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Physics Letters B

www.elsevier.com/locate/physletb

$SU(n)$ symmetry breaking by rank three and rank two antisymmetric tensor scalars



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ARTICLE INFO

Article history:

Received 25 March 2015

Received in revised form 7 April 2015

Accepted 9 April 2015

Available online 14 April 2015

Editor: M. Cvetič

ABSTRACT

We study $SU(n)$ symmetry breaking by rank three and rank two antisymmetric tensor fields. Using tensor analysis, we derive branching rules for the adjoint and antisymmetric tensor representations, and explain why for general $SU(n)$ one finds the same $U(1)$ generator mismatch that we noted earlier in special cases. We then compute the masses of the various scalar fields in the branching expansion, in terms of parameters of the general renormalizable potential for the antisymmetric tensor fields.

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1. Introduction

The most familiar case of symmetry breaking for grand unified theories, such as minimal $SU(5) \supset SU(2) \times SU(3) \times U(1)$, utilizes a scalar field in the adjoint representation, with a gauge singlet component with $U(1)$ generator zero that receives a vacuum expectation. The symmetry breaking mechanism is then straightforward: since the gauge fields and the symmetry breaking scalar are both in the adjoint representation, the same representations appear in their branching expansions. As a consequence, the massless gauge fields that pick up masses, and the scalars that supply their longitudinal components, have the same group theoretic quantum numbers.

We recently noted [1,2] that when the symmetry breaking scalar is in a totally antisymmetric representation, the situation is more complicated. Using as explicit examples $SU(8)$ broken by a rank three antisymmetric tensor scalar, and $SU(5)$ broken by a rank two antisymmetric tensor scalar, we showed that there is a mismatch between the $U(1)$ generator values of the massless gauge fields that obtain masses, and the scalars that supply their longitudinal components. We noted that this mismatch is related to the fact that the gauge singlet component of the antisymmetric tensor field that receives a vacuum expectation has a nonzero $U(1)$ generator N , requiring a modular ground state that is periodic in integer divisors p of N .

The purpose of this paper is twofold. First, we show that the mismatch found in [1,2] appears in the case of general $SU(n)$, and can be traced to the fact that invariant tensors lying in $SU(3)$ or $SU(2)$ subgroups are available to lower subgroup indices. This anal-

ysis is given in Section 2, where we use tensor methods to compute the relevant branching expansions and $U(1)$ generator values. The second aim of this paper is to calculate the masses of the various scalar field components in the branching expansions, obtained by expanding the general renormalizable scalar field potential around the generic symmetry breaking minimum. This analysis is given in Section 3, and a brief summary of our results follows in Section 4.

Our notation is to define the upper index totally antisymmetric tensor with R components to be a basis for the representation R , and the corresponding lower index tensor to be a basis for the conjugate representation \bar{R} . Thus in $SU(n)$ the tensor ϕ^α , $\alpha = 1, \dots, n$, is a basis for the fundamental representation n , and ϕ_α is a basis for the conjugate representation \bar{n} . In $SU(3)$, the tensor ϕ^α is a basis for the 3, and since the totally antisymmetric tensor $\epsilon_{\alpha\beta\gamma}$ is invariant and can be used to lower indices, both the tensors ϕ_α and $\phi^{[\alpha\beta]}$ give a basis for the $\bar{3}$, and the tensor $\phi^{[\alpha\beta\gamma]} \propto \epsilon^{\alpha\beta\gamma}$ is a singlet. Similarly, in $SU(2)$, since the invariant tensor $\epsilon_{\alpha\beta}$ can be used to lower indices the representations 2 and $\bar{2}$ are equivalent, and can be represented by either ϕ^α or ϕ_α , and the tensor $\phi^{[\alpha\beta]} \propto \epsilon^{\alpha\beta}$ is a singlet [3].

2. Branching rules for the $SU(n)$ antisymmetric tensor and adjoint representations

2.1. Branching under $SU(n) \supset SU(3) \times SU(n-3) \times U(1)$ for the rank three antisymmetric tensor and adjoint representations

We assume that $SU(n)$ is broken by the ground state expectation of a single component $\bar{\phi}^{[123]} = a \neq 0$, corresponding to the simplest case considered by Cummins and King [4], which applies

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for all n . The conditions on the scalar potential for this case to apply will be given in Section 3. Let us now divide the tensor indices into two classes,

$$\begin{aligned}\mathcal{A} &= \{1, 2, 3\}, \\ \mathcal{B} &= \{4, \dots, n\}.\end{aligned}\quad (1)$$

To get the needed branching expansions, we have to enumerate the possibilities for tensor indices to belong to these two classes. We use the notation $(R_{SU(3)}, R_{SU(n-3)})(g)$, with g the $U(1)$ generator eigenvalue. Writing the $U(1)$ generator G as

$$G = \text{Diag}(n-3, n-3, n-3, -3, -3, \dots, -3) \quad (2)$$

with $n-3$ entries -3 , the $U(1)$ generator value g is simply $n-3$ times the number of upper indices in \mathcal{A} plus -3 times the number of upper indices in \mathcal{B} ; for lower indices the $U(1)$ contributions are reversed in sign. Since the overall normalization of the $U(1)$ generator is arbitrary, normalization-independent statements refer only to *relative* values of the $U(1)$ generators for different representations.

We begin by deriving the branching expansion for the $SU(n)$ rank three antisymmetric tensor representation $n(n-1)(n-2)/6$, represented by the tensor $\phi^{[\alpha\beta\gamma]}$, enumerating cases as follows.

1. 3 indices in \mathcal{A} . This corresponds to the representation $(1, 1)(3n-9)$.
2. 2 indices in \mathcal{A} , 1 index in \mathcal{B} . Since the index in \mathcal{B} can be chosen $n-3$ ways, this corresponds to the representation $(\bar{3}, n-3)(2n-9)$.
3. 1 index in \mathcal{A} , 2 indices in \mathcal{B} . Since the indices in \mathcal{B} can be chosen $(n-3)(n-4)/2$ ways, this corresponds to the representation $(3, (n-3)(n-4)/2)(n-9)$.
4. 3 indices in \mathcal{B} . Since the indices in \mathcal{B} can be chosen $(n-3)(n-4)(n-5)/6$ ways, this corresponds to the representation $(1, (n-3)(n-4)(n-5)/6)(-9)$.

Thus we have the branching expansion, for $n > 8$,

$$\begin{aligned}\frac{n(n-1)(n-2)}{6} &= (1, 1)(3n-9) + (\bar{3}, n-3)(2n-9) \\ &\quad + \left(3, \frac{(n-3)(n-4)}{2}\right)(n-9) \\ &\quad + \left(1, \frac{(n-3)(n-4)(n-5)}{6}\right)(-9).\end{aligned}\quad (3)$$

As a check on the counting, we note the identity

$$\begin{aligned}\frac{n(n-1)(n-2)}{6} &= 1 + 3(n-3) + 3\frac{(n-3)(n-4)}{2} \\ &\quad + \frac{(n-3)(n-4)(n-5)}{6}.\end{aligned}\quad (4)$$

For the case $n=8$ discussed in [1], the $SU(5)$ three upper index antisymmetric tensor is equivalent, by use of the invariant tensor $\epsilon_{\alpha\beta\gamma\delta\epsilon}$, to the $SU(5)$ two lower index antisymmetric tensor, and so represents a $\bar{10}$ rather than a 10. Thus we get the expansion

$$56 = (1, 1)(15) + (\bar{3}, 5)(7) + (3, 10)(-1) + (1, \bar{10})(-9). \quad (5)$$

This agrees with the expansion given in [1] and the Slansky tables [5], apart from the fact that in this paper we have chosen the opposite sign convention for the $U(1)$ generator G . For $n < 8$, one makes similar conversions of upper index tensors to lower index ones in Eq. (3), when the number of lower indices can be made smaller than the number of upper indices, with the corresponding

replacement of the representation R by \bar{R} . We also note that when $n-3$ is divisible by 3, the $U(1)$ generator values can all be divided by 3, and this is the convention that is used in the Slansky tables (see e.g. the expansion for the 20 of $SU(6)$).

We turn next to the branching expansion for the $SU(n)$ adjoint representation n^2-1 , represented by the tensor ϕ_{β}^{α} , with $\sum_{\alpha} \phi_{\alpha}^{\alpha} = 0$, again enumerating cases.

1. Diagonal traceless part analogous to the $U(1)$ generator G . This corresponds to the representation $(1, 1)(0)$.
2. Upper index and lower index both in \mathcal{A} , traceless part. This corresponds to the representation $(8, 1)(0)$.
3. Upper index and lower index both in \mathcal{B} , traceless part. This corresponds to the representation $(1, (n-3)^2-1)(0)$.
4. Upper index in \mathcal{A} , lower index in \mathcal{B} . This corresponds to the representation $(3, n-3)(n)$.
5. Lower index in \mathcal{A} , upper index in \mathcal{B} . This corresponds to the representation $(\bar{3}, n-3)(-n)$.

Thus we have the branching expansion

$$\begin{aligned}n^2-1 &= (1, 1)(0) + (8, 1)(0) + (1, (n-3)^2-1)(0) \\ &\quad + (3, \overline{n-3})(n) + (\bar{3}, n-3)(-n).\end{aligned}\quad (6)$$

As a check on the counting, we note the identity

$$n^2-1 = 1 + 8 + (n-3)^2-1 + 6(n-3). \quad (7)$$

We now note the phenomenon discussed in the $n=8$ case in [1,2], that the $U(1)$ generator of the $(\bar{3}, n-3)$ is $-n$ in the branching expansion for the adjoint, whereas it is $2n-9$ in the branching expansion for the rank three antisymmetric tensor. The difference between these two $U(1)$ generators is $2n-9 - (-n) = 3n-9 = 3(n-3)$, which is just the $U(1)$ generator of the singlet $(1, 1)$ in the expansion of Eq. (3). This is a direct result of the fact that the $\bar{3}$ is represented by a two upper index antisymmetric tensor in the expansion of Eq. (3), and by a one lower index tensor in the expansion of Eq. (6), so the difference in $U(1)$ generator values is $(2 - (-1))(n-3) = 3(n-3)$. When we discuss the scalar potential in Section 3, we will see that the complex states $(\bar{3}, n-3)$ in Eq. (3) are zero mass Goldstone modes. When the rank three antisymmetric tensor is used to break the $SU(n)$ symmetry, the Goldstone modes are absorbed as longitudinal parts of the $(3, n-3) + (\bar{3}, n-3)$ in the adjoint. This is possible, even though the $U(1)$ generators do not match, because for the $(1, 1)(3n-9)$ to get a ground state expectation value, the ground state must have a periodic structure modulo an integer divisor of $3n-9$, and so the mismatch of the $U(1)$ generator values is equivalent to zero.

2.2. Branching under $SU(n) \supset SU(2) \times SU(n-2) \times U(1)$ for the rank two antisymmetric tensor and adjoint representations

In this case we shall assume that $SU(n)$ is broken by the ground state expectation of a single component $\bar{\phi}^{[12]} = a \neq 0$, corresponding to the case studied by Li [6]. We now define the index classes by

$$\begin{aligned}\mathcal{A} &= \{1, 2\}, \\ \mathcal{B} &= \{3, \dots, n\},\end{aligned}\quad (8)$$

and use the notation $(R_{SU(2)}, R_{SU(n-2)})(g)$, with g the $U(1)$ generator. Writing the $U(1)$ generator G as

$$G = \text{Diag}(n-2, n-2, -2, -2, \dots, -2) \quad (9)$$

with $n-2$ entries -2 , the $U(1)$ generator value g is simply $n-2$ times the number of upper indices in \mathcal{A} plus -2 times the number

of upper indices in \mathcal{B} ; for lower indices the $U(1)$ contributions are reversed in sign. Again, since the overall normalization of the $U(1)$ generator is arbitrary, normalization-independent statements refer only to relative values of the $U(1)$ generators for different representations.

Since the enumeration of cases parallels that in the rank three case, we go directly to the results. For the rank two antisymmetric tensor, we have for $n > 5$

$$\frac{n(n-1)}{2} = (1, 1)(2n-4) + (2, n-2)(n-4) + \left(1, \frac{(n-2)(n-3)}{2}\right)(-4), \quad (10)$$

with the three terms corresponding, respectively, to zero, one, and two upper indices in \mathcal{B} . As a check on the counting, we note the identity

$$\frac{n(n-1)}{2} = 1 + 2(n-2) + \frac{(n-2)(n-3)}{2}. \quad (11)$$

For the case $n=5$, since the $SU(3)$ two upper index antisymmetric tensor represents a $\bar{3}$, we get the expansion

$$10 = (1, 1)(6) + (2, 3)(1) + (1, \bar{3})(-4), \quad (12)$$

in agreement with the expansion given in the Slansky tables [5]. When $n-2$ is divisible by 2, the $U(1)$ generator values can all be divided by 2, and this is the convention used in the Slansky tables (see, e.g., the expansion for the 15 of $SU(6)$).

For the adjoint representation $n^2 - 1$ of $SU(n)$, we get the branching expansion

$$n^2 - 1 = (1, 1)(0) + (3, 1)(0) + (1, (n-2)^2 - 1)(0) + (2, \overline{n-2})(n) + (2, n-2)(-n), \quad (13)$$

and as a check on counting

$$n^2 - 1 = 1 + 3 + (n-2)^2 - 1 + 4(n-2). \quad (14)$$

We again see the mismatch discussed in [2] in the $n=5$ case. The $U(1)$ generator of the $(2, n-2)$ is $-n$ in the branching expansion of the adjoint, whereas it is $n-4$ in the branching expansion for the rank two antisymmetric tensor. The difference between these two $U(1)$ values is $n-4 - (-n) = 2n-4 = 2(n-2)$, which is the $U(1)$ generator of the singlet $(1, 1)$ in the expansion of Eq. (10). This results from the fact that the 2 is represented by a one upper index tensor in the expansion of Eq. (10), and by a one lower index tensor in the expansion of Eq. (13), with a resulting difference of $U(1)$ generator values $(1 - (-1))(n-2) = 2(n-2)$. When we discuss the scalar potential in Section 3, we will see that the complex states $(2, n-2)$ in Eq. (10) are zero mass Goldstone modes. When the rank two antisymmetric tensor is used to break the $SU(n)$ symmetry, the Goldstone modes are absorbed as longitudinal parts of the $(2, \overline{n-2}) + (2, n-2)$ in the adjoint. This is possible, despite the $U(1)$ generator mismatch, because for the $(1, 1)(2n-4)$ to get a ground state expectation value, the ground state must have a periodic structure modulo an integer divisor of $2n-4$, and so the mismatch of the $U(1)$ generator values is equivalent to zero.

3. Residual scalar masses

In this section we analyze the residual scalar masses arising from $SU(n)$ symmetry breaking with a general renormalizable scalar potential, first for a rank three antisymmetric tensor scalar, and then for a rank two antisymmetric tensor.

3.1. Residual scalar masses for $SU(n)$ symmetry breaking by a rank three antisymmetric tensor

The most general $SU(n)$ invariant fourth degree potential formed from $\phi^{[\alpha\beta\gamma]}$, where the indices all range from 1 to n , has the form [4]

$$V(\phi) = -\frac{1}{2}\mu^2 \sum_{\alpha\beta\gamma} \phi_{[\alpha\beta\gamma]}^* \phi^{[\alpha\beta\gamma]} + \frac{1}{4}\lambda_1 \left(\sum_{\alpha\beta\gamma} \phi_{[\alpha\beta\gamma]}^* \phi^{[\alpha\beta\gamma]}\right)^2 + \frac{1}{4}\lambda_2 \sum_{\alpha\beta\gamma\rho\kappa\tau} \phi_{[\alpha\beta\gamma]}^* \phi^{[\alpha\beta\tau]} \phi_{[\rho\kappa\tau]}^* \phi^{[\rho\kappa\gamma]}. \quad (15)$$

We assume $\mu^2 > 0$, so that the origin is a local maximum, and consider the case $\lambda_2 < 0$ studied in [4], for which the potential is bounded from below, for all n , when $3\lambda_1 + \lambda_2 > 0$,

$$V(\phi) \geq -\frac{3}{4} \frac{\mu^4}{3\lambda_1 + \lambda_2}. \quad (16)$$

This lower bound is attained when only one component of ϕ is nonzero, and as in our branching analysis we take the nonvanishing component to be $\bar{\phi}^{[123]} = a \neq 0$, where

$$|a|^2 = \frac{1}{2} \frac{\mu^2}{3\lambda_1 + \lambda_2}. \quad (17)$$

We will derive Eqs. (16) and (17) shortly.

Continuing to follow [4], we note that the potential of Eq. (15) can be rewritten in terms of

$$\theta_\gamma^\tau \equiv \sum_{\alpha\beta} \phi_{[\alpha\beta\gamma]}^* \phi^{[\alpha\beta\tau]}, \quad (18)$$

which obeys $(\theta_\gamma^\tau)^* = \theta_\tau^\gamma$, as

$$V(\phi) = -\frac{1}{2}\mu^2 \sum_\gamma \theta_\gamma^\gamma + \frac{1}{4}\lambda_1 \left(\sum_\gamma \theta_\gamma^\gamma\right)^2 + \frac{1}{4}\lambda_2 \sum_{\gamma\tau} \theta_\gamma^\tau \theta_\tau^\gamma = -\frac{1}{2}\mu^2 \sum_\gamma \theta_\gamma^\gamma + \frac{1}{4}\lambda_1 \left(\sum_\gamma \theta_\gamma^\gamma\right)^2 + \frac{1}{4}\lambda_2 \sum_\gamma (\theta_\gamma^\gamma)^2 + \frac{1}{2}\lambda_2 \sum_{\gamma<\tau} \theta_\gamma^\tau \theta_\tau^\gamma. \quad (19)$$

To expand the potential around its minimum, we substitute

$$\phi^{[\alpha\beta\gamma]} = \bar{\phi}^{[\alpha\beta\gamma]} + \sigma^{[\alpha\beta\gamma]}, \quad (20)$$

where $\bar{\phi}^{[\alpha\beta\gamma]} = a\epsilon^{\alpha\beta\gamma}$ is nonzero only when its tensor indices are some permutation of 1, 2, 3. For θ_γ^τ we find

$$\theta_\gamma^\tau = 2 \sum_{\alpha<\beta} (\bar{\phi}_{[\alpha\beta\gamma]}^* \bar{\phi}^{[\alpha\beta\tau]} + \bar{\phi}_{[\alpha\beta\gamma]}^* \sigma^{[\alpha\beta\tau]} + \sigma_{[\alpha\beta\gamma]}^* \bar{\phi}^{[\alpha\beta\tau]}) + \sigma_{[\alpha\beta\gamma]}^* \sigma^{[\alpha\beta\tau]}. \quad (21)$$

We consider first the term $\sum_{\gamma<\tau} \theta_\gamma^\tau \theta_\tau^\gamma$ in Eq. (19). The term in θ_γ^τ that is quadratic in $\bar{\phi}$ must have $\gamma = \tau$, and so does not contribute to this sum over $\gamma < \tau$. Hence the term in θ_γ^τ that is quadratic in σ makes a contribution to this sum that is third order in σ , and can be dropped in calculating the potential to second order in σ . Thus we get

$$\sum_{\gamma<\tau} \theta_\gamma^\tau \theta_\tau^\gamma = \sum_{\gamma<\tau} |\theta_\gamma^\tau|^2 = 4 \sum_{\gamma<\tau} \left| \sum_{\alpha<\beta} (\bar{\phi}_{[\alpha\beta\gamma]}^* \sigma^{[\alpha\beta\tau]} + \sigma_{[\alpha\beta\gamma]}^* \bar{\phi}^{[\alpha\beta\tau]}) \right|^2$$

$$= 4 \sum_{\gamma < \tau} \left| \sum_{\alpha < \beta} \bar{\phi}_{[\alpha\beta\gamma]}^* \sigma^{[\alpha\beta\tau]} \right|^2, \quad (22)$$

where in the final line we have used the fact that when α, β, τ are permutations of 1, 2, 3, then when $\tau = 1$ there is no γ obeying $\gamma < \tau$, and when $\tau = 2$ or $\tau = 3$, the γ obeying $\gamma < \tau$ must equal either α or β , and so the factor $\sigma_{[\alpha\beta\gamma]}^*$ multiplying $\bar{\phi}^{[\alpha\beta\tau]}$ vanishes. By similar reasoning, the sum over τ in the final line of Eq. (22) must range from 4 to n independent of the values of $\alpha < \beta$, since if $\tau \leq 3$, then $\gamma \leq 2$ and either the first or the second factor vanishes. Hence we get

$$\sum_{\gamma < \tau} \theta_\gamma^\tau \theta_\tau^\gamma = 4|a|^2 \sum_{(\alpha, \beta) = (1,2), (1,3), (2,3)} \sum_{\tau=4}^n |\sigma^{[\alpha\beta\tau]}|^2. \quad (23)$$

Since $\sigma^{[\alpha\beta\tau]}$ in this equation has $\alpha \in \mathcal{A}$, $\beta \in \mathcal{A}$, and $\tau \in \mathcal{B}$, it belongs to the representation $(\bar{3}, n-3)$, and so we can rewrite Eq. (23) as

$$\sum_{\gamma < \tau} \theta_\gamma^\tau \theta_\tau^\gamma = 4|a|^2 \sum_{k=1}^3 \sum_{l=1}^{n-3} |\sigma(\bar{3}, k; n-3, l)|^2. \quad (24)$$

The remaining terms in Eq. (19) all involve the diagonal element θ_γ^γ , which from Eq. (21) is given by

$$\theta_\gamma^\gamma = 2 \sum_{\alpha < \beta} (|\bar{\phi}^{[\alpha\beta\gamma]}|^2 + 2 \operatorname{Re}(\bar{\phi}_{[\alpha\beta\gamma]}^* \sigma^{[\alpha\beta\gamma]}) + |\sigma^{[\alpha\beta\gamma]}|^2). \quad (25)$$

From this, substituting $\bar{\phi}^{[\alpha\beta\gamma]} = a \epsilon^{\alpha\beta\gamma}$, splitting sums on γ into disjoint sums $\sum_{\gamma \in \mathcal{A}}$ and $\sum_{\gamma \in \mathcal{B}}$, and dropping terms of higher order than quadratic in σ , one finds

$$\begin{aligned} \sum_{\gamma} (\theta_\gamma^\gamma)^2 &= 12 \left(|a|^4 + 4|a|^2 \operatorname{Re}(a^* \sigma^{[123]}) + 2|a|^2 |\sigma^{[123]}|^2 \right. \\ &\quad \left. + 4(\operatorname{Re}(a^* \sigma^{[123]}))^2 \right), \\ \sum_{\gamma} \theta_\gamma^\gamma &= 6 \left(|a|^2 + 2 \operatorname{Re}(a^* \sigma^{[123]}) + |\sigma^{[123]}|^2 \right) \\ &\quad + \sum_{\alpha\beta} \sum_{\gamma \in \mathcal{B}} |\sigma^{[\alpha\beta\gamma]}|^2, \\ \left(\sum_{\gamma} \theta_\gamma^\gamma \right)^2 &= 36 \left(|a|^4 + 4|a|^2 \operatorname{Re}(a^* \sigma^{[123]}) + 2|a|^2 |\sigma^{[123]}|^2 \right. \\ &\quad \left. + 4(\operatorname{Re}(a^* \sigma^{[123]}))^2 \right) + 12|a|^2 \sum_{\alpha\beta} \sum_{\gamma \in \mathcal{B}} |\sigma^{[\alpha\beta\gamma]}|^2. \end{aligned} \quad (26)$$

Substituting Eqs. (24) and (26) into Eq. (19), and combining the first order terms in σ , we get

$$\operatorname{Re}(a^* \sigma^{[123]})(-6\mu^2 + 36\lambda_1 |a|^2 + 12\lambda_2 |a|^2), \quad (27)$$

which when equated to zero gives Eq. (17). Using this value of $|a|^2$, we find the lower bound of Eq. (16) for the value of the potential at the minimum. Splitting the sum $\sum_{\alpha\beta} \sum_{\gamma \in \mathcal{B}} |\sigma^{[\alpha\beta\gamma]}|^2$ into three pieces,

$$\begin{aligned} \sum_{\alpha\beta} \sum_{\gamma \in \mathcal{B}} |\sigma^{[\alpha\beta\gamma]}|^2 \\ = \left(\sum_{\alpha\beta \in \mathcal{A}} \sum_{\gamma \in \mathcal{B}} + 2 \sum_{\alpha \in \mathcal{A}} \sum_{\beta\gamma \in \mathcal{B}} + \sum_{\alpha\beta\gamma \in \mathcal{B}} \right) |\sigma^{[\alpha\beta\gamma]}|^2, \end{aligned} \quad (28)$$

and relabeling $\sigma^{[\alpha\beta\gamma]}$ in terms of the representations appearing in the branching expansion of Eq. (3), we get as the final result for

the expansion of the potential near the minimum through second order terms,

$$\begin{aligned} V(\bar{\phi} + \sigma) \\ = -\frac{3}{4} \frac{\mu^4}{3\lambda_1 + \lambda_2} \\ + \left(\operatorname{Re} \left(\frac{a^*}{|a|} \sigma(1, 1) \right) \right)^2 6\mu^2 \\ + \sum_{k=1}^3 \sum_{l=1}^{n-3} |\sigma(\bar{3}, k; n-3, l)|^2 \times 0 \\ + \sum_{k=1}^3 \sum_{l=1}^{(n-3)(n-4)/2} |\sigma(3, k; (n-3)(n-4)/2, l)|^2 2\mu^2 \frac{-\lambda_2}{3\lambda_1 + \lambda_2} \\ + \sum_{l=1}^{(n-3)(n-4)(n-5)/6} |\sigma(1; (n-3)(n-4)(n-5)/6, l)|^2 3\mu^2 \\ \times \frac{-\lambda_2}{3\lambda_1 + \lambda_2}. \end{aligned} \quad (29)$$

The remarks made in Section 2 about using the rank $n-3$ epsilon tensor to replace upper index tensors by lower index tensors in conjugate representations, when this reduces the number of indices, applies here. We see that as noted in Section 2, the Goldstone modes, with mass 0, are in the representation $(\bar{3}, n-3)$, which has a $U(1)$ generator mismatch with respect to the corresponding representation in the expansion of the adjoint representation.

3.2. Residual scalar masses for $SU(n)$ symmetry breaking by a rank two antisymmetric tensor

The most general $SU(n)$ invariant fourth degree potential formed from the rank two antisymmetric tensor scalar $\phi^{[\alpha\beta]}$, where the indices all range from 1 to n , has the form [6] for $n > 4$,¹

$$\begin{aligned} V(\phi) = -\frac{1}{2} \mu^2 \sum_{\alpha\beta} \phi_{[\alpha\beta]}^* \phi^{[\alpha\beta]} + \frac{1}{4} \lambda_1 \left(\sum_{\alpha\beta} \phi_{[\alpha\beta]}^* \phi^{[\alpha\beta]} \right)^2 \\ + \frac{1}{4} \lambda_2 \sum_{\alpha\beta\rho\tau} \phi_{[\alpha\rho]}^* \phi^{[\alpha\tau]} \phi_{[\beta\tau]}^* \phi^{[\beta\rho]}. \end{aligned} \quad (30)$$

Since the method of analysis parallels that used in the rank three case, we state only the final results. We assume that $\lambda_2 < 0$ and $2\lambda_1 + \lambda_2 > 0$, and as in our branching analysis we take the non-vanishing component of $\phi^{[\alpha\beta]}$ to be $\bar{\phi}^{[12]} = a \neq 0$. The potential minimum is at

$$|a|^2 = \frac{\mu^2}{2\lambda_1 + \lambda_2}, \quad (31)$$

and the value of the potential at the minimum is

$$-\frac{1}{2} \frac{\mu^4}{2\lambda_1 + \lambda_2}. \quad (32)$$

¹ A perceptive referee has pointed out that for $SU(4)$ there is an exception; one can construct the invariant $\phi^{\alpha\beta} \phi^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} + \text{adjoint}$, and so the most general renormalizable potential has a more complicated form than Eq. (30). For rank three antisymmetric tensors in $SU(6)$ the analog of this invariant vanishes by antisymmetry of the epsilon tensor, so there is not a similar exception to the potential of Eq. (15). The paper of Li [6] overlooked the rank two exception because it first treated rank two symmetric tensors, and then took the same potential for the antisymmetric tensor case.

For the expansion of the potential through second order terms, we find

$$\begin{aligned}
 V(\bar{\phi} + \sigma) &= -\frac{1}{2} \frac{\mu^4}{2\lambda_1 + \lambda_2} \\
 &+ \left(\operatorname{Re} \left(\frac{a^*}{|a|} \sigma(1, 1) \right) \right)^2 2\mu^2 \\
 &+ \sum_{k=1}^2 \sum_{l=1}^{n-2} |\sigma(2, k; n-2, l)|^2 \times 0 \\
 &+ \sum_{l=1}^{(n-2)(n-3)/2} |\sigma(1; (n-2)(n-3)/2, l)|^2 \mu^2 \frac{-\lambda_2}{2\lambda_1 + \lambda_2}. \quad (33)
 \end{aligned}$$

The remarks made in Section 2 about using the rank $n-2$ epsilon tensor to replace upper index tensors by lower index tensors in conjugate representations, when this reduces the number of indices, applies here. As noted in Section 2, the zero mass Goldstone modes are in the representation $(2, n-2)$, which has a $U(1)$ generator mismatch with respect to the corresponding representation in the expansion of the adjoint representation.

4. Summary

We have derived further properties of $SU(n)$ symmetry breaking by rank three and rank two antisymmetric tensor scalars, extending previous analyses in the literature. The $U(1)$ generator mismatch highlighted in [1,2] is seen to originate from the fact that the $SU(3)$ representation $\bar{3}$ can be represented by a two upper index antisymmetric tensor, or a one lower index tensor, the former occurring in the branching expansion for the rank three antisymmetric tensor, and the latter in the branching expansion for the adjoint. An analogous statement holds for the $SU(2)$ representation $\bar{2} \equiv 2$ in the rank two antisymmetric tensor case. The results of Eqs. (29) and (33) for residual scalar masses will be of use in model building in which $SU(n)$ symmetry is broken by a rank three or rank two antisymmetric tensor scalar field.

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