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SOME GENERATING FUNCTIONS FOR THE JACOBI POLYNOMIALS

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Abstract—A certain family of generating functions for the classical Jacobi polynomials, given earlier by R.K. Saxena and Z. Hussain [1], are observed here to be derivable directly from some known (and readily available) results in the literature. An interesting generalization of the aforementioned generating functions emerges naturally in the process. Some relevant connections of these results with various known generating functions are also pointed out.

1. INTRODUCTION AND PRELIMINARIES

For the classical Jacobi polynomials (cf., e.g., Szegö [2, p. 62])

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left[-n, \alpha+\beta+n+1; \alpha+1; \frac{1}{2}(1-x)\right], \tag{1.1}$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$, Jacobi's *original* notation:

$$\mathcal{F}_n(\lambda, \mu; z) = {}_2F_1[-n, \lambda+n; \mu; z] \tag{1.2}$$

can be found, among other places, in the early editions of the book by Magnus and Oberhettinger [3]. Making use of an *alternative* notation:

$$\begin{aligned}\Phi_n(\lambda; \mu; z) &= \frac{(\lambda)_n}{n!}\, {}_2F_1[-n, \lambda+n; \mu; z] \\ &= \frac{(\lambda)_n}{n!}\, \mathcal{F}_n(\lambda, \mu; z) \\ &= \frac{(\lambda)_n}{(\mu)_n}\, P_n^{(\mu-1,\lambda-\mu)}(1-2z),\end{aligned} \tag{1.3}$$

Saxena and Hussain [1] proved two interesting generating functions for the Jacobi polynomials. In terms of Kampé de Fériet's double hypergeometric function (*cf.* [4] and [5, p. 150]; see also [6, p. 63]):

$$F^{p:r;u}_{q:s;v}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p: & a_1,\ldots,a_r; & c_1,\ldots,c_u; \\ & & & x, y \\ \beta_1,\ldots,\beta_q: & b_1,\ldots,b_s; & d_1,\ldots,d_v;\end{matrix}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_{m+n}\prod_{j=1}^{r}(a_j)_m\prod_{j=1}^{u}(c_j)_n}{\prod_{j=1}^{q}(\beta_j)_{m+n}\prod_{j=1}^{s}(b_j)_m\prod_{j=1}^{v}(d_j)_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}, \tag{1.4}$$

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where, for convergence of the double hypergeometric series,

(i) $p+r<q+s+1, \quad p+u<q+v+1, \quad |x|<\infty, \quad \text{and} \quad |y|<\infty,$

or

(ii) $p+r=q+s+1, \quad p+u=q+v+1,$

and

$$\begin{cases} |x|^{1/(p-q)}+|y|^{1/(p-q)}<1, & \text{if } p>q, \\ \max\{|x|,|y|\}<1, & \text{if } p\le q, \end{cases} \tag{1.5}$$

we recall their main results in the following *equivalent* forms:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n(\alpha+\beta+1)_n}{(\alpha+1)_n(\alpha+\beta-\lambda+2)_n}P_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-\alpha-\beta-1}F_{1:1;0}^{1:2;1}\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1): & \lambda,\frac{1}{2}(\alpha+\beta)+1; & \frac{1}{2}(\alpha+\beta)-\lambda+1; \\ \alpha+\beta-\lambda+2: & \alpha+1; & \text{———};\end{matrix}\ X,Y\right] \tag{1.6}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n(\alpha+\beta+1)_n}{(\alpha+1)_n(\alpha+\beta-\lambda+2)_n(\alpha+\beta-\mu+2)_n}P_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-\alpha-\beta-1}F_{2:1;0}^{2:2;1}\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(\alpha+\beta)+1: & \lambda,\mu; & \alpha+\beta-\lambda-\mu+2; \\ \alpha+\beta-\lambda+2,\ \alpha+\beta-\mu+2: & \alpha+1; & \text{———};\end{matrix}\ X,Y\right], \tag{1.7}$$

where, and in what follows, we have set

$$X=-\frac{2(1-x)t}{(1-t)^2}, \quad Y=-\frac{4t}{(1-t)^2} \quad (|t|<1). \tag{1.8}$$

Formula (1.6) is a limiting case of the generating function (1.7) when $|\mu|\to\infty$. Indeed, since (*cf.*, *e.g.*, Erdélyi *et al.* [7, p. 47])

$$\frac{(\mu)_n}{(\alpha+\beta-\mu+2)_{m+n}}=(-1)^{m+n}\mu^{-m}\{1+O(\mu^{-1})\} \tag{1.9}$$
$$(|\mu|\to\infty;\ |\arg(\mu)|\le\pi-\epsilon, \quad 0<\epsilon<\pi; \quad m,n=0,1,2,3,\dots),$$

the limiting case of (1.7) when $|\mu|\to\infty$ yields

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n(\alpha+\beta+1)_n}{(\alpha+1)_n(\alpha+\beta-\lambda+2)_n}P_n^{(\alpha,\beta)}(x)(-t)^n = (1-t)^{-\alpha-\beta-1}F_{1:1;0}^{2:1;0}\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(\alpha+\beta)+1: & \lambda; & \text{——}; \\ \alpha+\beta-\lambda+2: & \alpha+1; & \text{——};\end{matrix}\ -X,Y\right]. \tag{1.10}$$

Now apply the definition (1.4) to rewrite this $F_{1:1;0}^{2:1;0}$ as an infinite series of the Gauss hypergeometric function:

$${}_2F_1\left[m+\frac{1}{2}(\alpha+\beta+1), m+\frac{1}{2}(\alpha+\beta)+1; m+\alpha+\beta-\lambda+2; Y\right] \quad (m=0,1,2,\dots),$$

and make use of the Pfaff-Kummer transformation [7, p. 64]:

$${}_2F_1[a,b;c;z]=(1-z)^{-a}\,{}_2F_1\left[a,c-b;c;\frac{z}{z-1}\right] \tag{1.11}$$

to each term of the series. Upon replacing t on both sides of the resulting equation by $-t$, (1.10) thus leads us to the generating function (1.6).

A further limiting case of the generating function (1.6) when $|\lambda| \to \infty$ (or, equivalently, of the generating function (1.7) when $\min\{|\lambda|, |\mu|\} \to \infty$) would similarly yield the well-known result (*cf.* [8, p. 102, Example 19]; see also [9, p. 256] and [6, p. 112]):

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-\alpha-\beta-1}\, {}_2F_1\left[\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta)+1; \alpha+1; X\right], \tag{1.12}$$

where X is given by (1.8).

Formula (1.12) follows also when we set

$$\lambda = \frac{1}{2}(\alpha+\beta)+1$$

in the generating function (1.6) or when we set

$$\lambda = \mu = \frac{1}{2}(\alpha+\beta)+1$$

in the more general result (1.7). As a matter of fact, the generating function (1.6) can be deduced also as an *obvious* special case of (1.7) by setting

$$\mu = \frac{1}{2}(\alpha+\beta)+1.$$

In view of such important consequences of the generating function (1.7) as (1.6) and (1.12), we first show how (1.7) can be derived *directly* from some known (and readily available) results in the literature. We also present an interesting further generalization of the generating function (1.7).

2. ALTERNATIVE DERIVATION OF THE GENERATING FUNCTION (1.7)

As one of the many special cases of his hypergeometric generating function [10, p. 76, Equation (3.1)], Srivastava [10] derived the following generating function for the Jacobi polynomials (see [10, p. 78] and [6, p. 145]):

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\gamma_j)_n}{(\alpha+1)_n(\beta+1)_n \prod_{j=1}^{q}(\delta_j)_n} P_n^{(\alpha,\beta)}(x)t^n = F_{q:1;1}^{p:0;0}\left[\begin{matrix}\gamma_1,\ldots,\gamma_p: & \text{——}; & \text{——}; \\ \delta_1,\ldots,\delta_q: & \alpha+1; & \beta+1;\end{matrix}\ \tfrac{1}{2}(x-1)t,\ \tfrac{1}{2}(x+1)t\right]. \tag{2.1}$$

Setting

$$\begin{cases} p-2=q=2, \gamma_1=\lambda, \gamma_2=\mu, \gamma_3=\alpha+\beta+1, \gamma_4=\beta+1, \\ \delta_1=\alpha+\beta-\lambda+2, \text{ and } \delta_2=\alpha+\beta-\mu+2, \end{cases} \tag{2.2}$$

the familiar result (2.1) readily yields

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n(\alpha+\beta+1)_n}{(\alpha+1)_n(\alpha+\beta-\lambda+2)_n(\alpha+\beta-\mu+2)_n} P_n^{(\alpha,\beta)}(x)t^n = F_{2:1;1}^{4:0;0}\left[\begin{matrix}\lambda,\ \mu,\ \alpha+\beta+1, & \beta+1: & \text{——}; & \text{——}; \\ \alpha+\beta-\lambda+2, & \alpha+\beta-\mu+2: & \alpha+1; & \beta+1;\end{matrix}\ \tfrac{1}{2}(x-1)t,\ \tfrac{1}{2}(x+1)t\right]. \tag{2.3}$$

The left-hand sides of (1.7) and (2.3) are identical. In order to show that their right-hand sides are also identical, let $\Omega(t)$ denote the second member of the generating function (2.3). Then it follows from the definition (1.4) and the hypergeometric identity [11, p. 28, Equation 1.3(30)] that

$$\begin{aligned}\Omega(t) = &\sum_{m=0}^{\infty} \frac{(\lambda)_m(\mu)_m(\alpha+\beta+1)_m(\beta+1)_m}{(\alpha+\beta-\lambda+2)_m(\alpha+\beta-\mu+2)_m(\alpha+1)_m} \frac{\{\frac{1}{2}(x-1)t\}^m}{m!} \\ &\times \sum_{n,p=0}^{\infty} \frac{(\lambda+m)_{n+p}(\mu+m)_{n+p}(\alpha+\beta+m+1)_{n+p}(\beta+m+1)_{n+p}}{(\alpha+\beta-\lambda+m+2)_{n+p}(\alpha+\beta-\mu+m+2)_{n+p}(\beta+1)_{n+p}} \\ &\times \frac{t^n}{n!}\frac{\{\frac{1}{2}(x-1)t\}^p}{p!},\end{aligned} \tag{2.4}$$

where we have used the elementary observation that

$$\frac{1}{2}(x+1)t = t + \frac{1}{2}(x-1)t.$$

Now replace the summation index m in (2.4) by $m-p$ and rearrange the resulting triple sum to the form:

$$\begin{aligned}\Omega(t) = &\sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n}(\mu)_{m+n}(\alpha+\beta+1)_{m+n}(\beta+1)_{m+n}}{(\alpha+\beta-\lambda+2)_{m+n}(\alpha+\beta-\mu+2)_{m+n}(\alpha+1)_m(\beta+1)_n} \\ &\times \;{}_2F_1[-m,-\alpha-m;\beta+n+1;1]\frac{\{\frac{1}{2}(x-1)t\}^m}{m!}\frac{t^n}{n!} \\ = &\sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n}(\mu)_{m+n}(\alpha+\beta+1)_{2m+n}}{(\alpha+\beta-\lambda+2)_{m+n}(\alpha+\beta-\mu+2)_{m+n}(\alpha+1)_m} \frac{\{\frac{1}{2}(x-1)t\}^m}{m!}\frac{t^n}{n!} \\ = &\sum_{m=0}^{\infty} \frac{(\lambda)_m(\mu)_m(\alpha+\beta+1)_{2m}}{(\alpha+\beta-\lambda+2)_m(\alpha+\beta-\mu+2)_m(\alpha+1)_m} \frac{\{\frac{1}{2}(x-1)t\}^m}{m!} \\ &\times \;{}_3F_2\left[\begin{matrix}\alpha+\beta+2m+1, & \lambda+m, \; \mu+m; \\ \alpha+\beta-\lambda+m+2, & \alpha+\beta-\mu+m+2;\end{matrix}\; t\right],\end{aligned} \tag{2.5}$$

where we have applied the Chu-Vandermonde theorem [8, p. 3] to sum the terminating Gauss hypergeometric series with argument 1.

Finally, by appealing to Whipple's transformation [8, p. 97, Example 4 (iv)]:

$$\begin{aligned}{}_3F_2\left[\begin{matrix}a, & b, & c; \\ & a-b+1, & a-c+1;\end{matrix}\; z\right] \\ = (1-z)^{-a}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}a, & \frac{1}{2}(a+1), & a-b-c+1; \\ & a-b+1, & a-c+1;\end{matrix}\; -\frac{4z}{(1-z)^2}\right] \quad (|z|<1),\end{aligned} \tag{2.6}$$

and interpreting the resulting double series by means of the definition (1.4), (2.5) leads us precisely to the second member of the generating function (1.7). This evidently completes our alternative derivation of the generating function (1.7) as a further special case of the familiar result (2.1).

3. FURTHER GENERALIZATIONS AND BASIC (OR q-) EXTENSIONS

Some interesting unifications (and generalizations) of several generating functions for hypergeometric polynomials (due, for example, to Chaundy [12], Burchnall [13], and Rainville [14]) were considered by Srivastava [15] who also gave the basic (or $q-$) extensions of his general results. Of our interest here is a very special case of one of Srivastava's results [15] which have since been reproduced systematically by Srivastava and Manocha [6, p. 142 *et seq.*]. By further particularizing this special case (*cf.* [15, p. 329, Equation (1.9)] and [6, p. 142, Equation 2.6(21)]) suitably, we obtain (see also [6, p. 177, Problem 32])

$$\sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{r}(c_j)_n}{\Pi_{j=1}^{s}(d_j)_n}\, {}_{p+2}F_q \left[\begin{matrix} -n, & \omega+n, & a_1,\ldots,a_p; \\ & & b_1,\ldots,b_q; \end{matrix}\; z\right] \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{(\omega+n)_n \,\Pi_{j=1}^{p}(a_j)_n\, \Pi_{j=1}^{r}(c_j)_n}{\Pi_{j=1}^{q}(b_j)_n\, \Pi_{j=1}^{s}(d_j)_n} \frac{(-zt)^n}{n!} \tag{3.1}$$
$$\times \quad {}_{r+1}F_{s+1} \left[\begin{matrix} \omega+2n, & c_1+n,\ldots,c_r+n; \\ \omega+n, & d_1+n,\ldots,d_s+n; \end{matrix}\; t\right]$$
$$(r \leq s, |t| < \infty; r = s+1, |t| < 1),$$

which may be looked upon as a generalization of the generating function (2.1).

Upon setting

$$\begin{cases} r-1 = s = 2, c_1 = \lambda, c_2 = \mu, c_3 = \omega, \\ d_1 = \omega-\lambda+1, d_2 = \omega-\mu+1, \end{cases} \tag{3.2}$$

and applying Whipple's transformation (2.6) to the resulting hypergeometric ${}_3F_2$ function on the right-hand side of (3.1), we arrive finally at the generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n(\omega)_n}{(\omega-\lambda+1)_n(\omega-\mu+1)_n}\, {}_{p+2}F_q \left[\begin{matrix} -n, & \omega+n, & a_1,\ldots,a_p; \\ & & b_1,\ldots,b_q; \end{matrix}\; z\right] \frac{t^n}{n!}$$
$$= (1-t)^{-\omega} F^{2:p+2;1}_{2:\ q;0} \left[\begin{matrix} \frac{1}{2}\omega, \frac{1}{2}(\omega+1): & \lambda,\mu, & a_1,\ldots,a_p; & \omega-\lambda-\mu+1; \\ \omega-\lambda+1, \omega-\mu+1: & & b_1,\ldots,b_q; & \text{———}; \end{matrix}\; -\frac{4zt}{(1-t)^2}, -\frac{4t}{(1-t)^2}\right] \quad (|t|<1). \tag{3.3}$$

In view of the definition (1.1), this last result (3.3) would readily yield the generating function (1.7) in the special case when

$$p = q-1 = 0, b_1 = \alpha+1, \omega = \alpha+\beta+1, \text{ and } z = \frac{1}{2}(1-x).$$

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