimality of \(n, u', v'\) whence \(y \neq z\). We have \(au'y, bu'y, cu'z, du'z \in F(s)\), for some letters \(a, b, c, d \in \text{Alph}(s), a \neq b, c \neq d\). If \(u' \neq s_{n-2}\) write \(u' = wu'', s_{n-1} = ws''\), \(e, f \in A, e \neq f, As a \neq b\) suppose for instance \(f \neq a\) and consider an order \(<\) of \(A\) such that \(a = \min(A)\), \(e < f\). Thus \(as_{n-2} > au'\), a contradiction. Thus \(u' = s_{n-2}\). Now set \(s_{n-1} = s_{n-2}x, x \in A\). As \(x \neq z\) we can suppose \(y \neq x\), for instance. Consider an order \(<\) of \(A\) such that \(a = \min(A)\) and \(y < x\). Then \(as_{n-1} = as_{n-2}z > as_{n-2}y = au'y\) in contradiction with \(as \leq \min(s)\).

Consequently, \(u = v\). Thus for all \(n, s\) has at most one left special factor of length \(n\).

Moreover, the argument above shows that all left special factors of \(s\) are prefixes of \(s\).
Let us show now that \( F(s) \) is closed under reversal. Suppose by contradiction that for some \( u = xvy, x, y \in A, u \in F(s), \tilde{u} = y\tilde{v}x \notin F(s) \) and suppose \(|u|\) minimal. Then \( y\tilde{v}, \tilde{v}x \in F(s) \) whence, for some letter \( t \neq x, y\tilde{v}t \in F(s) \). Thus by minimality of \(|u|\), \( tv \in F(s) \). As \( xv \in F(s) \), \( v \) is left special hence is a prefix of \( s \).

Now suppose that \( \tilde{v}x \) is not a prefix of \( s \). As it is a factor of \( s \), there exists \( m \in A \) such that \( m\tilde{v}x \in F(s) \), and \( m \neq y \). Thus as \( m\tilde{v}, y\tilde{v} \in F(s), \tilde{v} \) is left special hence is a prefix of \( s \). Thus \( \tilde{v} = v \) whence in particular \( x \neq y \). Thus, as we have supposed that \( \tilde{v}x \) is not a prefix of \( s, v \tilde{a} \) is a prefix of \( s \) for some letter \( a \neq x \). Thus \( y\tilde{v}, m\tilde{v}, x\tilde{v}, v\tilde{a} \in F(s) \) and \( y\tilde{v}x \notin F(s) \). Recall \( t \neq x, m \neq y, a \neq x, y \neq x \). At least one of the two following cases is realized.

\( (c) a \neq m. \) Consider an order \( < \) such that \( m = \min(A), x < a \). Then \( m\tilde{v}a > m\tilde{v}x \) contradicting \( ms \leq \min(s) \).

\( (c') a \neq y. \) Then with order \( < \) such that \( x = \min(A), y < a \) we get a similar contradiction \( x\tilde{a} > x\tilde{v}y \).

Consequently, \( \tilde{v}x \) is a prefix of \( s \). As \( v \) is left special, hence is a prefix of \( s, v = \tilde{v} \). Thus \( x\tilde{v}y, y\tilde{v}t \in F(s) \). Consider an order \( < \) such that \( \min(A) = y \) and \( t < x \) (possible as \( y \neq x \) and \( t \neq x \)), we get \( y\tilde{v}x > y\tilde{v}t \) contradicting \( y\tilde{v}s \leq \min(s) \). Consequently \( F(s) \) is closed under reversal and \( s \) satisfies the definition of episturmian words. Moreover, as remarked above all its left special factors are prefixes of it, hence it is standard episturmian. \( \square \)

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