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Dynamics of shallow water waves with Boussinesq equation

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Abstract This paper obtains the solitary wave solutions of two different forms of Boussinesq equations that model the study of shallow water waves in lakes and ocean beaches. The tanh method is applied to solve the governing equations. The travelling wave hypothesis is also utilized to solve the generalized case of coupled Boussinesq equations, and, thus, an exact 1-soliton solution is obtained. The results are also supported by numerical simulations.

1. Introduction

The dynamics of shallow water waves, that are seen in various places like sea beaches, lakes and rivers, are governed by the Boussinesq Equation (BE). The Korteweg–deVries (KdV) equation that models shallow water waves is definitely very well known. However, the BE gives a much better approximation to such waves. There are two forms of the BE that will be addressed in this paper, and both are with cubic nonlinearity [1]. The soliton solutions will be obtained for these equations. These solutions will be extremely useful in carrying out further analysis in the context of shallow water waves that arises in the context of oceanography.

There are various mathematical techniques to solve these forms of BE. Some of the commonly used techniques are; the variational iteration method, the semi-inverse variational principle, the G'/G-expansion method, the exp-function method, the Riccati equation approach, Fan's F-expansion, and many others [2–12]. In this paper, the tanh method will be used for obtaining the closed form solution of some variants of the BE equation.

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The traveling wave solutions will also be used to integrate the nonlinearly coupled BE. These analytical solutions are supported by numerical simulations. The symbolic calculations are made using the symbolic programming package, Mathematica, version 8.0 [13].

2. Outline of tanh method

The method is applied to find out exact solutions of nonlinear differential equations:

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0 \tag{1}$$

where P is a polynomial of the variable u and its derivatives. Considering the transformation variable, $\xi = kx - \lambda t$, so that $u(x, t) = U(\xi)$, we obtain the following relations:

$$\frac{\partial}{\partial t} = -\lambda \frac{d}{d\xi}, \qquad \frac{\partial}{\partial x} = k\frac{d}{d\xi}, \qquad \frac{\partial^2}{\partial x^2} = k^2\frac{d^2}{d\xi^2}, \qquad \frac{\partial^3}{\partial x^3} = k^3\frac{d^3}{d\xi^3}. \tag{2}$$

Thus, Eq. (1) reduces to the Ordinary Differential Equation (ODE):

$$Q(U, U', U'', \ldots) = 0. \tag{3}$$

With Q being another polynomial. Eq. (3) is then integrated and the integration constant is chosen to be zero, in view of the localized solutions. However, the nonzero constants can be used and handled as well [14]. Now, finding the traveling wave

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solution to Eq. (1) is equivalent to obtaining the solution to the reduced ODE given by Eq. (3).

We introduce the tanh method in a similar manner to that undertaken earlier in [15–17]. The tanh method is based on a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function. For the tanh method, the first step is to introduce the new independent variable:

$$Y(x,t)=\tanh(\xi), \tag{4}$$

that transforms the differential operators as follows:

$$\frac{d}{d\xi}=(1-Y^2)\frac{d}{dY},$$
$$\frac{d^2}{d\xi^2}=-2Y(1-Y^2)\frac{d}{dY}+(1-Y^2)^2\frac{d^2}{dY^2},$$
$$\frac{d^3}{d\xi^3}=2(1-Y^2)(3Y^2-1)\frac{d}{dY}-6Y(1-Y^2)^2\frac{d^2}{dY^2}+(1-Y^2)^3\frac{d^3}{dY^3}.$$

The next step is that the solution is expressed in the form:

$$u(x,t)=U(\xi)=\sum_{i=0}^{m}a_iY^i, \tag{5}$$

where the parameter, m, can be found by balancing the highest-order linear term with the nonlinear terms in Eq. (2), and $k,\lambda,a_0,a_1,\ldots,a_m$ are to be determined. Substituting Eq. (5) into Eq. (2) will yield a set of algebraic equations for $k,\lambda,a_0,a_1,\ldots,a_m$ because all coefficients of Y have to vanish. From these relations, $k,\lambda,a_0,a_1,\ldots,a_m$ can be obtained. Having determined these parameters, knowing that m,n are positive integers, in most cases, and using Eq. (5), we obtain analytic solutions, $u(x,t)$, in a closed form. The tanh method seems to be a powerful tool in dealing with nonlinear physical models.

It needs to be noted that the tanh method is not a unique approach to carrying out the integration of these nonlinear evolution equations. There are several other powerful mathematical tools of integration that can be very easily and effectively applied to integrate these NLEEs. Some of these well studied approaches are; the G'/G-expansion method, the exp-function approach, the homotopy perturbation method, the variational iteration method, and the Adomian decomposition method. One of the most powerful methods is the Lie symmetry approach, which yields a lot of additional information in addition to retrieving solutions to these NLEEs, such as obtaining the conservation laws of these equations [18]. This paper will, however, focus on the tanh approach and the traveling wave solutions.

3. Examples

In this section, the technique of the tanh method, that was developed in the previous section, will be applied to various forms of BE. These studies will be conducted in the following couple of subsections.

3.1. Cubic Boussinesq equation

A well-known model of nonlinear dispersive waves, which was proposed by Boussinesq, is formulated as [1]:

$$u_{tt}-u_{xx}-u_{xxxx}+2(u^3)_{xx}=0. \tag{6}$$

The Boussinesq equation (6) that was considered by Wazwaz [1] describes motions of long waves in shallow water under gravity and in a one dimensional nonlinear lattice.

Note that a similar equation is considered in [19], but with u_{xxxt} instead of the term u_{xxxx}. In order to implement the Tanh method, the starting point is the traveling wave hypothesis as given by:

$$u(x,t)=U(\xi), \tag{7}$$

where:

$$\xi=kx-\lambda t. \tag{8}$$

Using Eqs. (7) and (8), the nonlinear Partial Differential Equation (PDE) Eq. (6) is transformed to the ODE:

$$(\lambda^2-k^2)U''-k^4U''''+6k^2(U^2U')'=0. \tag{9}$$

Integrating the previous equation twice and setting the integration constants to zero yields:

$$(\lambda^2-k^2)U-k^4U''+2k^2U^3=0. \tag{10}$$

Based on the tanh method, we assume that solution $U(\xi)$ is given by:

$$U(\xi)=\sum_{i=0}^{m}a_i\Upsilon^i,\qquad \Upsilon(x,t)=\tanh(\xi).$$

Hence, Eq. (10) reduces to:

$$(\lambda^2-k^2)U-k^4\left(-2Y(1-Y^2)\frac{dU}{dY}+(1-Y^2)^2\frac{d^2U}{dY^2}\right)+2k^2U^3=0. \tag{11}$$

To determine the index, m, we balance the linear term of the highest order with the highest order nonlinear terms. Therefore, in Eq. (11) we balance U^3 with U'', so that $3m=4+m-2$, and this gives us $m=1$. Hence, the expression for $U(\xi)$ now simplifies to:

$$U(\Upsilon)=a_0+a_1\Upsilon,\quad a_1\neq 0. \tag{12}$$

Substituting Eq. (12) into Eq. (11) and equating the coefficients of Y^i; $i=0;1;2;3$ to zero, leads to the following nonlinear system of algebraic equations:

$$(\lambda^2-k^2)a_0+2k^2a_0^3=0,$$
$$(\lambda^2-k^2)a_1+2k^4a_1+6k^2a_0^2a_1=0,$$
$$6k^2a_1^2a_0=0,$$
$$-2k^4a_1+2k^2a_1^3=0. \tag{13}$$

Solving system (13) gives:

$$a_0=0,\qquad a_1=\mp k,\qquad \lambda=\mp k\sqrt{1-2k^2}.$$

Hence, the 1-soliton solution of the starting Eq. (6) is given by:

$$u(x,t)=\mp k\tanh\left(k\{x\pm\sqrt{1}-2k^2t\}\right), \tag{14}$$

where:

$$|k|\leq\frac{1}{\sqrt{2}}.$$

However, for $k=1/2$

$$u(x,t)=\mp\frac{1}{2}\tanh\left(\frac{1}{2}\left\{x\pm\frac{t}{\sqrt{2}}\right\}\right). \tag{15}$$

The solitary wave and behavior of the solution $u(x,t)=\mp\frac{1}{2}\tanh\left(\frac{1}{2}\{x\pm\frac{t}{\sqrt{2}}\}\right)$ are shown in Figure 1 for $0\leq t\leq 1$ and $-10\leq x\leq 10$.

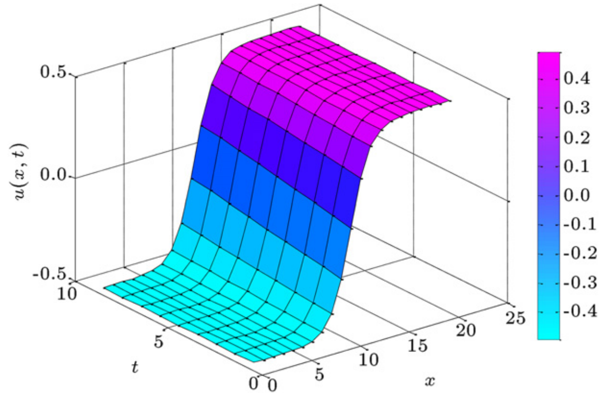

Figure 1: The solution $u(x,t) = \frac{1}{2}\tanh\left[\frac{1}{2}\left\{x + \frac{\tau}{\sqrt{2}}\right\}\right]$ for $0 \le t \le 1$ and $-10 \le x \le 10$.

3.2. Coupled Boussinesq equations

Consider the following $(1+1)$-dimensional nonlinear Boussinesq equations:

$$u_t + v_x + uu_x = 0, \tag{16}$$

$$v_t + (vu)_x + u_{xxx} = 0. \tag{17}$$

This coupled BE also arises in shallow water waves for two-layered fluid flow. This situation occurs when there is an accidental oil spill from a ship which results in a layer of oil floating above the layer of water. In this case, also, we assume that:

$$u(x,t) = U(\xi) = \sum_{i=0}^{m} a_i Y^i, \tag{18}$$

$$u(x,t) = U(\xi) = \sum_{i=0}^{m} a_i Y^i, \tag{19}$$

where:

$$\xi = kx - \lambda t. \tag{20}$$

The nonlinear system of PDEs (16)–(17) leads to the system of ODEs:

$$-\lambda \frac{dU}{d\xi} + k\frac{dV}{d\xi} + kU\frac{dU}{d\xi} = 0,$$

$$-\lambda \frac{dV}{d\xi} + kV\frac{dU}{d\xi} + kU\frac{dV}{d\xi} + k^3\frac{d^3U}{d\xi^3} = 0. \tag{21}$$

Now, introducing the same change of variable $Y = \tanh(\xi)$, the first equation in Eq. (21) implies:

$$-\lambda(1-Y^2)\frac{dU}{dY} + k(1-Y^2)\frac{dV}{dY} + kU(1-Y^2)\frac{dU}{dY} = 0, \tag{22}$$

while the second equation leads to:

$$-\lambda(1-Y^2)\frac{dV}{dY} + kV(1-Y^2)\frac{dU}{dY} + kU(1-Y^2)\frac{dV}{dY} + k^3\left[2(1-Y^2)(3Y^2-1)\frac{d}{dY} - 6Y(1-Y^2)^2\frac{d^2}{dY^2} + (1-Y^2)^3\frac{d^3}{dY^3}\right] = 0. \tag{23}$$

To determine the parameters, m and n, we balance the linear term of the highest-order with the highest order nonlinear terms. Hence, in Eq. (22) we balance V' with UU' (derivatives are with respect to Y), to obtain $2+n-1 = 2+m+m-1$, which yields $n = 2m$. Also, in Eq. (23), we balance U''' with UV', to obtain $6+m-3 = 2+m+n-1$, which, therefore, gives $n = 2, m = 1$. Hence, we can write the following expressions for $U(\xi)$ and $V(\xi)$:

$$U(\Upsilon) = a_0 + a_1\Upsilon, \quad a_1 \neq 0, \tag{24}$$

$$V(\Upsilon) = b_0 + b_1\Upsilon + b_2\Upsilon, \quad b_2 \neq 0. \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (22), then equating the coefficient of Y^i $(i = 0, 1, 2, 3)$ to zero and the removal of redundant equations, leads to the following system:

$$-\lambda a_1 + kb_1 + ka_0a_1 = 0,$$

$$2b_2 + a_1^2 = 0. \tag{26}$$

Again, substituting Eqs. (24) and (25) into Eq. (23), then equating the coefficient of Y, $(i = 0, 1, 2, 3, 4)$ and similarly removing redundant equations, leads to the following system:

$$-\lambda b_1 + kb_0a_1 + ka_0b_1 - 2k^3a_1 = 0,$$

$$-\lambda b_2 + ka_1b_1 + ka_0b_2 = 0, \tag{27}$$

$$a_1b_2 + 2k^2a_1 = 0.$$

Solving the nonlinear system of Eqs. (26) and (27), we get:

$$a_0 = \frac{\lambda}{k}, \qquad a_1 = \mp 2k, \qquad b_0 = 2k^2, \qquad b_1 = 0,$$

$$b_2 = -2k^2.$$

Then:

$$u(x,t) = \frac{\lambda}{k} \mp 2k\tanh(kx - \lambda t), \tag{28}$$

and:

$$v(x,t) = 2k^2 \sec h^2(kx - \lambda t). \tag{29}$$

The solitary wave and behavior for $\lambda = 0.5, k = 1$ of the solutions $u(x,t) = 1/2 + 2\tanh(x - t/2)$ and $v(x,t) = 2\,\mathrm{sech}^2(x - t/2)$ are shown in Figures 2 and 3, respectively, for $0 \le t \le 1$ and $-10 \le x \le 10$.

4. Travelling waves

In this section, the coupled BE, given by Eqs. (16)–(17), will be revisited in a more general setting. This coupled system will be rewritten in the form:

$$u_t + av_x + buu_x = 0, \tag{30}$$

$$v_t + c(vu)_x + buu_x = 0. \tag{31}$$

Applying the tanh method to the system (30)–(31) does not give us a solution. Therefore, we can conclude that there is no solution of the form (18)–(19) in this case.

Hence, we use the traveling wave hypothesis to carry out the integration of this form of coupled BE. The hypothesis is taken to be:

$$u(x,t) = g(x - \lambda t), \tag{32}$$

and:

$$v(x,t) = h(x - \lambda t), \tag{33}$$

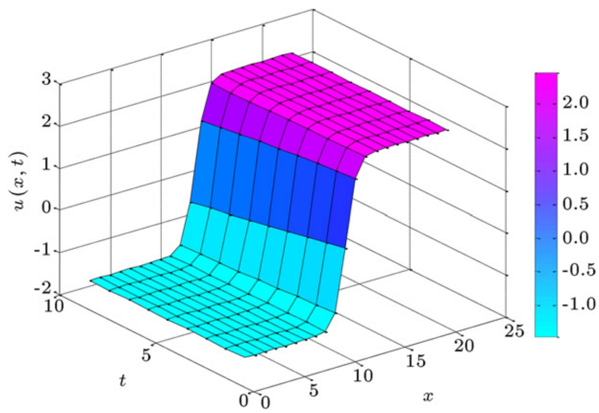

Figure 2: The solitary wave and behavior of the solutions $u(x, t) = 0.5 + 2\tanh(x - t/2)$ for $0 \le t \le 1$ and $-10 \le x \le 10$.

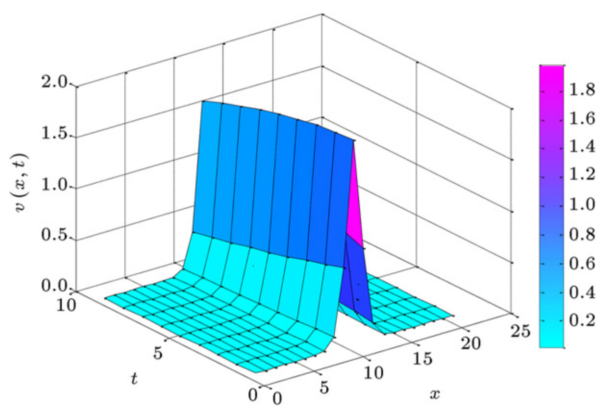

Figure 3: The solitary wave and behavior of the solutions $v(x, t) = 2\,\mathrm{sech}^2(x - t/2)$ for $0 \le t \le 1$ and $-10 \le x \le 10$.

where, in Eqs. (32) and (33), the functions, g and h, represent the solitary wave profile, while λ represents the velocity of the waves. Substituting Eqs. (32) and (33) into Eq. (30) gives the ODE:

$$-\lambda g' + ah' + bgg' = 0. \tag{34}$$

Here, we denoted $g' = dg/d\xi$ and $h' = dh/d\xi$, where:

$$\xi = x - \lambda t. \tag{35}$$

Integrating Eq. (34) and choosing the integration constant to be zero (since the search is for soliton solutions) gives:

$$h = \frac{2\lambda g - bg^2}{2a}. \tag{36}$$

Now, by replacing Eqs. (32) and (33) into Eq. (31), we obtain:

$$-\lambda h' + c(gh)' + dg''' = 0. \tag{37}$$

Then, integrating Eq. (37) and again taking the integration constant to be zero gives:

$$(cg - \lambda)h + dg'' = 0. \tag{38}$$

Thus, by virtue of Eqs. (36) and (38) reduces to:

$$2adg'' - cbg^3 + \lambda(2c + b)g^2 - 2\lambda^2 g = 0. \tag{39}$$

Multiplying both sides by g' and integrating once, we obtain:

$$(g')^2 = k^2 g^2[g^2 - \alpha g + \beta], \tag{40}$$

where:

$$k = \sqrt{\frac{bc}{4ad}}, \qquad \alpha = \frac{4}{3}\frac{\lambda(2c+b)}{cb},$$
$$\beta = \frac{4\lambda^2}{cb}. \tag{41}$$

Provided $abcd > 0$. Note that Eq. (40) reduces into the following two equations:

$$g' = kg\sqrt{g^2 - \alpha g + \beta}, \tag{42}$$
$$g' = -kg\sqrt{g^2 - \alpha g + \beta}. \tag{43}$$

Both of these equations can be easily solved by separating variables, and its solutions are given by:

$$g_1(\xi) = \frac{4\beta e^{\sqrt{\beta}(c_1 + k\xi)}}{(\alpha^2 - 4\beta)e^{2\sqrt{\beta}(c_1+k\xi)} + 2\alpha e^{2\sqrt{\beta}(c_1+k\xi)} + 1}, \tag{44}$$

and:

$$g_2(\xi) = \frac{4\beta e^{\sqrt{\beta}(c_1 + k\xi)}}{(\alpha^2 - 4\beta)e^{2\sqrt{\beta}(c_1+k\xi)} + 2\alpha e^{2\sqrt{\beta}(c_1+k\xi)} + e^{2\sqrt{\beta}(k\xi)}}, \tag{45}$$

where c_1 is an arbitrary constant. Corresponding expressions for $h_1(\xi)$ and $h_2(\xi)$ can be obtained directly from Eq. (36). Together with Eq. (41), they define two pairs of solutions;

$$u_i(x, t) = g_i(x - \lambda t),$$
$$v_i(x, t) = h_i(x - \lambda t), \qquad u_i(x, t) = g_i(x - \lambda t),$$
$$i = 0, 1, \tag{46}$$

of the initial system. In a similar way, we can obtain analogous solutions for $abcd < 0$. In Section 3.2, we considered the system (30)–(31) for $a = b = c = d = 1$. As we already noted, that system is also considered in [1]. In such a case, our solutions (by taking $c_1 = 0$) reduce to:

$$u_1(x, t) = \frac{16\lambda^2 e^{\lambda(x-\lambda t)}}{8\lambda e^{\lambda(x-\lambda t)} + 1},$$
$$v_1(x, t) = \frac{16\lambda^3 e^{\lambda(x-\lambda t)}}{(8\lambda e^{\lambda(x-\lambda t)} + 1)^2}, \tag{47}$$

and:

$$u_2(x, t) = \frac{16\lambda^2}{e^{\lambda(x-\lambda t)} + 8\lambda},$$
$$v_2(x, t) = \frac{16\lambda^3 e^{\lambda(x-\lambda t)}}{(e^{\lambda(x-\lambda t)} + 8\lambda)^2}. \tag{48}$$

These solutions are independent of those obtained in Section 3.2 (i.e. Eqs. (28) and (29)). However, for the purpose of simplification, we had to choose integration constants to be zero. For Eq. (40), the other choice would lead to the differential equation which contains a square root of the fourth-order polynomial, and is probably not integrable in closed form.

Now, let us consider when function $u(x,t)$, in the soliton solution Eq. (46), has a sech profile. By choosing $c_1 = 0$ and $\alpha = 0$ (i.e. $b = -2c$), Eqs. (44)–(45) reduce to:

$$g_1(\xi) = -\frac{\frac{8\lambda^2}{c^2} e^{\frac{\lambda}{\sqrt{ad}}\xi}}{\frac{8\lambda^2}{c^2} e^{\frac{2\lambda}{\sqrt{ad}}\xi} + 1},$$

$$g_2(\xi) = -\frac{\frac{8\lambda^2}{c^2} e^{\frac{\lambda}{\sqrt{ad}}\xi}}{e^{\frac{2\lambda}{\sqrt{ad}}\xi} + \frac{8\lambda^2}{c^2}}. \tag{49}$$

In order to have a sech profile, we choose $8\lambda^2 = c^2$, i.e. $\lambda = \mp c/2\sqrt{2}$. This gives:

$$g_1(\xi) = g_2(\xi) = -\frac{1}{2}\operatorname{sech}\left(\frac{c\xi}{2\sqrt{2ad}}\right). \tag{50}$$

And, hence:

$$u_1(\xi) = u_2(\xi) = -\frac{1}{2}\operatorname{sech}\left(\frac{c}{2\sqrt{2ad}}\left(x \mp \frac{c}{2\sqrt{2}}t\right)\right). \tag{51}$$

Corresponding expressions for $h_1(\xi) = h_2(\xi)$ and $v_1(x,t) = v_2(x,t)$ are obtained from Eq. (36), which are given by;

$$h_1(\xi) = h_2(\xi) = -\frac{c\operatorname{sech}\left(\frac{c\xi}{2\sqrt{2ad}}\right)\left[\sqrt{2} - 2\operatorname{sech}\left(\frac{c\xi}{2\sqrt{2ad}}\right)\right]}{8a}, \tag{52}$$

and:

$$v_1(\xi) = v_2(\xi) = \frac{-c\operatorname{sech}\left(\frac{c(\mp ct - 2\sqrt{2}x)}{8\sqrt{ad}}\right)\left[\sqrt{2} - 2\operatorname{sech}\left(\frac{c(\mp ct - 2\sqrt{2}x)}{8\sqrt{ad}}\right)\right]}{8a}. \tag{53}$$

Additionally, we also had to assume that $ad \neq 0$. If, however, $ad < 0$, then all sech functions reduce to sec functions which are periodic singular waves.

5. Conclusions

This paper studied the application of the tanh method to two different forms of Boussinesq equation. It is shown that these two variants have solutions that are expressed as a polynomial in the tanh function, where the independent variable is related to the traveling wave variable. Since that was not the case in the generalization of the second variant (coupled BE), we used a direct approach in this situation. Besides that, the tanh method is useful, since its application is usually not difficult and it is efficient whenever an equation permits a soliton solution, namely, a polynomial of the tanh function that is dependent on the traveling wave variable.

The future of this research holds a lot of promise. These results will be analyzed further. Later, the semi-inverse variational approach will be utilized to integrate these equations where closed form analytical soliton solutions will be obtained. Additionally, the soliton perturbation theory will be employed to obtain the adiabatic dynamics of the soliton parameters. Furthermore, these models will be studied with time-dependent coefficients. This represents situations that are closer to reality. The stochastic perturbation terms will be taken into consideration too. The results of this research will be reported in future publications.

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