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Factorization of Laurent series over commutative rings

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ABSTRACT

We generalize the Wiener–Hopf factorization of Laurent series to more general commutative coefficient rings, and we give explicit formulas for the decomposition. We emphasize the algebraic nature of this factorization.

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0. Introduction

Suppose that we have a Laurent series $a(z)$ with rapidly decreasing coefficients and it is multiplicatively invertible, so its multiplicative inverse $a(z)^{-1}$ allows a Laurent series expansion also with rapidly decreasing coefficients; say

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \quad \text{and} \quad a(z)^{-1} = \sum_{n \in \mathbb{Z}} b_n z^n.$$

Then it is an easy exercise in complex function theory that there exists a factorization

$$a(z) = a^-(z) \tilde{a}(z) a^+(z)$$

such that

$$a^-(z) = \sum_{n \in \mathbb{N}} a_n^- z^{-n}, \quad \tilde{a}(z) = \tilde{a}_p z^p, \quad a^+(z) = \sum_{n \in \mathbb{N}} a_n^+ z^n,$$

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$$a^-(z)^{-1} = \sum_{n \in \mathbb{N}} b_n^- z^{-n}, \quad \tilde{a}(z)^{-1} = \tilde{a}_p^{-1} z^{-p}, \quad a^+(z)^{-1} = \sum_{n \in \mathbb{N}} b_n^+ z^n,$$

$$a_0^- = 1, \quad b_0^- = 1, \quad p \in \mathbb{Z}, \quad a_0^+ = 1, \quad b_0^+ = 1,$$

where $a_n^+, a_n^-, b_n^+, b_n^-$ are rapidly decreasing.

If G denotes the unit group of Laurent series with rapidly decreasing coefficients then we find that there is an inner direct product decomposition $G = G^- \times \tilde{G} \times G^+$ to strictly antiholomorphic, monomial, and strictly holomorphic parts, respectively.

In this paper we generalize this situation to coefficients from certain commutative topological rings, and we give explicit formulas for the decomposition in terms of $\{a_n\}_{n \in \mathbb{Z}}$ and $\{b_n\}_{n \in \mathbb{Z}}$.

1. The statement of theorem

1.1. We say that the topological ring \mathfrak{A} is a strong polymetric ring if

- (a) its topology is induced by a family of “seminorms” $p : \mathfrak{A} \rightarrow [0, +\infty)$ such that $p(0) = 0$, $p(-X) = p(X)$, $p(X + Y) \leq p(X) + p(Y)$,
- (b) for each “seminorm” p there exists a “seminorm” \tilde{p} such that for any $n \in \mathbb{N}$ $p(X_1 \cdot \dots \cdot X_n) \leq \tilde{p}(X_1) \cdot \dots \cdot \tilde{p}(X_n)$ holds.

In what follows \mathfrak{A} is assumed to be a commutative, separated, sequentially complete strong polymetric ring with $1 \in \mathfrak{A}$.

1.2. Let $\mathfrak{A}[z^{-1}, z]$ denote the algebra of Laurent series with rapidly decreasing coefficients with the usual topology. (We say that a sequence is rapidly decreasing if it is bounded multiplied by any function of polynomial growth. The Laurent series $a(z)$ is rapidly decreasing if the series $p(a_n)$ is rapidly decreasing for any seminorm p .) Let $G = \mathfrak{A}[z^{-1}, z]^\star$ be the unit group, ie. the group of multiplicatively invertible elements with topology induced from the topology of $\mathfrak{A}[z^{-1}, z]$ via the identity inclusion and the multiplicative inverse jointly.

Let $a(z) \in \mathfrak{A}[z^{-1}, z]^\star$ such that

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^n, \quad \text{and} \quad a(z)^{-1} = \sum_{n \in \mathbb{Z}} b_n z^n.$$

- (a) We say that $a(z)$ is strictly holomorphic, if $a_0 = 1$ and $a_n = 0$ for $n < 0$. It is easy to see that $a(z)^{-1}$ is also strictly holomorphic. In fact, the strictly holomorphic Laurent series form a subgroup $G^+ = \mathfrak{A}[z^{-1}, z]^+$ of the unit group. Similar comment applies to strictly antiholomorphic Laurent series forming the subgroup $G^- = \mathfrak{A}[z^{-1}, z]^-$.
- (b) We say that $a(z)$ is an orthogonal Laurent series if for $n, m \in \mathbb{Z}$, $n \neq m$ the identity $a_n a_m = 0$ holds. In this case it is immediate that $\Pi_n^a = a_n b_{-n}$ is an idempotent, and the idempotents Π_n^a give a pairwise orthogonal decomposition of 1. Then, $a_n \Pi_m^a = \delta_{n,m} a_n$ holds, for what we can say that $a(z)$ is subordinated to the orthogonal decomposition $\{\Pi_n^a\}_{n \in \mathbb{Z}}$. Conversely, if a general $a(z)$ is invertible and it is subordinated to an orthogonal decomposition then $a(z)$ is also orthogonal, and it is subordinated only to a unique decomposition. It is easy to see that $\sum_{n \in \mathbb{Z}} \Pi_n^a b_{-n} z^{-n}$ is an inverse to $a(z)$, hence, from the unicity of the inverse $\Pi_{-n}^a b_m = \delta_{n,m} b_m$ follows. Consequently, $a(z)^{-1}$ is also orthogonal but it is subordinated to $\{\Pi_{-n}^a\}_{n \in \mathbb{Z}} = \{\Pi_n^{a^{-1}}\}_{n \in \mathbb{Z}}$. Furthermore, if $a_1(z)$ is orthogonal subordinated to $\{\Pi_n^{a_1}\}_{n \in \mathbb{Z}}$ and $a_2(z)$ is orthogonal subordinated to $\{\Pi_n^{a_2}\}_{n \in \mathbb{Z}}$ then $a(z) = a_1(z)a_2(z)$ will be orthogonal subordinated to $\{\Pi_n\}_{n \in \mathbb{Z}}$ where $\Pi_n = \sum_{m \in \mathbb{Z}} \Pi_m^{a_1} \Pi_{n-m}^{a_2}$. From that it is easy to see that the orthogonal Laurent series form a subgroup $\tilde{G} = \mathfrak{A}[z^{-1}, z]^\sim$ of the unit group. Clearly, in the case of indecomposable rings, like $\mathfrak{A} = \mathbb{C}$, the orthogonal Laurent series are exactly the (invertible) monomial ones.

$$U^+(z^{-n}, t, w) = \left[\begin{array}{ccc|ccc} \ddots & & & & & \\ & 1 & -tw & & & \\ & & 1 & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 1 & -tw \\ \hline & & & & & 1 \\ \hline & & & & & \ddots \end{array} \right]$$

and similar matrices show up for $U^-(z^n, t, w)$ and $U^-(z^{-n}, t, w)$. We can see that $U^X(a(z), t, w)$ is a rapidly decreasing perturbation of $U(a(z))$. Furthermore, we may also notice that it is perfectly legal to substitute $t = 1$ into $U^X(a(z), t, w)$ even if it is illegal to substitute $t = 1$ into $F_{\mathbb{Z}}^X(t, w)^{-1}$.

2.6. We also define

$$\tilde{U}(a(z), w) = \left[\begin{array}{ccc|ccc} \ddots & \ddots & \vdots & & \vdots & \ddots & \ddots \\ \ddots & a_0 & a_{-1} & -wa_{-2} & -wa_{-3} & \ddots & \\ \cdots & a_1 & a_0 & -wa_{-1} & -wa_{-2} & \cdots & \\ \hline & & & 1 & & & \\ \hline \cdots & -w^{-1}a_2 & -w^{-1}a_1 & & a_0 & a_{-1} & \cdots \\ \ddots & -w^{-1}a_3 & -w^{-1}a_2 & & a_1 & a_0 & \ddots \\ \ddots & \ddots & \vdots & & \vdots & \ddots & \ddots \end{array} \right]$$

This is also a rapidly decreasing perturbation of $U(a(z))$.

2.7. **Lemma.** We have

$$U^R(a(z), 1, w) = \tilde{U}(a(z), w) \left[\begin{array}{ccc|ccc} \ddots & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline \cdots & c_2(w) & c_1(w) & a(w) & c_{-1}(w) & c_{-2}(w) & \cdots \\ \hline & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{array} \right],$$

where the $c_\lambda(w)$ are rapidly decreasing in λ (in both directions).

Proof. By inspection we see that $U^R(a(z), 1, w)$ differs from $\tilde{U}(a(z), w)$ only in the 0th row, and that row can be brought out by a multiplier as above. \square

3. Auxiliaries: determinants

3.1. If A is a rapidly decreasing $S \times S$ matrix then $\det(\mathbf{1}_S + A)$ can be taken as usual, as an alternating sum of products associated to finite permutations of S . This yields a multiplicative operation. In such computations what we have to worry about is that the sum of the seminorms of the components of the permanent of the matrix should be convergent, that provides convergence. Sometimes this naive definition of a determinant gives a convergent result even if the argumentum W is not so nice. Being careful, we use the separate notation $\det W$ in those cases.

Let us use the notation $[T_1, T_2] = T_1 T_2 T_1^{-1} T_2^{-1}$ if it makes sense. If T_1 and T_2 are invertible Toeplitz matrices (meaning that the inverses are also Toeplitz matrices) with commuting symbols (as it is obvious in our case) then we see that $[T_1, T_2]$ is a rapidly decreasing perturbation of the identity matrix.

3.2. Lemma. (a) If C is a rapidly decreasing perturbation of $\mathbf{1}_{\mathbb{Z}}$ or $\mathbf{1}_{\mathbb{X}}$ and T is an invertible Toeplitz matrix then

$$\det C = \det TCT^{-1}.$$

(b) If T_1, T_2, T_3 are invertible Toeplitz matrices and T_1 commutes with T_2 then

$$\det[T_1 T_2, T_3] = \det[T_1, T_3] \det[T_2, T_3].$$

(c) If T_1, T_2 are invertible Toeplitz matrices, and T'_1, T'_2 are rapidly decreasing perturbations, respectively, then

$$\det \left((T'_1 T'_2)(T_1 T_2)^{-1} \right) = \det T'_1 T_1^{-1} \det T'_2 T_2^{-1}.$$

Proof. (a) If A is a rapidly decreasing matrix and T is any matrix of Toeplitz type then $\det(\mathbf{1}_S + AT) = \det(\mathbf{1}_S + TA)$ is easy to see. If T is invertible then after putting AT^{-1} to the place of A we see that $\det(\mathbf{1}_S + A) = \det(\mathbf{1}_S + TAT^{-1}) = \det T(\mathbf{1}_S + A)T^{-1}$. (b) Regarding the first part: According to the multiplicativity of the determinant and part (a) we see that (LHS) = $\det T_2[T_1, T_3]T_2^{-1} \det[T_2, T_3] =$ (RHS). [c]: It follows from multiplicativity and (a). \square

3.3. The orthogonal Laurent series $a(z)$ is called orthonormal if $a(1) = 1$. From the discussion earlier it is clear that any orthogonal $a(z)$ allows a unique decomposition to a constant and an orthonormal part:

$$a(z) = a(1) \sum_{n \in \mathbb{Z}} \Pi_n^a z^n.$$

There is a very natural way in which orthonormal Laurent series occur:

Suppose that P is a $\mathbb{X} \times \mathbb{X}$ matrix such that it is a rapidly decreasing perturbation of $\mathbf{1}_{\mathbb{X}^-}$ and $P^2 = P$. Then, we claim,

$$N_P(z) = \det \left((\mathbf{1}_{\mathbb{X}} - P + zP)(\mathbf{1}_{\mathbb{X}^+} + z\mathbf{1}_{\mathbb{X}^-})^{-1} \right)$$

is an orthonormal Laurent series. This immediately follows from the identities $N_P(1) = 1$ and $N_P(z_1 z_2) = N_P(z_1) N_P(z_2)$; where z_1 and z_2 are independent Laurent series variables. Of these only the second is nontrivial although it follows from Lemma 3.2d with the choices $T'_i = \mathbf{1}_{\mathbb{X}} - P + z_i P$ and $T_i = \mathbf{1}_{\mathbb{X}^+} + z_i \mathbf{1}_{\mathbb{X}^-}$. (Conversely, every orthonormal Laurent series occurs in this special form. Indeed, if $\Pi(z)$ is orthonormal then $P = \sum_{n \in \mathbb{Z}}^{\text{weak}} \Pi_n \mathbf{1}_{\mathbb{X}^- + n}$ can be considered. Here the sum was taken in weak sense, ie. entry-wise in the matrix. Then one can check that $\Pi(z) = N_P(z)$.)

4. The proof of the theorem

4.1. Let

$$\tilde{\pi}(a(z), w) = \det \left(U(a(z)) \tilde{U}(a(z), w)^{-1} \right),$$

$$\pi^+(a(z), t, w) = \det \left(U^+(a(z), t, w) U(a(z))^{-1} \right),$$

$$\pi^-(a(z), t, w) = \det \left(U^-(a(z), t, w) U(a(z))^{-1} \right),$$

$$\pi^\pm(a(z), t, w) = \det \left(U^R(a(z), t, w) U(a(z))^{-1} \right).$$

4.2. **Lemma.** *The following identities hold:*

(a)

$$\pi^\pm(a(z), t, w) = \pi^+(a(z), t, w)\pi^-(a(z), t, w).$$

(b)

$$\pi^\pm(a(z), 1, w) = \pi^+(a(z), 1, w)\pi^-(a(z), 1, w).$$

(c)

$$\pi^\pm(a(z), 1, w) = a(w)\tilde{\pi}(a(z), w)^{-1}.$$

(d)

$$\begin{aligned} \tilde{\pi}(a(z), w)\tilde{\pi}(a(z), 1)^{-1} &= \det(\tilde{U}(a(z), 1)\tilde{U}(a(z), w)^{-1}) \\ &= \det\left(\mathbf{U}_{\mathbb{K}}(a(z))(w\mathbf{1}_{\mathbb{K}^-} + \mathbf{1}_{\mathbb{K}^+})\mathbf{U}_{\mathbb{K}}(a(z)^{-1})(w\mathbf{1}_{\mathbb{K}^-} + \mathbf{1}_{\mathbb{K}^+})^{-1}\right). \end{aligned}$$

Proof. (a) It follows from Lemma 3.2b with the choices $T_1 = F^{R^+}(t, w), T_2 = F^{R^-}(t, w), T_3 = U(a(z))$. (b) We can substitute $t = 1$ into (a). (c) It follows from Lemma 2.7d. The first equality follows immediately from the multiplicativity of the determinant. Regarding the second, take $\tilde{U}(a(z), 1)\tilde{U}(a(z), w)^{-1}$. Remove the single element 1 in the center. Then we obtain a $(\mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \setminus \{0\})$ matrix. Relabel the elementary matrices $\mathbf{e}_{n,m}$ into $\mathbf{e}_{n-\frac{1}{2}\text{sgn}n, m-\frac{1}{2}\text{sgn}m}$. Conjugate by $\mathbf{1}_{\mathbb{K}^-} - \mathbf{1}_{\mathbb{K}^+}$. Then the determinant does not change, yet we obtain $\mathbf{U}_{\mathbb{K}}(a(z))(w\mathbf{1}_{\mathbb{K}^-} + \mathbf{1}_{\mathbb{K}^+})\mathbf{U}_{\mathbb{K}}(a(z)^{-1})(w\mathbf{1}_{\mathbb{K}^-} + \mathbf{1}_{\mathbb{K}^+})^{-1}$ at the end. \square

4.3. **Lemma.** *The following identities/statements hold:*

(a)

$$a(w) = \pi^-(a(z), 1, w)\tilde{\pi}(a(z), w)\pi^+(a(z), 1, w),$$

(b)

$$\begin{aligned} \pi^-(a(z), 1, w) &\in \mathfrak{A}[w, w^{-1}]^-, \\ \tilde{\pi}(a(z), w) &\in \mathfrak{A}[w, w^{-1}]^\sim, \\ \pi^+(a(z), 1, w) &\in \mathfrak{A}[w, w^{-1}]^+. \end{aligned}$$

(c)

$$\begin{aligned} \pi^+(a_1(z)a_2(z), 1, w) &= \pi^+(a_1(z), 1, w)\pi^+(a_2(z), 1, w), \\ \tilde{\pi}(a_1(z)a_2(z), w) &= \tilde{\pi}(a_1(z), w)\tilde{\pi}(a_2(z), w), \\ \pi^-(a_1(z)a_2(z), 1, w) &= \pi^-(a_1(z), 1, w)\pi^-(a_2(z), 1, w). \end{aligned}$$

(d)

$$\begin{aligned} \tilde{\pi}(a(z), w) = \pi^+(a(z), 1, w) &= 1 \quad \text{if } a(z) \in \mathfrak{A}[z, z^{-1}]^-, \\ \pi^-(a(z), 1, w) = \pi^+(a(z), 1, w) &= 1 \quad \text{if } a(z) \in \mathfrak{A}[z, z^{-1}]^\sim, \\ \pi^-(a(z), 1, w) = \tilde{\pi}(a(z), w) &= 1 \quad \text{if } a(z) \in \mathfrak{A}[z, z^{-1}]^+. \end{aligned}$$

Proof. (a) It is a combination of Lemma 4.2b and c. (b) The first and third lines follow from the special form of the matrices involved, which can be understood as special power series in w and w^{-1} respectively. The second line follows from Lemma 4.2d and the discussion about orthonormal Laurent series applied with the choice $P = U(a(z)^{-1})\mathbf{1}_{\mathbb{K}^-}U(a(z))$. (c) They follow from Lemma 3.2c. (d) They follow from that we take the determinant of upper or lower triangular matrices with 1's in the diagonals. \square

4.4. Then we can define the operations π^- , $\tilde{\pi}$, π^+ , respectively, for $a(z)$ by taking $\pi^-(a(z), 1, w)$, $\tilde{\pi}(a(z), w)$, $\pi^+(a(z), 1, w)$, respectively, and replacing w by z in the result. Now, the content of the previous lemma is exactly that these operations serve an inner direct product decomposition of as indicated in Theorem 1.3. By this we have proved the theorem.

5. Explicit formulas

5.1. The operations π^- , $\tilde{\pi}$, π^+ , or rather $\pi^-(a(z), 1, w)$, $\tilde{\pi}(a(z), w)$, $\pi^+(a(z), 1, w)$ are already given in a sufficiently explicit form. Yet, it is useful to have transparent and uniform formulas as much as it is possible.

5.2. **Lemma.** Suppose that A is a rapidly decreasing $\mathbb{Z} \times \mathbb{Z}$ matrix. Then

$$\widetilde{\det} \left(F^{R^+}(t, w) + A \right) = \det \left((F^{R^+}(t, w) + A)F^{R^+}(t, w)^{-1} \right).$$

Similar statement holds with $F^{R^-}(t, w)$.

Proof. For finite matrices A the statement is relatively easy to see. Due to the special form of the matrices the permanents are sufficiently controlled, hence the statement extends. \square

5.3. Proposition

$$\begin{aligned} \pi^+(a(z), 1, w) &= \widetilde{\det} \left(U(a(z)^{-1})F^{R^+}(1, w)U(a(z)) \right) \\ &= \widetilde{\det} \left(\mathbf{1}_{\mathbb{Z}} - wU(a(z)^{-1})\mathbf{1}_{\mathbb{Z}^-}U(a(z))U(z^{-1}) \right), \\ \tilde{\pi}(a(z), w) &= \det \left(U(a(z))\tilde{U}(a(z), 1)^{-1} \right) \\ &\quad \cdot \det \left(U_{\mathbb{X}}(a(z)^{-1})(w\mathbf{1}_{\mathbb{X}^-} + \mathbf{1}_{\mathbb{X}^-})^{-1}U_{\mathbb{X}}(a(z))(w\mathbf{1}_{\mathbb{X}^-} + \mathbf{1}_{\mathbb{X}^-}) \right), \\ \pi^-(a(z), 1, w) &= \widetilde{\det} \left(U(a(z)^{-1})F^{R^-}(1, w)U(a(z)) \right) \\ &= \widetilde{\det} \left(\mathbf{1}_{\mathbb{Z}} - w^{-1}U(a(z)^{-1})\mathbf{1}_{\mathbb{Z}^+}U(a(z))U(z) \right). \end{aligned}$$

Proof. The second equality follows from Lemma 4.2d, while the other ones follow from Lemma 5.2. \square

5.4. The point is that the matrices $U(a(z)^{-1})\mathbf{1}_{\mathbb{Z}^-}U(a(z))$, $U(a(z)^{-1})\mathbf{1}_{\mathbb{Z}^+}U(a(z))$ are relatively transparent perturbations of $\mathbf{1}_{\mathbb{Z}^-}$ and $\mathbf{1}_{\mathbb{Z}^+}$. In particular, if $a(z)$ is finite then these perturbations restrict to finitely many columns, hence, as a result of the block triangular form, effectively reducing the computations to finite matrices. Cf. like when

$$U(a(z)^{-1})F^{R^+}(1, w)U(a(z)) = \mathbf{1}_{\mathbb{Z}} - wU(a(z)^{-1})\mathbf{1}_{\mathbb{Z}^-}U(a(z))U(z^{-1})$$

$$= \left[\begin{array}{ccc|ccc|cc} \ddots & & & \vdots & \vdots & \vdots & & & \\ & \ddots & & * & * & * & & & \\ & & 1 & * & * & * & & & \\ & & & * & * & * & & & \\ & & & & & & & & \\ \hline & & & * & * & * & & & \\ & & & * & * & * & & & \\ & & & * & * & * & & & \\ \hline & & & * & * & * & 1 & & \\ & & & * & * & * & & 1 & \\ & & & \vdots & \vdots & \vdots & & & \ddots \end{array} \right]$$

and taking determinant reduces to the block in the middle. Even there, the determinant will have the shape of a characteristic polynomial $\det(\mathbf{1}_S - wB)$.

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