New Changhee $q$-Euler numbers and polynomials associated with $p$-adic $q$-integrals

Taekyun Kim$^b$, Seog-Hoon Rim$^a$,*

$^a$Department of Mathematics Education, Kyungpook National University, Daegu 702-701, Republic of Korea
$^b$EECS, Kyungpook National University, Taegu 702-701, Republic of Korea

Received 25 July 2006; accepted 12 December 2006

Abstract


© 2007 Elsevier Ltd. All rights reserved.

Keywords: $p$-Adic $q$-Integral; $q$-Euler numbers and polynomials

1. Introduction

Let $p$ be a fixed odd prime in this paper. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ denote the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $v_p(p)$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one speaks of a $q$-extension, $q$ can be regarded as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$; it is always clear from the context. If $q \in \mathbb{C}$, then one usually assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then one usually assumes that $|q - 1|_p < p^{-\frac{1}{p-1}}$, and hence $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. In this paper, we use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad (a : q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad \text{cf. [3–8]}.$$
Note that \(\lim_{q \to 1}[x]_q = x\) for any \(x\) with \(|x|_p \leq 1\) in the \(p\)-adic case. For a fixed positive integer \(d\) with \((p, d) = 1\), set

\[
X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z},
\]

\[
X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp\mathbb{Z}_p,
\]

\[
a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^N}\},
\]

where \(a \in \mathbb{Z}\) satisfies the condition \(0 \leq a < dp^N\) (cf. [10,11]). We say that \(f\) is a uniformly differentiable function at a point \(a \in \mathbb{Z}_p\), and write \(f \in UD(\mathbb{Z}_p)\), if the difference quotients \(F_f(x, y) = \frac{f(x) - f(y)}{x - y}\) have a limit \(f'(a)\) as \((x, y) \to (a, a)\). For \(f \in UD(\mathbb{Z}_p)\), let us begin with the expression

\[
\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p), \quad \text{cf. [4–6,8]},
\]

which represents a \(q\)-analogue of Riemann sums for \(f\). The integral of \(f\) on \(\mathbb{Z}_p\) is defined as the limit of those sums (as \(n \to \infty\)) if this limit exists. The \(p\)-adic \(q\)-integral of a function \(f \in UD(\mathbb{Z}_p)\) is defined by

\[
I_q(f) = \int_X f(x) d\mu_q(x) = \int_{X_d} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x.
\]

Recently, many mathematicians studied Bernoulli and Euler numbers (see [1–24]). Using non-archimedean \(q\)-integrals on \(\mathbb{Z}_p\) defined in [15,16], we define new Changhee \(q\)-Euler polynomials and numbers which are different from those of Kim [7] and Carlitz [2]. We define generating functions of multiple \(q\)-Euler numbers and polynomials. Furthermore we construct a multivariate Hurwitz type zeta function which interpolates the multivariate \(q\)-Euler numbers or polynomials at negative integers.

2. Multivariate \(q\)-Euler numbers and polynomials

Using \(p\)-adic \(q\)-integrals on \(\mathbb{Z}_p\), we now define new \(q\)-Euler polynomials as follows:

\[
\int_{\mathbb{Z}_p} q^x (x + y)^n d\mu_{-1}(y) = E_{n, q}(x).
\]

Note that \(\lim_{q \to 1} E_{n, q}(x) = E_n(x)\), where \(E_n(x)\) are Euler polynomial which are defined by \(\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}\). In the case \(x = 0\), \(E_{n, q} = E_{n, q}(0)\) are called new \(q\)-Euler numbers. And note that \(\lim_{q \to 1} E_{n, q} = E_n\), where \(E_n\) are classical Euler numbers. Let \(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r\) be positive integers. Then we consider a multivariate integral as follows:

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{b_1x_1 + \cdots + b_rx_r} (x + a_1x_1 + \cdots + a_rx_r)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
= E_{n, q}^{(r)}(x|a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r).
\]

Here \(E_{n, q}^{(r)}(x|a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r)\) are called multivariate \(q\)-Euler polynomials of order \(r\).

In the case \(x = 0\) in (1), \(E_{n, q}(0|a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r) = E_{n, q}(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r)\) will be called multivariate \(q\)-Euler numbers of order \(r\).

From (1), we derive the following generating function for multivariate \(q\)-Euler polynomials:

\[
F_q^{(r)}(t, x|a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r)
\]

\[
= \sum_{n=0}^{\infty} E_{n, q}^{(r)}(x|a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r) \frac{t^n}{n!}
\]
Note that \( E_{0,q}^{(r)}(x|a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r) = \frac{2^r}{(q^{b_1} e^{a_1} + 1)(q^{b_2} e^{a_2} + 1) \cdots (q^{b_r} e^{a_r} + 1)} e^{xt} \). (2)

Let \( \chi \) be the Dirichlet character with conductor \( f (=\text{odd}) \in \mathbb{N} \). Then we define the multivariate generalized \( q \)-Euler numbers attached to \( \chi \) as follows:

\[
\int_X \cdots \int_X \chi(x_1) \cdots \chi(x_r) q^{b_1 x_1 + \cdots + b_r x_r} (a_1 x_1 + \cdots + a_r x_r)^n \, d\mu(-1)(x_1) \cdots d\mu(-1)(x_r)
\]

\[= E_{n,\chi,q}(a_1, \ldots, a_r; b_1, \ldots, b_r). \] (3)

From (3), we can derive generating functions for the multivariate generalized \( q \)-Euler numbers attached to \( \chi \),

\[
F_{\chi,q}^{(r)}(t|a_1, \ldots, a_r; b_1, \ldots, b_r) = \sum_{n=0}^{\infty} E_{n,\chi,q}(a_1, \ldots, a_r; b_1, \ldots, b_r) \frac{t^n}{n!}
\]

\[= \sum_{n=0}^{f-1} \sum_{n_1, \ldots, n_r=0}^{r} \left( -1 \right)^{n_1+\cdots+n_r} q^{b_1 n_1 + \cdots + b_r n_r} e^{(a_1 n_1 + \cdots + a_r n_r) t} \chi(n_1) \cdots \chi(n_r) \] (4)

where \( n \) is odd.

Let \( n \) be an odd positive integer. From the definition of \( E_{n,\chi,q}^{(r)}(a_1, \ldots, a_r; b_1, \ldots, b_r) \) in (3), we have the following:

\[
E_{n,\chi,q}^{(r)}(a_1, \ldots, a_r; b_1, \ldots, b_r)
\]

\[= \lim_{N \to \infty} \sum_{n_1, \ldots, n_r=0}^{p^N-1} \chi(n_1) \cdots \chi(n_r)(-1)^{n_1+\cdots+n_r} q^{b_1 n_1 + \cdots + b_r n_r} (a_1 x_1 + \cdots + a_r x_r)^n
\]

\[= \lim_{N \to \infty} \sum_{n_1, \ldots, n_r=0}^{f-1} \sum_{x_1, \ldots, x_r=0}^{p^N-1} \chi(n_1 + f n_2)(-1)^{n_1+\cdots+n_r} q^{b_1 n_1 + \cdots + b_r n_r} (a_1 x_1 + \cdots + a_r n_r)^n
\]

\[= \lim_{N \to \infty} \sum_{x_1, \ldots, x_r=0}^{p^N-1} \left( \sum_{j=1}^{f} \frac{a_j x_j}{f} + \sum_{j=1}^{r} a_j x_j \right)^n
\]

\[= f^f \sum_{n_1, \ldots, n_r=0}^{f-1} (1)^{n_1+\cdots+n_r} \prod_{j=1}^{r} \chi(n_j) q^{-1} b_j n_j
\]

\[\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{b_j x_j} \left( \sum_{j=1}^{f} \frac{a_j x_j}{f} + \sum_{j=1}^{r} a_j x_j \right)^n \, d\mu(-1)(x_1) \cdots d\mu(-1)(x_r)
\]

\[= f^f \sum_{n_1, \ldots, n_r=0}^{f-1} (1)^{n_1+\cdots+n_r} \prod_{j=1}^{r} \chi(n_j) q^{-1} b_j n_j E_{n,\chi,q}^{(r)} \left( \sum_{j=1}^{f} \frac{a_j n_j}{f} \right) a_1, \ldots, a_r; b_1, \ldots, b_r
\]

Therefore we obtain:
Theorem 1. Let \( a_1, \ldots, a_r, b_1, \ldots, b_r \) be positive integers; then we have
\[
E_{n, \chi, q}^{(r)}(a_1, \ldots, a_r; b_1, \ldots, b_r)
\]
\[
= \sum_{n_1, \ldots, n_r = 0}^{f-1} (-1)^{j-1} \prod_{j=1}^{r} \chi(j)^{q^{j-1}} \sum_{j=1}^{r} b_j n_j \left( \frac{\sum_{j=1}^{r} a_j n_j}{f} \right)_{a_1, \ldots, a_r; b_1, \ldots, b_r},
\]
where \( f, n \) are odd positive integers.

3. Multivariate \( q \)-zeta functions in \( \mathbb{C} \)

In this section we assume that \( q \in \mathbb{C} \) with \( |q| < 1 \). Let us assume that \( a_1, \ldots, a_r, b_1, \ldots, b_r \) are positive integers.

The purpose of this section is to study a multivariate Hurwitz type zeta function which interpolates multivariate \( q \)-Euler polynomials of order \( r \) at negative integers. By (2), we easily see that
\[
F_q^{(r)}(t, x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) = 2^r \sum_{n_1, \ldots, n_r = 0}^{\infty} (-1)^{j-1} \sum_{j=1}^{r} b_j n_j \left( \frac{\sum_{j=1}^{r} a_j n_j + x}{q} \right)^{j-1}.
\]

By taking derivatives of order \( k \), on both sides of (6) we obtain the following:

Theorem 2. Let \( k \) be positive odd integer and let \( a_1, \ldots, a_r, b_1, \ldots, b_r \) be positive integers. Then we have
\[
E_{k, q}^{(r)}(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) = 2^r \sum_{n_1, \ldots, n_r = 0}^{\infty} (-1)^{j-1} \sum_{j=1}^{r} b_j n_j \left( \sum_{j=1}^{r} a_j n_j + x \right)^{j-1}.
\]

By the above theorem, we may now construct the complex multivariate \( q \)-zeta functions as follows:

Definition 1. For \( s \in \mathbb{C} \), we define
\[
\zeta_r(s, x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) = \sum_{n_1, \ldots, n_r = 0}^{\infty} \left( \frac{\sum_{j=1}^{r} a_j n_j + x}{\sum_{j=1}^{r} b_j n_j} \right)^{s}.
\]

Thus we note that this function in (8) is an analytic continuation in the whole complex plane. And we see that this multivariate \( q \)-zeta function interpolates \( q \)-Euler polynomials at negative integers.

Theorem 3. Let \( n \) be an odd positive integer. Then we have
\[
\zeta_r(-n, x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) = E_{n, q}^{(r)}(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r).
\]

We now give the complex integral representation of \( \zeta_r(s, x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) \). Using (6), we have the following:
\[
\frac{1}{\Gamma(s)} \int_{\mathbb{C}} F_q^{(r)}(-t, x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) t^{s-1} dt
\]
\[
= 2^r \sum_{n_1, \ldots, n_r = 0}^{\infty} (-1)^{j-1} \sum_{j=1}^{r} b_j n_j \int_{\mathbb{C}} \frac{1}{\Gamma(s)} e^{-\sum_{j=1}^{r} a_j n_j + x} t^{s-1} dt
\]
\[
= 2^r \sum_{n_1, \ldots, n_r = 0}^{\infty} (-1)^{j-1} \sum_{j=1}^{r} b_j n_j \int_{\mathbb{C}} e^{-\sum_{j=1}^{r} n_j a_j + x} y^{s-1} dy
\]
Therefore we have the following:

\[ L \]

Thus by (10) and (11), we have the following:

\[ \zeta_r(s, x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) = \sum_{m=0}^{\infty} (-1)^m \frac{E_m^{(r)}(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r)}{m!} \frac{1}{\Gamma(s)} \int \mathbb{C}^{m+s-1} dt. \] 

Thus we have

\[ \zeta_r(-n, x \mid a_1, \ldots, a_r; b_1, \ldots, b_r) = E_n^{(r)}(x \mid a_1, \ldots, a_r; b_1, \ldots, b_r). \] 

To construct the multivariate Dirichlet \( L \)-function we investigate the generating function of generalized multivariate \( q \)-Euler numbers attached to \( \chi \), which is derived in (4).

\[
2^r \sum_{n_1, \ldots, n_r = 0}^{f-1} \left( \sum_{j=1}^{r} n_j \sum_{q=1}^{r} b_{jn} \sum_{j=1}^{r} a_{jn} \right) \prod_{j=1}^{r} \chi(n_j) \\
= 2^r \sum_{n_1, \ldots, n_r = 0}^{f-1} \left( \sum_{j=1}^{r} n_j \sum_{q=1}^{r} b_{jn} \prod_{j=1}^{r} \chi(n_j) \sum_{x_1, \ldots, x_r = 0}^{\infty} \left( \sum_{j=1}^{r} n_j \sum_{q=1}^{r} f_{bx_j} \sum_{e=1}^{r} f_{ax_j} \right) \prod_{j=1}^{r} \chi(n_j + f x_j) \right) \\
= 2^r \sum_{n_1, \ldots, n_r = 0}^{\infty} \left( \sum_{j=1}^{r} n_j \sum_{q=1}^{r} b_{jn} \prod_{j=1}^{r} \chi(n_j) \right) \left( \sum_{j=1}^{r} a_{jn} \right) .
\]

Thus we can write

\[
F_{\chi, q}^{(r)}(t \mid a_1, \ldots, a_r; b_1, \ldots, b_r) = 2^r \sum_{n_1, \ldots, n_r = 1}^{\infty} \left( \sum_{j=1}^{r} n_j \sum_{q=1}^{r} b_{jn} \prod_{j=1}^{r} \chi(n_j) \right) \left( \sum_{j=1}^{r} a_{jn} \right) \\
(\text{since } \chi(0) = 0, \text{ we can start at } n_1 = 1, n_2 = 1, \ldots, n_r = 1) \\
= \sum_{n=1}^{\infty} E_{n, \chi, q}(a_1, \ldots, a_r; b_1, \ldots, b_r) t^n \frac{1}{n!}.
\] 

From (13), we can derive the following:

\[
2^r \sum_{n_1, \ldots, n_r = 1}^{\infty} \left( \sum_{j=1}^{r} n_j \sum_{q=1}^{r} b_{jn} \prod_{j=1}^{r} \chi(n_j) \right) (a_1 n_1 + \cdots + a_r n_r)^k = E_{k, \chi, q}(a_1, \ldots, a_r; b_1, \ldots, b_r).
\]

Therefore we have the following:
Definition 2. For $s \in \mathbb{C}$, define multivariate Dirichlet $L$-function as follows:

$$L_r(s, \chi | a_1, \ldots, a_r; b_1, \ldots, b_r) = 2^r \sum_{n_1, \ldots, n_r = 1}^{\infty} \frac{(-1)^{n_1 + \cdots + n_r} q^{b_1 n_1 + \cdots + b_r n_r} \chi(n_1) \cdots \chi(n_r)}{(a_1 n_1 + \cdots + a_r n_r)^s}. \quad (15)$$

Note that $L_r(s, \chi | a_1, \ldots, a_r; b_1, \ldots, b_r)$ is also an analytic function in the whole complex plane. By (12)–(15), we see that the $q$-analogue multivariate Dirichlet $L$-function interpolates multivariate generalized $q$-Euler numbers attached to $\chi$ at negative integers as follows:

Theorem 4. Let $k$ be a positive integer. Then we have

$$L_r(-k, \chi | a_1, \ldots, a_r; b_1, \ldots, b_r) = E_{k, \chi, q}(a_1, \ldots, a_r; b_1, \ldots, b_r).$$

Acknowledgements

This paper was supported by Jangjeon Research Institute for Mathematical Science (JRIMS-06-C0001) and Jangjeon Mathematical Society.

References