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The Jordan Regular Ring Associated to a Finite JBW-Algebra

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INTRODUCTION

If A is a finite von Neumann algebra, then there exists a *-regular ring R (in the sense of von Neumann) whose lattice of principal left ideals is isomorphic to the lattice of projections of A (it is said that R coordinatizes A); this ring was constructed by Murray and von Neumann [14] by enlarging A to contain certain unbounded operators defined on dense linear subspaces of the Hilbert space on which A acts. By using an abstract version of the Murray–von Neumann construction, Berberian showed in [4] that a finite AW^*-algebra A is always contained in a continuous *-regular ring R such that R has no new projections. Later, Hafner [7] and Pyle [16] showed that the regular ring constructed by Berberian is the maximal ring of quotients of A.

The analogous problem for Rickart *-rings was considered by Handelman [9], who constructs, for a finite Rickart C^*-algebra A, a *-regular ring R containing A such that R has no new projections. Later, Ara and Menal [3] showed that the regular ring constructed by Handelman is the classical ring of quotients of A.

In the case of Jordan algebras, Ayupov [1] has given an enlargement of a JW-algebra, similar to the one of Murray and von Neumann for W^*-algebras, without going into the problem of regularity or ring of quotients in the finite case.

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In this paper, following the abstract construction of Berberian, we show that every finite *JBW*-algebra A is contained in a von Neumann regular Jordan ring $\hat{A}$ such that $\hat{A}$ has no new idempotents. Moreover, we show that every finite *JBW*-algebra has the common multiple property (non-associative analogous to the Ore condition) and that $\hat{A}$ is the (unique) total ring of quotients of A.

1. FINITE *JBW*-ALGEBRAS

Recall that a *JB*-algebra A is a real Jordan algebra which is also a Banach space with respect to a norm $\| \ \|$ having the following properties:

(i) $\|a^2\| = \|a\|^2$,

(ii) $\|a^2\| \leqq \|a^2 + b^2\|$,

(iii) $\|a \cdot b\| \leqq \|a\| \ \|b\|$, $\quad a, b \in A$.

When the *JB*-algebra A has an identity element then A is said to be a *unital JB-algebra.* We denote the identity by 1. For the general theory of *JB*-algebras, the reader is referred to [2, 8]. The treatment of *JB*-algebras in [8] is mainly based on [2], except that many proofs are altered. The standard reference for the algebraic theory of Jordan algebras is [10].

If φ is a positive operator on a unital *JB*-algebra A, it is clear that for any positive linear form f, the mapping $(x, y) \to \langle f, \varphi(x \cdot y)\rangle$ is a positive symmetric bilinear mapping. From the Cauchy–Schwarz inequality, it follows that $|\langle f, \varphi(x \cdot 1)\rangle|^2 \leqq \langle f, \varphi(x^2)\rangle \cdot \langle f, \varphi(1)\rangle$. Thus, $\varphi = 0$ if and only if $\varphi(1) = 0$. In particular, since for all elements a in A, U_a is a positive operator [2, Proposition 2.7], we have $U_{a_1} \dots U_{a_n} = 0$ iff $U_{a_1} \dots U_{a_n}(1) = 0$, with $a_1 \dots a_n$ in A. On the other hand, if a, b, c are elements in a unital *JB*-algebra A, it is not difficult to show, using the identity $[U_x(y)]^2 = U_x U_y(x^2)$ [8, Identity 2.40], that $U_a U_b(c^2) = 0$ if and only if $U_c U_b(a^2) = 0$. Thus, in a unital *JB*-algebra, the following are equivalent: (i) $U_a U_b U_c = 0$: (ii) $U_a U_b(c^2) = 0$; (iii) $U_c U_b(a^2) = 0$; (iv) $U_c U_b U_a = 0$. In particular, $U_a(b^2) = 0$ iff $U_b(a^2) = 0$.

From here, using the identity 2.33 in [2], we have the following

LEMMA 1.1. *If e is an idempotent element in a unital JB-algebra A, then for all a in A the following conditions are equivalent*: (i) $e \cdot a = 0$; (ii) $e \cdot a^2 = 0$; (iii) $U_e(a^2) = 0$.

A *JBW*-algebra A is a *JB*-algebra which is the dual of a Banach space. In [6; Lemma 1] it is shown that every *JBW*-algebra is unital.

In this paper the following theorem is essential.

THEOREM 1.2. *If a and b are elements in a unital JB-algebra (resp. JBW-algebra) A, then the closed (resp. w^*-closed) subalgebra of A generated by a, b and 1 is isometrically isomorphic to a JC-algebra (resp. JW-algebra).*

Proof. The assertion referring to JB-algebras is known [20, Proposition 2.1]. Now, if A is a JBW-algebra, the subalgebra B of A generated by a, b and 1 is (by the Shirshov–Cohn Theorem) a special Jordan algebra; since the product is w^*-continuous in each variable [18, Lemma 2.2], the w^*-closure of B is a w^*-closed subalgebra of A which satisfies all the s-identities. Thus, from [2, Lemma 9.4] it is a JC-algebra and by [18, Corollary 2.4] a JW-algebra.

Let A be a JBW-algebra; it is shown in [2, Proposition 4.9] that the set $P(A)$ of idempotent elements in A, with the ordering $e \leqq f$ iff $e \cdot f = e$ (for idempotent elements this ordering is equivalent to $e \leqq f$ iff $f - e \in A^2$), is a complete, orthomodular, and complemented lattice. We denote $e \vee f = \sup(e, f)$ and $a \wedge f = \inf(e, f)$, e and f in $P(A)$.

LEMMA 1.3. *Let A be a JBW-algebra and consider two idempotent elements e, f in A. Then $e \vee f$ and $e \wedge f$ are contained in any w^*-closed subalgebra of A containing e and f.*

Proof. Let B be the w^*-closed subalgebra of A generated by e and f; B is a JBW-algebra and so B has an identity element p. It is straightforward to verify that $e \vee f = p$; hence, if C is any w^*-closed subalgebra of A containing e and f, we have $e \vee f \in C$. We divide the second part of the proof into three steps:

(i) If C is a w^*-closed subalgebra of A containing e, f, and 1 then, since $e \wedge f = 1 - (1 - e) \vee (1 - f)$, it is obvious that $e \wedge f \in C$.

(ii) If C contains e and f, and is of the form $U_h(A)$, h idempotent, we have $e \cdot h = e$, $f \cdot h = f$, so $e, f \leqq h$. Thus, $(e \wedge f) \cdot h = e \wedge f$ and we have $e \wedge f \in C$.

(iii) Now, the w^*-closed subalgebra B of A generated by e and f is a w^*-closed subalgebra of the JBW-algebra $U_p(A)$, where $p = e \vee f$ is the identity element. Thus, by (i) $e \wedge f$ (in $U_p(A)$) $\in B$, and from (ii) we have $e \wedge f$ (in A) $= e \wedge f$ (in $U_p(A)$) $\in B$.

Following [2, p. 39] we say that two idempotents e, f in a JBW-algebra A are *equivalent* and write $e \sim f$ if there exists a finite family $s_1 \ldots s_n$ of symmetries in A such that $U_{s_n} \ldots U_{s_1}(e) = f$. Recall that a lattice L is called *modular* if $e \leqq f$ implies $(e \vee f) \wedge g = e \vee (f \wedge g)$, $e, f, g \in L$. Following [19], an idempotent e in a JBW-algebra is called *finite* if $f \leqq e$ and $f \sim e$ imply $f = e$. A JWB-algebra A is called *finite* if all idempotents in A are finite. Likewise, if A is a JWB-algebra, by a *center-valued trace* we shall mean a mapping $T: A \to Z(A)$ ($Z(A)$ denotes the center of A) such that:

T_1: T is linear.

T_2: $T(z \cdot a) = z \cdot T(a)$, a in A, z in $Z(A)$.

T_3: $a \geqq 0$ implies $T(a) \geqq 0$, $a \in A$.

T_4: $T(U_s(a)) = T(a)$, for a in A and s a symmetry in A.

T_5: $T(1) = 1$.

If $a > 0$ implies $T(a) > 0$, we say that T is *non-degenerate*. From [15, p. 371] it follows that T_4 is equivalent to

$$T_4': T(a \cdot (b \cdot c)) = T((a \cdot b) \cdot c) \text{ for all elements } a, b, c \text{ in } A.$$

It is not difficult to prove that the usual trace on the exceptional Jordan algebra M_3^8, with the obvious normalization, is a w^*-continuous and non-degenerate center-valued trace. (See [17].)

THEOREM 1.4. *Let A be a JBW-algebra. Then the following conditions are equivalent*:

(i) *A is a finite JBW-algebra.*

(ii) *$P(A)$ is a modular lattice.*

(iii) *A has a w^*-continuous and non-degenerate center-valued trace.*

(iv) *A has a non-degenerate center-valued trace.*

For the proof we shall need the following

LEMMA 1.5. *Let X be a non-empty compact set and let A be a unital JB-algebra. If f is a continuous function from X into A, then the following conditions are equivalent*:

(i) *f is a positive element in the JB-algebra $C(X, A)$.*

(ii) *$f(x) \geqq 0$ for all x in X.*

Proof. It is obvious that (i) implies (ii). Reciprocally, if f is an element in $C(X, A)$ such that $f(x) \geqq 0$ for all x in X, then $f(x)$ is a square in A, for all x in X. Thus, there exists a unique positive element y in A such that $y^2 = f(x)$. Hence, we can define a mapping $g: X \to A$ such that $g(x) = y$. It is obvious that $g^2 = f$ and $g \in B(X, A)$ (algebra of bounded functions from X into A); thus, f is a positive element in $B(X, A)$. Since $C(X, A) \subset B(X, A)$ and an element in a unital JB-algebra is positive if and only if it is positive in a closed subalgebra in which it is contained (see [2, pp. 14–15]), it follows that f is a positive element in $C(X, A)$.

Proof of the theorem. It is clear that (iii) implies (iv) and (iv) implies (i).

(i) $\Rightarrow$ (ii). Let e, f, g be idempotents in A with $e \leqq g$; set $a = g - e + \frac{1}{2}e$. Thus, $\{a^n\} \to^{w^*} g - e$. Let B be the w^*-closed subalgebra of A generated by a, f and 1; from Theorem 1.1, B is a JW-algebra which contains $g - e$; moreover, $g - \frac{1}{2}e = a \in B$. Thus, g and e are contained in B. Since it is clear that $p \sim q$ (in B) implies $p \sim q$ (in A), it follows that B is finite. Hence, by [19, Proposition 14], $(e \vee f) \wedge g = e \vee (f \wedge g)$ and so $P(A)$ is a modular lattice.

(ii) $\Rightarrow$ (iii). Using [18, Theorem 3.9] and [19, Theorem 26] the proof is reduced to proving the existence of a w^*-continuous and non-degenerate center-valued trace on $C(X, M_3^8)$ (algebra of continuous functions from a hyperstonean compact space X into M_3^8). Let f be an element in $C(X, M_3^8)$; if t is the usual (normalized) trace on M_3^8, the mapping $\hat{f}: x \to t(f(x))$ is a center-valued continuous function from X into the center of M_3^8. That is, $\hat{f}$ is an element in the center of $C(X, M_3^8)$; thus, $T: f \to \hat{f}$ is a center-valued mapping defined on $C(X, M_3^8)$. It is clear that T verifies T_1, T_2, T'_4, T_5; T_3 is an immediate corollary of Lemma 1.5. In order to prove the w^*-continuity of T we recall that the Banach space $C(X, M_3^8)$ is identified in a natural way with the dual of the Banach space $C(X)_* \otimes_\gamma M_{3*}^8$, where $C(X)_*$ and M_{3*}^8 denote the preduals of $C(X)$ and M_3^8, respectively, and γ denotes the greatest cross norm on the (algebraic) tensor product (see [18, pp. 362 and 375]). Now, it is not difficult to show that, in this identification, our trace T is just the transpose mapping of the operator T_* on $C(X)_* \otimes M_{3*}^8$ defined by $T_*(a \otimes b) = a \otimes t_*(b)$, $a \in C(X)_*$, $b \in M_{3*}^8$, where t_* denotes the pretranspose mapping of the usual (normalized) center-valued operator t on M_3^8. Therefore T is w^*-continuous.

2. The Support

If a is an element in a JBW-algebra A, the w^*-closed subalgebra B of A generated by a is a JBW-algebra and so B has an identity element e and obviously $e \cdot a = a$. If f is an idempotent in A such that $f \cdot a = a$, then a is contained in the subspace $A_1^{(f)}$ of the Peirce decomposition of A relative to f (see [10, pp. 118–119]). $A_1^{(f)}$ is a w^*-closed subalgebra of A; thus, B is contained in $A_1^{(f)}$ and so $e \in A_1^{(f)}$; thus, $e \cdot f = e$; that is, $e \leqq f$.

Thus, we have shown that for every element a in a JBW-algebra A there exists a smallest idempotent e such that $e \cdot a = a$; e is called the *support* of a, and denoted by $s(a)$. It is clear that $1-s(a)$ is the largest idempotent e such that $e \cdot a = 0$. From Lemma 1.1 it is trivial that $s(a) = s(a^2)$ for all $a \in A$. Moreover, since the operators U_x are positive we have that if $a \leqslant b$ are positive elements, then $s(a) \leqslant s(b)$ explain.

PROPOSITION 2.1. *If a is an element in a JBW-algebra A, such that $0 \leqslant a \leqslant 1$, then $s(a) = w^*\text{-}\lim\{1-(1-a)^n\}$.*

Proof. Since $0 \leqq a \leqq 1$, it is clear that $\{1-(1-a)^n\}$ is a norm bounded increasing sequence; thus, $\{1-(1-a)^n\}$ w^*-converges to $y = \sup\{1-(1-a)^n\}$ [2, Lemma 4.1]; hence, $\{(1-a)^n\} \to^{w^*} 1-y$. From [2, Lemma 4.1] it follows that $\{(1-a)^{2n}\} \to^{w^*} (1-y)^2$; thus, $(1-y)^2 = 1-y$ and y is an idempotent. On the other hand, $\{(1-a)\cdot(1-a)^n\} = \{(1-a)^{n+1}\} \to^{w^*} 1-y$; thus, $(1-a)\cdot(1-y) = 1-y$ and so $y \cdot a = a$. Now, if p is an idempotent in A such that $p \cdot a = a$, we have $(1-p)\cdot a = 0$, which implies $(1-p)\cdot a^n = 0$ for all n in N, and so $[1-(1-a)^n]\cdot(1-p) = 0$ for all n; since $\{1-(1-a)^n\} \to^{w^*} y$ it follows that $y \cdot p = y$, and so $y \leqq p$. This completes the proof.

PROPOSITION 2.2. *Let a, b be elements in a finite JBW-algebra A. Denote by T the w^*-continuous and non-degenerate center-valued trace on A. Then,*

$$T(s(U_a(b^2))) = T(s(U_b(a^2))).$$

Proof. Since the trace form is associative, $T(U_a(b)\cdot c) = T(b \cdot U_a(c))$. In particular $T(a^2 \cdot b) = T(U_a(b))$. On the other hand, from the Shirshov–Cohn Theorem, it follows that $U_a\, U_b(U_b(a^2))^{n-1} = (U_a(b^2))^n$ for all n in N, a, b in A. Then, $T((U_b(a^2))^n) = T(U_b(a^2)\cdot(U_b(a^2))^{n-1}) = T(a^2 \cdot U_b(U_b(a^2))^{n-1}) = T(U_a\, U_b(U_b(a^2))^{n-1}) = T((U_a(b^2))^n)$ for all n in N, a, b in A. Moreover, we can assume without loss of generality that $\|U_a(b^2)\| \leqq \|U_b(a^2)\| \leqq 1$. Now, the required equality follows from Proposition 2.1 and the w^*-continuity of the trace.

3. THE CONSTRUCTION OF THE RING

Let A be a finite JBW-algebra. Following Berberian [4], if (e_n) is a sequence of idempotents in A, $e_n \uparrow$ means that $e_n \leqq e_{n+1}$. If moreover $\sup(e_n) = e$, we write $e_n \uparrow e$. In case $e_n \uparrow 1$ we say that (e_n) is an SDD (Strongly Dense Domain).

LEMMA 3.1. *If (e_n), (f_n),..., (k_n) are* SDD's, *then $(e_n \wedge f_n \wedge \cdots \wedge k_n)$ is* a SDD.

Proof. First, note that if e and f are idempotents in a JBW-algebra A then, by Theorem 1.2, Lemma 1.3, and [19, Corollary 8], the Parallelogram law $e \vee f - e \sim f - e \wedge f$ holds. Thus, $T(e) + T(f) = T(e \vee f) + T(e \wedge f)$. Now, to prove the Lemma it is sufficient to consider two SDD's (e_n), (f_n). Set $g_n = e_n \wedge f_n$, $g = \sup(g_n)$; evidently $g_n \uparrow g$; since $1 - g \leqq 1 - g_n = (1 - e_n) \vee (1 - f_n)$, we have $T(1 - g) \leqq T(1 - e_n) + T(1 - f_n)$. Since T is w^*-continuous and non-degenerate we have $g = 1$.

DEFINITION 3.2. Let A be a JBW-algebra and let $\{x_n\}$ be a sequence of elements in A; a SDD (e_n) is said to be *admissible* for $\{x_n\}$ if

$$m \leqq n \text{ implies } (x_n - x_m) \cdot e_m = 0.$$

Our purpose is to show that the set A_C of the sequences $\{x_n\}$ in A for which there exists an admissible SDD is a subalgebra of the Jordan algebra of all sequences in A. It is obvious that all constant sequences are in A_C.

LEMMA 3.3. *If* $\{x_n\}$, $\{y_n\}$ *are in* A_C *and* $\lambda \in R$, *then* $\{x_n + y_n\}$ *and* $\{\lambda x_n\}$ *are in* A_C.

Proof. It is clear that if e and f are idempotents in a unital JB-algebra A and $e \leqslant f$, the subalgebra of A generated by e, f, and 1 is $R1 + Re + Rf$ which, from [2, Lemma 2.11], is strongly associative. Hence, by [5, Satz 3.7], we have $U_e U_f = U_e$. From this observation and Lemma 1.1 it follows that if $f \cdot a = 0$, $e \leqq f$, then e.a $= 0$. Thus, if (e_n) is an admissible SDD for $\{x_n\}$ and (f_n) is an admissible SDD for $\{y_n\}$ then $(e_n \wedge f_n)$ is an admissible SDD for $\{x_n + y_n\}$. The second part of the Lemma is trivial.

The discussion of product requires another concept:

DEFINITION 3.4. *If* $x \in A$ *and* e *is an idempotent in* A, *we write*

$$x^{-1}(e) = 1 - s(U_x(1 - e)).$$

It is clear that $x^{-1}(e)$ is the largest idempotent f such that $f \cdot U_x(1 - e) = 0$.

LEMMA 3.5. *Let* A *be a finite* JBW-*algebra. If* $x \in A$ *and* e *is an idempotent in* A, *then*

$$T(e) \leqq T(x^{-1}(e)).$$

Proof. $T(1 - x^{-1}(e)) = T(s(U_x(1 - e)))$; from Proposition 2.2 it follows that $T(1 - x^{-1}(e)) = T(s(U_{1-e}(x^2)))$, since it is clear that $s(U_{1-e}(x^2)) \leqq 1 - e$, we conclude $T(1 - x^{-1}(e)) \leqq T(1 - e)$, and so $T(e) \leqq T(x^{-1}(e))$.

LEMMA 3.6. *Let* (e_n) *be an admissible* SDD *for the sequence* $\{x_n\}$. *Then, for any* SDD (f_n), $(e_n \wedge x_n^{-1}(f_n))$ *is an* SDD *also.*

Proof. If a and b are elements in any Jordan algebra J, and e is an idempotent in J, it is not difficult, using the Peirce decomposition of J relative to the idempotent e, to show that if $a \cdot e = b \cdot e$ then $U_a(e) = U_b(e)$ (see [10, pp. 118–119]). Now, set $g_n = e_n \wedge x_n^{-1}(f_n)$. If $m \leqq n$ we have $g_m \leqq 1 - s(U_{x_m}(1 - f_m))$ (Definition 3.4); thus, $g_m \cdot U_{x_m}(1 - f_m) = 0$ and so $U_{g_m}(U_{x_m}(1 - f_m)) = 0$; hence, $U_{1-f_m} U_{x_m}(g_m) = 0$ and so $U_{1-f_n} U_{1-f_m} U_{x_m}(g_m) = 0$. Since $f_m \leqq f_n$, we have $U_{1-f_n} U_{x_m}(g_m) = 0$. Since (e_n) is an admissible SDD for $\{x_n\}$ we have $(x_n - x_m) \cdot e_m = 0$ and so $(x_n - x_m) \cdot g_m = 0$. Thus, $U_{x_m}(g_m) = U_{x_n}(g_m)$; hence, we have $U_{1-f_n} U_{x_n}(g_m) = 0$ and so $U_{g_m} U_{x_n}(1 - f_n) = 0$. From [2, Proposition 2.8] it follows that $g_m \cdot U_{x_n}(1 - f_n) = 0$ and so $g_m \leqq 1 - s(U_{x_n}(1 - f_n))$. That is, $g_m \leqq x_n^{-1}(f_n)$; hence, $g_n \uparrow$.

Moreover, from Lemma 3.5, we have $T(f_n) \leqq T(x_n^{-1}(f_n))$ for all n; indeed, $1 - g_n = (1 - e_n) \vee (1 - x_n^{-1}(f_n))$; thus, $T(1 - g_n) \leqq T(1 - e_n) + T(1 - x_n^{-1}(f_n)) \leqq T(1 - e_n) + T(1 - f_n)$. It follows that $g_n \uparrow 1$.

PROPOSITION 3.7. *If* $\{x_n\}$ *and* $\{y_n\}$ *are in* A_C, *then* $\{x_n \cdot y_n\}$ *is in* A_C.

Proof. If (e_n) is an admissible SDD for $\{x_n\}$ and (f_n) is an admissible SDD for $\{y_n\}$, we shall prove that $([f_n \wedge y_n^{-1}(e_n)] \wedge [e_n \wedge x_n^{-1}(f_n)])$ is an admissible SDD for $\{x_n \cdot y_n\}$. Setting $k_n = [f_n \wedge y_n^{-1}(e_n)] \wedge [e_n \wedge x_n^{-1}(f_n)]$, it is clear, by Lemmas 3.6 and 3.1, that (k_n) is an SDD. Moreover, if $m \leqq n$ we have

(i) $(x_n - x_m) \cdot e_m = 0$,

(ii) $(y_n - y_m) \cdot f_m = 0$,

(iii) $k_m \leqq 1 - s(U_{x_m}(1 - f_m))$,

(iv) $k_m \leqq 1 - s(U_{y_m}(1 - e_m))$,

(v) $k_m \leqq e_m$,

(vi) $k_m \leqq f_m$.

Since $(x_n - x_m) \cdot e_m = 0$, we have $(x_n - x_m)^2 \cdot e_m = 0$, and so $(1 - e_m) \cdot (x_n - x_m)^2 = (x_n - x_m)^2$. Thus, $U_{1-e_m}((x_n - x_m)^2) = (x_n - x_m)^2$; since $k_m \leqq 1 - s(U_{y_m}(1 - e_m))$ we have $k_m \cdot U_{y_m}(1 - e_m) = 0$; thus, $U_{k_m} U_{y_m}(1 - e_m) = 0$ and $U_{k_m} U_{y_m} U_{1-e_m} = 0$. So $U_{k_m} U_{y_m} U_{1-e_m}((x_n - x_m)^2) = 0$; hence, $U_{k_m} U_{y_m}((x_n - x_m)^2) = 0$. Thus, by [2, Proposition 2.8], we have $k_m \cdot U_{y_m}((x_n - x_m)^2) = 0$, and so $k_m \leqq 1 - s(U_{y_m}((x_n - x_m)^2))$. In an analogous way, we obtain $k_m \leqq 1 - s(U_{x_m}((y_n - y_m)^2))$. Since $k_m \leqq e_m$, $k_m \leqq f_m$, we can write

(i) $k_m \leqq 1 - s(x_n - x_m)$,

(ii) $k_m \leqq 1 - s(y_n - y_m)$,

(iii) $k_m \leqq 1 - s(U_{y_m}((x_n - x_m)^2))$,

(iv) $k_m \leqq 1 - s(U_{x_m}((y_n - y_m)^2))$.

Let B be the w^*-closed subalgebra of A generated by $x_n - x_m$, y_m, and 1. By Theorem 1.2., B is a JW-algebra which contains $1 - s(x_n - x_m)$ and $1 - s(U_{y_m}((x_n - x_m)^2))$ and therefore $h = (1 - s(x_n - x_m)) \wedge (1 - s(U_{y_m}(x_n - x_m)^2))$ (Lemma 1.3); we have $h \cdot (x_n - x_m) = 0$ and $h \cdot U_{y_m}((x_n - x_m)^2) = 0$ which, in B, are equivalent to $h(x_n - x_m) = 0$, $hy_m(x_n - x_m) = 0$ where the juxtaposition denotes the associative product; thus, $h(x_n - x_m)\, y_m = 0 = hy_m(x_n - x_m)$ and so $h((x_n - x_m) \cdot y_m) = 0 = ((x_n - x_m) \cdot y_m)\, h$. Hence, $((x_n - x_m) \cdot y_m) \cdot h = 0$. Since $k_m \leqq h$, we have $((x_n - x_m) \cdot y_m) \cdot k_m = 0$. Similarly, we obtain $((y_n - y_m) \cdot x_m) \cdot k_m = 0$; hence $((x_n - x_m) \cdot y_m + (y_n - y_m) \cdot x_m) \cdot k_m = 0$. Since $(x_n - x_m) \cdot k_m = 0 = (y_n - y_m) \cdot k_m$, we have $((x_n - x_m) \cdot (y_n - y_m)) \cdot k_m = 0$; therefore, $((x_n - x_m) \cdot y_m + (y_n - y_m) \cdot x_m + (x_n - x_m) \cdot (y_n - y_m)) \cdot k_m = 0$ and so $(x_n \cdot y_n - x_m \cdot y_m) \cdot k_m = 0$. This completes the proof.

Definition 3.8. If $\{x_n\}$ and $\{y_n\}$ are contained in the Jordan algebra A_C, we say that $\{x_n\}$ and $\{y_n\}$ are *equivalent*, written $\{x_n\} \equiv \{y_n\}$, if there exists a SDD (g_n) such that $(x_n - y_n) \cdot g_n = 0$ for all n.

The equivalent is said to be "implemented" via the SDD (g_n). Reflexivity and symmetry of equivalence are obvious. Moreover, if $\{x_n\} \equiv \{y_n\}$ via (g_n) and $\{y_n\} \equiv \{z_n\}$ via (h_n), then $\{x_n\} \equiv \{z_n\}$ via $(g_n \wedge h_n)$. Thus, the relation defined above is an equivalence relation.

Proposition 3.9. *The set* $A_{CO} = \{\{x_n\} \in A_C : \{x_n\} \equiv 0\}$ *is an ideal of* A_C.

Proof. It is clear that if $\{x_n\} \equiv 0$ via (h_n) and $\{y_n\} \equiv 0$ via (k_n), then $\{x_n + y_n\} \equiv 0$ via $(h_n \wedge k_n)$. It is obvious that if $\{x_n\} \in A_{CO}$, $\lambda \in R$ then $\{\lambda x_n\} \in A_{CO}$. If $\{x_n\} \equiv 0$ via (e_n) it is clear that (e_n) is an admissible SDD for $\{x_n\}$; let $\{y_n\}$ be an element in A_C and let (f_n) be an admissible SDD for $\{y_n\}$; set $g_n = (f_n \wedge y_n^{-1}(e_n)) \wedge (e_n \wedge x_n^{-1}(f_n))$; (g_n) is a SDD and we have

(i) $x_m \cdot e_m = 0$,

(ii) $(y_n - y_m) \cdot f_m = 0$,

(iii) $g_m \leqq 1 - s(U_{y_m}(1 - e_m))$,

(iv) $g_m \leqq 1 - s(U_{x_m}(1 - f_m))$,

(v) $g_m \leqq e_m$,

(vi) $g_m \leqq f_m$,

thus (see the proof of Proposition 3.7) we obtain $(x_m \cdot y_m) \cdot g_m = 0$ for all m. This completes the proof.

The quotient Jordan algebra A_C/A_{CO} is denoted $\hat{A}$. We denote the elements of $\hat{A}$ by capitals letters X, Y,.... If $x \in A$, we write $\bar{x} = \{x\} + A_{CO}$. The mapping $x \to \bar{x}$ is injective. For, if (e_n) is a SDD such that $x \cdot e_n = 0$ for all n, since the product is w^*-continuous, we have $x = 0$. Thus, we have the following.

THEOREM 3.10. *Let A be a finite JBW-algebra. Define $\hat{A}$ as indicated above. Then,*

(i) *$\hat{A}$ is a real Jordan algebra with identity element $\bar{1}$.*

(ii) *The mapping $x \to \bar{x}$ is an isomorphism from A onto a subalgebra $\bar{A}$ of $\hat{A}$. $\hat{A}$ is said to be the regular Jordan ring associated to A.*

Remark. $\hat{A}$ is called regular for, as we shall see later, $\hat{A}$ is regular in the sense of von Neumann. In view of the above Theorem we will identify A with $\bar{A}$ and therefore the notation $\bar{x}$, for x in A, will be avoided.

We conclude this section by showing the behaviour of the w^*-closed subalgebras of A with respect to the extension constructed above.

PROPOSITION 3.11. *Let A be a finite JBW-algebra and let B be a w^*-closed subalgebra of A such that $1 \in B$. Then, B is a finite JBW-algebra and the mapping $\{x_n\} + B_{CO} \to \{x_n\} + A_{CO}$ is an isomorphism from $\hat{B}$ in $\hat{A}$.*

Proof. It is clear that B is a finite JBW-algebra and every SDD (e_n) in B is also a SDD in A. Thus, $\{x_n\} + B_{CO} \to \{x_n\} + A_{CO}$ is a well-defined mapping; it is obvious that it is a ring homomorphism. Suppose $\{x_n\} \in B_C$ and $\{x_n\} \in A_{CO}$; then, there exists a SDD (k_n) (in A) such that $x_n \cdot k_n = 0$ for all n. Since B is a w^*-closed subalgebra of A, we have $s(x) \in B$ for all $x \in B$. Thus, $1 - s(x_n) \in B$ for all n; set $g_n = 1 - s(x_n)$. Denote $f_m = \inf\{g_n, n \geq m\}$. It is easy to see that $k_n \leq f_n \leq g_n$ and so (f_n) is a SDD (in B) such that $x_n \cdot f_n = 0$ for all n; hence, $\{x_n\} \in B_{CO}$ and the mapping is injective.

Remark. In what follows we will consider $\hat{B} \subset \hat{A}$.

Let C be a finite W^*-algebra; in particular, C is a finite AW^*-algebra. Let $\hat{C}$ be the *-regular ring associated to C by Berberian's construction. The set Sym(C) of self-adjoint elements in C is a finite JBW-algebra. Let $\widehat{\mathrm{Sym}(C)}$ be the Jordan regular ring associated to Sym(C). Then, we obtain the following result:

PROPOSITION 3.12. *If C is a finite W^*-algebra and $\hat{C}$ denotes its associated *-regular ring, then* $\mathrm{Sym}(\hat{C}) = \widehat{\mathrm{Sym}(C)}$.

Proof. It is clear that our construction of the Jordan regular ring associated to a finite JBW-algebra A is similar, except for notation, to Berberian's construction of the *-regular ring associated to a finite AW^*-algebra (note that for a projection e and a self-adjoint element x in a C^*-algebra $x \cdot e = 0$ is equivalent to $xe = ex = 0$). The Proposition is a consequence of the characterization of self-adjoint elements of the *-regular ring associated to a finite AW^*-algebra [4, Lemma 3.2].

4. THE REGULARITY OF THE RING

PROPOSITION 4.1. *If $X = \{x_n\} + A_{CO}$ and the x_n are invertible for all n, then X is invertible and $X^{-1} = \{x_n^{-1}\} + A_{CO}$.*

Proof. First, note that if e and x are elements in a JBW-algebra A, where e is idempotent and x is not a zero divisor (that is, U_x is injective), then $s(U_e(x^2)) = e$; for, it is clear that $e \cdot U_e(x^2) = U_e(x^2)$ and moreover if f is an idempotent such that $f \cdot U_e(x^2) = U_e(x^2)$, then $(1-f) \cdot U_e(x^2) = 0$, and so $U_{1-f}\, U_e(x^2) = 0$. Thus, $U_x\, U_e(1-f) = 0$ and so $U_e(1-f) = 0$; hence, $e \cdot (1-f) = 0$. That is, $e \leqq f$.

Now, let (e_n) be an admissible SDD for $\{x_n\}$. Set $f_n = s(U_{x_n}(e_n))$. If $m \leqq n$, we have $(x_n - x_m) \cdot e_m = 0$, so $U_{x_m}(e_m) = U_{x_n}(e_m) \leqq U_{x_n}(e_n)$; it follows that $f_n \uparrow$. Moreover, by Proposition 2.2, $T(f_n) = T(s(U_{x_n}(e_n))) = T(s(U_{e_n}(x_n^2))) = T(e_n)$; so $f_n \uparrow 1$.

If $m \leqq n$, set $e_0 = 1 - s(x_n - x_m)$. By Theorem 1.2, the w^*-closed subalgebra B of A generated by x_n, x_m, and 1 is a JW-algebra which contains e_0; moreover, from [2, Proposition 2.4] it follows that x_n^{-1} and x_m^{-1} lie in B, in which it is not difficult to show that $(x_n^{-1} - x_m^{-1}) \cdot s(U_{x_m}(e_0)) = 0$. Since $(x_n - x_m) \cdot e_m = 0$ we have $e_m \leqq e_0$; so $U_{x_m}(e_m) \leqq U_{x_m}(e_0)$. Thus, $s(U_{x_m}(e_m)) \leqq s(U_{x_m}(e_0))$ and so $(x_n^{-1} - x_m^{-1}) \cdot f_m = 0$. Hence, (f_n) is an admmissible SDD for $\{x_n^{-1}\}$. It is obvious that $X^{-1} = \{x_n^{-1}\} + A_{CO}$.

PROPOSITION 4.2. *If $X = \{x_n\} + A_{CO}$, and $\|x_n\| \leqslant M$ for all n, then X lies in A.*

Proof. Let x be a w^*-limit point of the sequence $\{x_n\}$. Let (e_n) be an admissible SDD for $\{x_n\}$; for a fixed m, we have $(x_n - x_m) \cdot e_m = 0$ if $n \leqq m$. Thus, $(x - x_m) \cdot e_m = 0$. Since m is arbitrary, we have $\{x\} \equiv \{x_n\}$ and so $\{x_n\} + A_{CO} = \{x\} + A_{CO}$.

Remark 4.3. It is clear that for any SDD (h_n) we have $\{h_n\}+A_{CO}=1+A_{CO}$; thus, if $X=\{x_n\}+A_{CO}$, then $X=U_1(X)=\{U_{h_n}(x_n)\}+A_{CO}$. Hence, if $X=\{x_n\}+A_{CO}$ and $\|U_{h_n}(X_n)\|\leqq M$ for all n and some SDD (h_n), then X lies in A.

PROPOSITION 4.4. *If $X\in\hat{A}$, then*

(i) $1+X^2$ *is invertible in* $\hat{A}$,

(ii) $z=(1+X^2)^{-1}$ *lies in* A,

(iii) $y=X\cdot(1+X^2)^{-1}$ *lies in* A,

(iv) $X=y\cdot z^{-1}$.

Proof. If $X=\{x_n\}+A_{CO}$, then $1+X^2=\{1+x_n^2\}+A_{CO}$. Now, our result follows from [2, Proposition 2.3] and Propositions 4.1 and 4.2.

Remark. Assertion (iv) in the above Proposition leads us to believe that the regular Jordan ring $\hat{A}$ associated to a finite *JBW*-algebra A may be the total ring of quotients of A. This will be proved in the following section.

PROPOSITION 4.5. *If a is an element in a finite JBW-algebra A, then the following conditions are equivalent*:

(i) $s(a)=1$,

(ii) *a is invertible in* $\hat{A}$,

(iii) *a is not a zero divisor in A.*

Proof. It is obvious that (ii) implies (iii). Likewise, (iii) implies (i), since $1-s(a)$ is in Ker U_a. Now, denote by B the w^*-closed subalgebra of A generated by 1 and a; then, B is the self-adjoint part of the $B_{\mathbb{C}}$ (complexifixation of B) which is an associative and commutative finite W^*-algebra. Corollary 7.5 of [4] and Propositions 3.11 and 3.12 complete the proof.

COROLLARY 4.6. *If A is a finite JBW-algebra and B is a w^*-closed subalgebra of A containing the identity element, then $b\in B\cap\operatorname{inv}(\hat{A})$ implies $b^{-1}\in\hat{B}$.*

THEOREM 4.7. *If A is a finite JBW-algebra, $\hat{A}$ is regular (in the sense of von Neumann).*

Proof. Let X be an element in $\hat{A}$; set $X=y\cdot z^{-1}$, y and z as in Proposition 4.4. Let B be the w^*-closed subalgebra of A generated by y, z and 1. Then, B is associative. Since $z\in B\cap\operatorname{Inv}(\hat{A})$, Corollary 4.6 implies $z^{-1}\in\hat{B}$. So $X=y\cdot z^{-1}\in\hat{B}$. From Proposition 3.12 and [4, Corollary 7.1] we have that X is regular.

We conclude this section by showing that $\hat{A}$ has no new idempotents. We shall need two lemmas.

LEMMA 4.8. *If a and b are elements in a unital JB-algebra A (resp. JBW-algebra), then the set* $H=\{c\in A: U_a U_b(c^2)=0\}$ *is a closed (resp. w*-closed) quadratic ideal in A.*

Proof. It is not difficult to show that H is a quadratic ideal. Moreover, it is clear that for any positive linear form f on A, the mapping $(x, y)\to \langle f, U_a U_b(x\cdot y)\rangle$ is a positive, symmetric bilinear mapping; thus, the Cauchy–Schwarz inequality gives $c\in H$ if and only if $U_a U_b(c\cdot d)=0$ for all $d\in A$. Then, the continuity (resp. w^*-continuity) of the product completes the proof.

Using the same techniques it is easy to prove the following:

LEMMA 4.9. *Let F be a positive operator on a unital JB-algebra A. Let a, b be elements in A, b positive. Then, if* $F(U_a(b))=0$ *we have* $F(a\cdot b)=0$.

PROPOSITION 4.10. *For every X in* $\hat{A}$, *there exist a (unique) idempotent f in A such that*

(i) $f\cdot X=X$,

(ii) $U_Y(X^2)=0$ *if and only if* $Y\cdot f=0$, $Y\in\hat{A}$.

Proof. (i) Set $X=\{x_n\}+A_{CO}$ and let (e_n) be an admissible SDD for $\{x_n\}$. Set $f_n=s(U_{x_n}(e_n))$; we have $f_n\uparrow f$. Set $h_n=1-(f-f_n)$; it is clear that (h_n) is an admissible SDD for the sequence $\{f_n\}$. Set $F=\{f_n\}+A_{CO}$; it is trivial that $F^2=F$. Setting $1=\{e_n\}+A_{CO}$, we have $F\cdot X=U_{F,1}(X)=\{U_{f_n,e_n}(x_n)\}+A_{CO}$. For every n, the w^*-closed subalgebra B_n of A generated by x_n, e_n, and 1 is a JW-algebra and $f_n\in B_n$. It is not difficult to prove that $U_{f_n,e_n}(x_n)=x_n\cdot e_n$. Thus, $F\cdot X=\{x_n\cdot e_n\}+A_{CO}=(\{x_n\}+A_{CO})\cdot(\{e_n\}+A_{CO})=X\cdot 1=X$. Moreover, we have $\{f_n\}\equiv\{f\}$ (via (h_n)). Hence, $F=\{f\}+A_{CO}$.

(ii) Set $X=\{x_n\}+A_{CO}$ as in (i). Let $Y=\{y_n\}+A_{CO}$ be such that $U_Y(X^2)=0$; there exists a SDD (k_n) such that $k_n\cdot U_{y_n}(x_n^2)=0$ for all n; thus, $U_{k_n}U_{y_n}(x_n^2)=0$ and so $U_{k_n}U_{y_n}U_{x_n}=0$; thus, $U_{k_n}U_{y_n}U_{x_n}U_{e_n}(x_n^2)=0$ and so $U_{k_n}U_{y_n}(U_{x_n}(e_n))^2=0$. From Lemma 4.8, we have $U_{k_n}U_{y_n}(s(U_{x_n}(e_n)))^2=0$. That is, $U_{k_n}U_{y_n}(f_n)=0$. Hence, $U_{k_n}(y_n\cdot f_n)=0$ by Lemma 4.9. Since $\{k_n\}+A_{CO}=1+A_{CO}$, then $Y\cdot f=\{y_n\cdot f_n\}+A_{CO}=U_1(Y\cdot f)=\{U_{k_n}(y_n\cdot f_n)\}+A_{CO}=0$.

Conversely, let $Y\in\hat{A}$ such that $Y\cdot f=0$. Then, Y is in the subspace $\hat{A}_0$ of the Peirce decomposition of the Jordan algebra $\hat{A}$ relative to the idempotent f; since $X\in\hat{A}_1$, and $\hat{A}_0$, $\hat{A}_1$ are orthogonal subalgebras of $\hat{A}$, we have $X^2 \in \hat{A}_1$, $Y \in \hat{A}_0$ and so $U_Y(X^2) = 0$. The uniqueness of f is clear, for if $g \in A$ is another idempotent with the same properties as f, we have $(1-g) \cdot X = 0$, which implies $(1-g) \cdot X^2 = 0$, and so $U_{1-g}(X^2) = 0$; thus, $(1-g) \cdot f = 0$. Analogously, we have $(1-f) \cdot g = 0$. Thus, $f = g$.

THEOREM 4.11. *Let A be a finite JBW-algebra. Let $\hat{A}$ be the regular Jordan ring associated to A. Then, $\hat{A}$ has no new idempotents.*

Proof. Let E be an idempotent in $\hat{A}$. From Proposition 4.9, there exists a (unique) idempotent $e \in A$ such that $E \cdot e = E$ and $U_Y(E) = 0$ if and only if $Y \cdot e = 0$. Since $U_{1-E}(E) = 0$, we have $(1-E) \cdot e = 0$; thus, $e = E \cdot e = E$.

5. CHARACTERIZATION OF $\hat{A}$

DEFINITION 5.1. *Let A be a Jordan ring with identity element. If $\hat{A}$ is a Jordan ring containing A and with the same identity element as A, $\hat{A}$ is said to be the total ring of quotients of A if:*

(i) *Every non-zero divisor s in A is invertible in $\hat{A}$.*

(ii) *Every morphism f from A into a Jordan ring B, having the property that $f(s)$ is invertible in B whenever s is not a zero divisor in A, extends in a unique way to a morphism from $\hat{A}$ into B.*

Recall that a Jordan ring A is said to have the *common multiple property* if for any $a, s \in A$, $a \neq 0$, and s not a zero divisor, there exist $a', s' \in A$, s' not a zero divisor, such that $U_a(s') = U_s(a') \neq 0$. (See [13, p. 155].)

Recall that for associative rings, this corresponds to the Ore condition, which is sufficient for an associative ring to have a total ring of quotients. However, it is unknown in general whether a Jordan ring with the common multiple property has a ring of quotients. (There is a paper by N. Jacobson, K. McCrimmon, and M. Parvathi [11] on localization of Jordan algebras. Also, there is a forthcoming article by E. Zel'manov on Goldie theory.)

Now, our purpose is to show that the Jordan regular ring $\hat{A}$ associated to a finite JBW-algebra A is the total Jordan ring of quotients of A. In fact, we shall prove the following

THEOREM 5.2. *Let A be a finite JBW-algebra. Let $\hat{A}$ denote the Jordan regular ring associated to A. Then:*

(i) *Every element X in $\hat{A}$ is of the form $X = U_{s^{-1}}(a)$ with $a, s \in A$, and s is not a zero divisor in A. Moreover, the subalgebra of A generated by a and s is strongly associative.*

(ii) *A has the common multiple property.*

(iii) *$\hat{A}$ is the (unique) total Jordan ring of quotients of A.*

To prove the theorem we shall need three lemmas.

LEMMA 5.3. *If M is a unital C^*-algebra and $u, v \in \mathrm{Sym}(M)$, then $1 + vu^2v$ is invertible and moreover $\|uv(1+vu^2v)^{-1}vu\| \leqslant 1$.*

Proof. Since $1 + vu^2v = 1 + (vu)(vu)^*$, it is clear that $1 + vu^2v$ is invertible. Likewise, $1 + uv^2u$ is invertible also. Moreover, it is not difficult to show that $uv(1+vu^2v)^{-1}vu = (1+uv^2u)^{-1}uv^2u$. Since $\mathrm{Sym}(M)$ is a unital JB-algebra, from [2, p. 18] it follows that $\|uv(1+vu^2v)^{-1}vu\| \leqslant 1$.

LEMMA 5.4. *Let A be a finite JBW-algebra. Let $\hat{A}$ denote the Jordan regular ring associated to A. If $b \in A$, $X \in \hat{A}$, then*

(i) *$1 + U_b(X^2)$ is invertible in $\hat{A}$,*

(ii) *$(1 + U_b(X^2))^{-1}$ lies in A,*

(iii) *$U_X U_b((1 + U_b(X^2))^{-1})$ lies in A.*

Proof. (i) Follows from [2, p. 17] and Proposition 4.1.

(ii) Follows from [2, p. 18] and Proposition 4.2.

(iii) If $X = \{x_n\} + A_{CO}$, then $U_X U_b((1+U_b(X^2))^{-1}) = \{U_{x_n} U_b((1+U_b(x_n^2))^{-1})\} + A_{CO}$. For every n, let B_n be the w^*-closed subalgebra of A generated by x_n, b, and 1; B_n is a JW-algebra (Theorem 1.2) and, in B_n, $U_{x_n}((1+U_b(x_n^2))^{-1}) = x_n b(1+bx_n^2b)^{-1}bx_n$. From Lemma 5.3 and Proposition 4.2 it follows that $U_X U_b((1+U_b(X^2))^{-1}) \in A$.

Recall that a subalgebra of a Jordan algebra with identity element is called a *full subalgebra* if it contains the identity element and the inverses of all its invertible elements. A Jordan algebra J, with identity element, is said to be *fully generated* by a subalgebra B if J is the smallest full subalgebra of J containing B.

LEMMA 5.5. *Let A and B be Jordan algebras with identity element and let f be a morphism from A into B, such that B is fully generated by $f(A)$. If a and b operators commute in A then $f(a)$ and $f(b)$ operators commute in B.*

Therefore, if C is a strongly associative subalgebra of A, then $f(C)$ is a strongly associative subalgebra of B.

Proof. For every element x in A, we have $[a, x, b] = 0$; thus, $[f(a), f(x), f(b)] = 0$. That is, $f(A)$ is contained in the kernel of the derivation $y \to [f(a), y, f(b)]$, which is a full subalgebra of B [10, p. 54].

Proof of the Theorem. (i) If $X \in \hat{A}$, set $s = (1 + X^2)^{-1} \in A$, $z = X \cdot (1 + X^2)^{-1} \in A$. It is obvious that s and z are in the full subalgebra of $\hat{A}$ generated by X, which is strongly associative from a result by Jacobson (see [12]), which states that if J is a Jordan algebra with identity element, then the full subalgebra of J generated by a strongly associative subalgbra B of J is strongly associative also. (It follows that for every element j in J, there is a strongly associative full subalgebra of J containing j.) Thus, $X = s^{-1} \cdot z = U_{s^{-1}}(a)$ with $a = s \cdot z$ and it is clear that the subalgebra of A generated by s and a is strongly associative.

(ii) Let a, s be elements in A such that $a \neq 0$ and s is not a zero divisor. By Proposition 4.5, s is invertible in $\hat{A}$. Setting $a' = U_{s^{-1}} U_a((1 + U_a(s^{-2}))^{-1})$, $s' = (1 + U_a(s^2))^{-1}$ we have $s, a' \in A$ (Lemma 5.4). It is obvious that s' is not a zero divisor and we have $U_s(a') = U_a(s') \neq 0$, for if $U_a(s') = 0$, since s' is a positive element, we have $U_a(h^2) = 0$, where h denotes the positive square root of s' (note that h is not a zero divisor also); hence, $U_h(a^2) = 0$ and so $a = 0$. Contradiction.

(iii) Let B be a Jordan ring with identity element, and let $f\colon A \to B$ be a morphism of A into B, having the property that $f(s)$ is invertible in B whenever s is not a zero divisor in A. It is clear that if there exists an extension g of f to $\hat{A}$ it should be of the form $g(U_{s^{-1}}(a)) = U_{f(s)}^{-1}(f(a))$. We will prove that g is a well-defined mapping: If $U_{s^{-1}}(a) = U_{s'^{-1}}(a')$, there exist s_1, $s'_1 \in A$, s_1 is not a zero divisor, such that $U_s(s_1) = U_{s'}(s'_1)$; s and s_1 are invertible in $\hat{A}$; thus, $U_s(s_1)$ is invertible and so $U_{s'}(s'_1)$ is invertible also. Hence, s'_1 is invertible in $\hat{A}$. Thus, s, s_1, s', s'_1 are all invertible in $\hat{A}$. Using the Fundamental Identity [10, p. 52] we have

$$U_{s^{-1}}(a) = U^{-1}_{U_s(s_1)}(U_s U_{s_1}(a))$$

$$U_{s'^{-1}}(a') = U^{-1}_{U_{s'}(s'_1)}(U_{s'} U_{s'_1}(a')) = U^{-1}_{U_s(s_1)}(U_{s'} U_{s'_1}(a'))$$

thus

$$U^{-1}_{U_s(s_1)}(U_s U_{s_1}(a)) = U^{-1}_{U_s(s_1)}(U_{s'} U_{s'_1}(a'))$$

and so

$$U_s U_{s_1}(a) = U_{s'} U_{s'_1}(a')$$

thus

$$U_{f(s)} U_{f(s_1)}(f(a)) = U_{f(s')} U_{f(s'_1)}(f(a'))$$

and so

$$U^{-1}_{U_{f(s)}(f(s_1))}(U_{f(s)}U_{f(s_1)}(f(a))) = U^{-1}_{U_{f(s)}(f(s_1))}(U_{f(s')}U_{f(s_1')}(f(a')))$$
$$= U^{-1}_{U_{f(s')}(f(s_1'))}(U_{f(s')}U_{f(s_1')}(f(a'))).$$

Thus

$$U^{-1}_{f(s)}(f(a)) = U^{-1}_{f(s')}(f(a')),$$

Using the same techniques it is easy to see that $g(X+Y)=g(X)+g(Y)$, X and Y in $\hat{A}$. To complete the proof, it remains to show that $g(X^2)=(g(X))^2$ for all X in $\hat{A}$. Set $X=U_{s^{-1}}(a)$, where a and s generate a strongly associative subalgebra C of A. We can assume, without loss of generality, that B is fully generated by $f(A)$. Setting $m=f(s)$, $n=f(a)$, we have $g(X^2)=U_{m^{-2}}(n^2)$, $(g(X))^2=(U_{m^{-1}}(n))^2$ with m, $n \in f(C)$, which is a strongly associative subalgebra of B by Lemma 5.3. Let D be the full subalgebra of B generated by $f(C)$; D is a strongly associative subalgebra and we have m, n, $m^{-1} \in D$. Thus, $(g(X))^2=(U_{m^{-1}}(n))^2=U_{m^{-2}}(n^2)=g(X^2)$ and the proof is complete.

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