Symmetric quivers, invariant theory, and saturation theorems for the classical groups

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Abstract

Let $G$ denote either a special orthogonal group or a symplectic group defined over the complex numbers. We prove the following saturation result for $G$: given dominant weights $\lambda_1, \ldots, \lambda_r$ such that the tensor product $V_{N\lambda_1} \otimes \cdots \otimes V_{N\lambda_r}$ contains nonzero $G$-invariants for some $N \geq 1$, we show that the tensor product $V_{2\lambda_1} \otimes \cdots \otimes V_{2\lambda_r}$ also contains nonzero $G$-invariants. This extends results of Kapovich–Millson and Belkale–Kumar and complements similar results for the general linear group due to Knutson–Tao and Derksen–Weyman. Our techniques involve the invariant theory of quivers equipped with an involution and the generic representation theory of certain quivers with relations.

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1. Introduction

Throughout, we fix an algebraically closed field $K$. We shall assume that $K$ is of characteristic 0 in the introduction. However, as some results in this paper extend to positive characteristic, we will mention what assumptions we make on the characteristic within each section of the paper.

When $G$ is a reductive group defined over $K$, and $\lambda$ is a dominant weight of $G$, the notation $V_\lambda$ denotes an irreducible representation of $G$ with highest weight $\lambda$. Also, if $W$ is a representation
of $G$, we use $W^G$ to denote the subspace of $G$-invariants. The following theorem is the main result of the paper.

**Theorem 1.1.** Let $G$ be either a special orthogonal or symplectic group, and let $\lambda^1, \ldots, \lambda^r$ be dominant weights of $G$. If $(V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^r})^G \neq 0$ for some $N \geq 1$, then $(V_{2\lambda^1} \otimes \cdots \otimes V_{2\lambda^r})^G \neq 0$.

We say that 2 is a saturation factor for the special orthogonal and symplectic groups. We will use the notation $\text{SO}(m)$, $\text{O}(m)$, and $\text{Sp}(2n)$ to mean special orthogonal, orthogonal, and symplectic groups, respectively.

**Corollary 1.2.** Let $G$ be the spin group $\text{Spin}(m)$, and let $\lambda^1, \ldots, \lambda^r$ be dominant weights of $G$. If $(V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^r})^G \neq 0$ for some $N \geq 1$, then $(V_{4\lambda^1} \otimes \cdots \otimes V_{4\lambda^r})^G \neq 0$.

**Proof.** If $\lambda$ is a dominant weight of $\text{Spin}(m)$, then $2\lambda$ is a dominant weight of $\text{SO}(m)$, and the action of $\text{Spin}(m)$ factors through $\text{SO}(m)$ on each $V_{4\lambda}$.

1.1. History and related results

Before we give an outline for the proof of Theorem 1.1, we mention some historical context for the theorem and some results that have previously been proven in this direction.

The results start with the so-called saturation conjecture proven by Knutson and Tao [12] and Derksen and Weyman [6].

**Theorem (Knutson–Tao, Derksen–Weyman).** Let $\lambda^1, \ldots, \lambda^r$ be dominant weights of $G = \text{GL}(n)$. If $(V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^r})^G \neq 0$, then $(V_{\lambda^1} \otimes \cdots \otimes V_{\lambda^r})^G \neq 0$.

This problem itself was inspired by Klyachko’s solution [11] of Horn’s problem of characterizing the possible eigenvalues of Hermitian matrices $A_1, \ldots, A_r$ whose sum is 0. We leave the details out and refer to Fulton’s paper [8] for a survey and further references.

When $r = 3$, this theorem can be restated in terms of the Littlewood–Richardson rule (see [21, Theorem 2.3.4] or [17, §12.5]), which gives an explicit combinatorial recipe for calculating the dimension of the $G$-invariant subspace of a triple tensor product, or equivalently, for calculating tensor product multiplicities. However, the formulation of this rule is not conducive to proving the saturation property. The proof of Knutson and Tao involves formulating a new combinatorial rule which more manifestly possesses the saturation property. However, this approach seems to be difficult to generalize. Our paper will follow the ideas of Derksen and Weyman. Before reviewing the ideas from that paper, we mention some other saturation results to put Theorem 1.1 into perspective. We refer the reader to [13] for more results and conjectures related to tensor product multiplicities.

**Theorem (Kapovich–Millson).** Let $G$ be a simple connected group over $K$, and let $\lambda^1, \ldots, \lambda^r$ be dominant weights of $G$ such that $\gamma^1 + \cdots + \gamma^r$ is in the root lattice of $G$. Let $k$ be the least common multiple of the coefficients of the highest root of $G$ written in terms of simple roots. If $(V_{N\gamma^1} \otimes \cdots \otimes V_{N\gamma^r})^G \neq 0$ for some $N \geq 1$, then $(V_{k\gamma^1} \otimes \cdots \otimes V_{k\gamma^r})^G \neq 0$.

See [10, Corollary 7.3 and Remark 7.2]. For the special orthogonal and symplectic groups, this gives a saturation factor of 4, which our Theorem 1.1 improves to 2 (and drops the assumption...
that $\lambda_1 + \cdots + \lambda_r$ be in the root lattice). The improvements for the odd orthogonal groups and symplectic groups have previously been shown by Belkale and Kumar [3, Theorems 6 and 7]. So Theorem 1.1 provides a new result for $G = \text{SO}(2n)$. Even in the known cases of the odd orthogonal groups and symplectic groups, we believe that our proof still has merit in that the ideas are uniform with respect to the classical groups and in some sense are more elementary. Corollary 1.2 provides a slight improvement to the general result of Kapovich and Millson as the next example shows.

**Example 1.3.** Let $G = \text{Spin}(10)$, label the Dynkin diagram $D_5$ as

```
4
1 — 2 — 3 — 5
```

and let $\omega_1, \ldots, \omega_5$ be the corresponding fundamental weights. Then $V_{\omega_1} \otimes \cdots \otimes V_{\omega_5}$ contains a nonzero $G$-invariant, but $2(\omega_2 + \omega_4 + \omega_5) + (2\omega_1 + \omega_2 + \omega_5)$ is not in the root lattice. Furthermore, none of these weights are sums of minuscule weights.

The relevance of the condition that $\lambda_1 + \cdots + \lambda_r$ be in the root lattice is that $(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r})^G$ can only be nonzero if this condition holds: $\lambda_1 + \cdots + \lambda_r$ and 0 would be weights of the tensor product, and any weights of a representation are equal modulo the root lattice. Conjecturally, the saturation constant in the general result of Kapovich and Millson for the even spin group can be shown to be 1. The index of the root lattice in the weight lattice for $\text{SO}(2n + 1)$, $\text{Sp}(2n)$, and $\text{SO}(2n)$, is 2, 1, and 2, respectively, so this conjecture includes the statement of Theorem 1.1. This more general statement has been proven for $\text{Spin}(8)$, see [9].

**Example 1.4.** Theorem 1.1 cannot be strengthened by replacing even saturation with actual saturation because the condition that $\lambda_1 + \cdots + \lambda_r$ be in the root lattice is not a linear condition. More explicitly, we have the following tensor product decompositions in types B, C, and D:

$$
V_{(1)} \otimes V_{(1)} = V_{(2)} \oplus V_{(1,1)} \oplus V_{(0)}
$$
$$
V_{(2)} \otimes V_{(2)} = V_{(4)} \oplus V_{(3,1)} \oplus V_{(2,2)} \oplus V_{(2)} \oplus V_{(1,1)} \oplus V_{(0)}
$$

where we have identified weights with partitions as in Section 4.1. In other words, $(V_{(1)}^\otimes 3)^G = 0$ but $(V_{(2)}^\otimes 2)^G \neq 0$. Note that the weight $(3)$ is in the root lattice in type B, but not in types C and D, so even taking into account the root lattice condition, the saturation factor cannot be 1 for type B. Counterexamples are also known for type C.

**1.2. Outline of the paper**

We first outline the proof of saturation for the general linear group due to Derksen and Weyman [6]. That paper is a study of the semi-invariants of quivers without oriented cycles, see Section 2.1 for definitions. The connection to tensor product multiplicities is as follows. Given dominant weights $\lambda_1, \ldots, \lambda_r$ for $\text{GL}(n)$, one can produce a quiver $Q$, a dimension vector $\beta$, and a weight $\theta$ such that
for any $N \geq 1$. So the saturation problem can be reduced to proving a related saturation problem for semi-invariants of the quiver $Q$. In fact, they prove this saturation result for any quiver without oriented cycles. More precisely, one proves the following equivalences:

$$\text{SI}(Q, \beta)_{N\theta} \neq 0 \iff \text{Ext}^1(N\alpha, \beta) = 0 \iff \text{Ext}^1(\alpha, \beta) = 0$$

(1.6)

see Section 3.1.3 for definitions.

The benefit from working in this more general context is that it allows reductions to smaller quivers or smaller dimension vectors that are not necessarily related to tensor product multiplicities. There are actually two key inductions. Given a representation $W$ of $Q$, Schofield [18] introduced the determinantal semi-invariant $c_W$ which is a nonzero function if and only if $\text{Ext}^1(W, \beta)$ generically vanishes. The main result of [6] is that the spaces of semi-invariants are linearly spanned by the $c_W$ for various representations $W$ of dimension $\alpha$. This provides the first and third equivalences above. Their proof involves a series of reductions to smaller quivers with the base case being the generalized Kronecker quiver on two vertices. The second equivalence was provided by Schofield [19], who showed that the dimension of the generic extension group $\text{Ext}^1(\alpha, \beta)$ can be calculated recursively from smaller dimension vectors, and the form of this recursion shows that the dimension is 0 if and only if $\alpha$ satisfies a finite system of linear inequalities that depend only on $\beta$ and $\theta$.

So one may hope that the ideas in the above proof can be generalized to the other classical groups. In trying to get an analogue of (1.5) for the orthogonal and symplectic groups, one needs to introduce two complications to the quiver $Q$. First, one introduces an involution on $Q$ and restricts to studying the representations compatible with this involution. We call these symmetric quivers and their symmetric representations. Second, one replaces the path algebra $KQ$ by a certain quotient ring $KQ/I$. Geometrically, both of these complications amount to restricting to certain subvarieties of the representation varieties of $Q$.

With regard to the first complication, the spaces of semi-invariants are no longer spanned by the determinantal semi-invariants. The fact responsible for this is that the determinant of a generic skew-symmetric matrix is the square of its Pfaffian. Motivated by this, we introduce Pfaffian semi-invariants as square roots of determinantal semi-invariants and show that they linearly span the space of semi-invariants for symmetric quivers. This is the content of Section 2. We remark that we know of no general criterion for a determinantal semi-invariant to possess a square root.

The second complication has the following effect. Any submodule of a projective module over $KQ$ is also projective. This is the same as saying that the global dimension of $KQ$ is at most 1. This fact was used extensively in Schofield’s proofs. In general, the global dimension of $KQ/I$ is bigger than 1. In our case, it is 2, so the problem is not so bad, but Schofield’s results no longer apply. To get around this, we extend Schofield’s results in Section 3 in the case of global dimension 2 under certain assumptions which are sufficient for our applications.

Finally, in Section 4, we combine these two generalizations to prove the analogue of (1.6).

1.3. Conventions

All topological notions refer to the Zariski topology. For us, a variety is a separated finite type scheme over $K$ which need not be irreducible nor reduced. Any field that is implicitly used in
this paper refers to the algebraically closed field $K$. The set of integers is denoted by $\mathbb{Z}$, and the set of nonnegative integers is denoted by $\mathbb{N}$.

2. Semi-invariants of symmetric quivers

2.1. Quivers

As a general reference for quivers, we refer to [2]. In this section, no assumption on the characteristic of $K$ is made. A quiver $Q$ is the data $(Q_0, Q_1, t, h)$ where $Q_0$ is the vertex set, $Q_1$ is the arrow set, and $t, h : Q_1 \to Q_0$ are functions. For $a \in Q_1$ we call $ta$ and $ha$ its tail and head and depict it by the following diagram:

$$ta \xrightarrow{a} ha,$$

so that the definitions of paths, cycles, etc. should be self-evident. The path algebra $KQ$ is defined as a vector space to be the finite linear combinations of paths in $Q$. The product of two paths $p_1$ and $p_2$ is defined to be the concatenation $p_1 p_2$ if this is a well-defined path (the sequence $a_n \cdots a_1$ means the path that starts with $a_1$ and ends with $a_n$) and is 0 otherwise.

We will always assume that $Q_0$ is finite and that $Q$ has no directed cycles.

2.1.1. Representations

Elements $\beta \in \mathbb{N}^{Q_0}$ are dimension vectors. The Euler form $\langle \cdot, \cdot \rangle _Q : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is

$$\langle \alpha, \beta \rangle _Q = \sum_{x \in Q_0} \alpha(x) \beta(x) - \sum_{a \in Q_1} \alpha(ta) \beta(ha).$$

When $Q$ is clear from context, we will drop the subscript. We define the representation variety

$$\text{Rep}(Q, \beta) = \bigoplus_{a \in Q_1} \text{Hom}(K^{\beta(ta)}, K^{\beta(ha)}),$$

and the groups

$$\text{GL}(Q, \beta) = \prod_{x \in Q_0} \text{GL}(K^{\beta(x)}), \quad \text{SL}(Q, \beta) = \prod_{x \in Q_0} \text{SL}(K^{\beta(x)})$$

which act on $\text{Rep}(Q, \beta)$ via

$$(gx)_{x \in Q_0} \cdot (\varphi_a)_{a \in Q_1} = (g_{ha} \varphi_a g^{-1}_{ta})_{a \in Q_1}.$$
2.1.2. Semi-invariants

For an affine variety \( X \), we denote its coordinate ring by \( K[X] \). For a quiver \( Q \) and dimension vector \( \beta \in \mathbb{N}^{Q_0} \), we define the ring of semi-invariants as the invariants

\[
\text{SI}(Q, \beta) = K[\text{Rep}(Q, \beta)]^{\text{SL}(Q, \beta)}.
\]

We grade it by the characters \( \chi \) of \( \text{GL}(Q, \beta) \)

\[
\text{SI}(Q, \beta)_\chi = \{ f \in \text{SI}(Q, \beta) \mid g \cdot f = \chi(g)f \text{ for all } g \in \text{GL}(Q, \beta) \}.
\]

We call \( \chi \) the weight of these semi-invariants. The characters of \( \text{GL}(Q, \beta) \) are of the form

\[
(\varphi_x)_{x \in Q_0} \mapsto \prod_{x \in Q_0} (\det g_x)^{\sigma(x)}
\]

for \( \sigma \in \mathbb{Z}^{Q_0} \). Hence we can identify weights with elements of \( \mathbb{Z}^{Q_0} \).

For \( x, y \in Q_0 \), let \([x,y]\) denote the \( K \)-vector space whose basis is the paths from \( x \) to \( y \). The indecomposable projective representations of \( Q \) are indexed by \( Q_0 \): for \( x \in Q_0 \), set \( P_x \) to be the representation with \( P_x(y) = [x,y] \) for \( y \in Q_0 \) and \( P_{x,a} : [x,ta] \to [x,ha] \) is the natural map which appends the arrow \( a \) to the end of a path from \( x \) to \( ta \). There is a canonical resolution for each representation \( V \) of \( Q \)

\[
0 \to \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha} \xrightarrow{d^V} \bigoplus_{x \in Q_0} V(x) \otimes P_x \to V \to 0,
\]

where the differential \( d^V \) is described as follows. Given \( v \otimes p \in V(ta) \otimes P_{ha} \), send it to \((V(ta) \otimes P_{ta}) \oplus (V(ha) \otimes P_{ha})\), where the map to the first factor is induced by the inclusion \( P_{ha} \subset P_{ta} \) given by appending \( a \) to the beginning of a path, and the map to the second factor is \(-V_a \otimes 1_{P_{ha}}\).

Given another representation \( W \), we define \( d^V_W = \text{Hom}(d^V, W) \). We can also define \( d^V_W \) by

\[
\bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) \xrightarrow{d^V_W} \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(ha)),
\]

\[
(\varphi_x)_{x \in Q_0} \mapsto (\varphi_{ha} V_a - W_a \varphi_{ta})_{a \in Q_1}.
\]

Let \( \alpha = \dim V \) and \( \beta = \dim W \). In the case that \( \langle \alpha, \beta \rangle = 0 \), \( d^V_W \) is a map between vector spaces of the same dimension. By fixing bases, we can define its determinant \( c^V_W = \det d^V_W \). This is only well defined up to a nonzero scalar multiple, but this choice will not be important for us. Then \( c^V \) gives a polynomial function on \( \text{Rep}(Q, \beta) \), which is a semi-invariant of weight \( \sigma^\alpha \) defined by

\[
\sigma^\alpha(x) = \langle \alpha, \varepsilon_x \rangle
\]

(2.1)

where \( \varepsilon_x \) is the vector defined by \( \varepsilon_x(x) = 1 \) and \( \varepsilon_x(y) = 0 \) for \( y \neq x \). These semi-invariants are called determinantal semi-invariant. See [18] for some basic properties. We recall a fundamental result on semi-invariants of quivers due to Derksen and Weyman [6, Theorem 1].
Theorem (Derksen–Weyman). Let $Q$ be a quiver without oriented cycles and let $\beta \in \mathbb{N}Q_0$. Then $SI(Q, \beta)$ is linearly spanned by determinantal semi-invariants $c_V$ where $\langle \dim V, \beta \rangle = 0$.

2.2. Symmetric quivers

Suppose that the characteristic is different from 2.

Let $\tau$ denote an involution $\tau : Q_0 \to Q_0$ and $\tau : Q_1 \to Q_1$ such that $t \tau(a) = \tau(ha)$ and $h \tau(a) = \tau(ta)$. We pick a sign function defined on the $\tau$-fixed vertices and arrows $s : Q_0^* \cup Q_1^* \to \{\pm 1\}$. The data $(Q_0, Q_1, \tau, s)$ is a symmetric quiver. If $Q$ is a symmetric quiver, then $Q^\circ$ denotes the underlying quiver. Our definition of symmetric quiver is called a signed quiver in [20].

For $i = 0, 1$, we can partition $Q_i = Q_i^+ \cup Q^\tau_i \cup Q_i^-$, such that $Q^\tau_i$ is the fixed point set of $\tau$ and $Q_i^- = \tau(Q_i^+)$. Also, we set $Q^\tau_{0}^\pm$ to be the set of $x \in Q^\tau_0$ such that $s(x) = \pm 1$, so $Q^\tau_0 = Q^\tau_0^+ \cup Q^\tau_0^-$. 

2.2.1. Representations

A dimension vector $\beta \in \mathbb{N}Q_0$ is symmetric if $\beta(\tau(x)) = \beta(x)$ for all $x \in Q_0$ and $\beta(x)$ is even whenever $x \in Q^\tau_0$. Given a symmetric dimension vector $\beta$, a symmetric representation $V$ of dimension $\beta$ is defined by assigning a vector space $V(x)$ of dimension $\beta(x)$ to each $x \in Q^+_0 \cup Q^\tau_0$. For $x \in Q^+_0$, we assign the dual vector space $V(\tau(x)) = V(x)^*$ to $\tau(x)$. For $x \in Q^\tau_0^+$, we endow $V(x)$ with a nondegenerate symmetric bilinear form, and for $x \in Q^\tau_0^-$, we endow $V(x)$ with a nondegenerate skew-symmetric bilinear form. In particular, the form gives an isomorphism $J_x : V(x) \to V(x)^*$ that we fix. Note that $J_{x}^{-1} = J_{x}^* = \varepsilon J_{x}$ where $x \in Q^\tau_0$.

For each $a \in Q_1$, we assign a linear map $V_{a} : V(ta) \to V(ha)$ such that

1. If $a \in Q_1^\tau$, then $V_{a} = \varepsilon V_{a}^*$ where we are identifying $V(ta)$ and $V(ta)^{**}$ in the canonical way.
2. If $a \in Q^+_1$ and $ta, ha \notin Q^\tau_0$, then $V_{a} = V_{\tau(a)}^*$. 
3. If $a \in Q^+_1$ and $ha \in Q^\tau_0$, then $V_{\tau(a)} = V_{a}^* J_{ha}$, and similarly if instead $ta \in Q^\tau_0$.

We define the symmetric representation variety

$$\text{SRep}(Q, \beta) = \bigoplus_{a \in Q^+_1} \text{Hom}(K^{\beta(ta)}, K^{\beta(ha)}) \oplus \bigoplus_{a \in Q^\tau_1^+} \text{Sym}^2(K^{\beta(ta)})^* \oplus \bigoplus_{a \in Q^\tau_1^-} \bigwedge^2(K^{\beta(ta)})^*$$

and the corresponding groups

$$G(Q, \beta) = \prod_{x \in Q^+_0} GL(K^{\beta(x)}) \times \prod_{x \in Q^\tau_0 / s(x)=1} O(K^{\beta(x)}) \times \prod_{x \in Q^\tau_0 / s(x)=-1} Sp(K^{\beta(x)})$$

$$SG(Q, \beta) = \prod_{x \in Q^+_0} SL(K^{\beta(x)}) \times \prod_{x \in Q^\tau_0 / s(x)=1} SO(K^{\beta(x)}) \times \prod_{x \in Q^\tau_0 / s(x)=-1} Sp(K^{\beta(x)})$$
These groups act on the symmetric representation variety as follows (here we interpret \(g_\tau(x) = (g^{-1}_\tau x)^t\) where the transpose is with respect to the natural bilinear form on \(V(x) \oplus V(\tau(x))\) for \(x \in Q^+_0\):

\[
(g_x)_{x \in Q^+_0 \cup Q^+_1} \cdot (\varphi_a)_{a \in Q^+_1 \cup Q^+_1} = (g_{ha}g^{-1}_{\tau a})_{a \in Q^+_1 \cup Q^+_1}.
\]

We have a natural identification \(G(Q, \beta) \subset GL(Q^\circ, \beta)\) such that \(S\text{Rep}(Q, \beta)\) is a \(G(Q, \beta)\)-invariant subvariety of \(\text{Rep}(Q^\circ, \beta)\). The symmetric representations of \(Q\) correspond to the \(K\)-points of \(S\text{Rep}(Q, \beta)\), and isomorphism of two symmetric representations is defined as being in the same \(G(Q, \beta)\)-orbit.

### 2.2.2. Semi-invariants

As in Section 2.1.2, when \(Q\) is a symmetric quiver and \(\beta\) is a symmetric dimension vector, we define the ring of symmetric semi-invariants

\[
SSI(Q, \beta) = K[S\text{Rep}(Q, \beta)]^{SG(Q, \beta)},
\]

which has a grading by characters of \(G(Q, \beta)\). The determinant is the only nontrivial character of the orthogonal group and has order 2, while the symplectic group has no nontrivial characters. So we can identify weights with elements of \(Z^{Q^+_0} \times (Z/2)^{Q^+_0}\). Now we describe the class of semi-invariants that we study in this article.

Let \(V\) be a representation of \(Q^\circ\) of dimension \(\alpha\). Recall the definition of the polynomial function \(c^V\) on \(\text{Rep}(Q^\circ, \beta)\) given in Section 2.1.2. We are interested in the restriction of \(c^V\) to \(S\text{Rep}(Q, \beta)\). The weight \(\sigma_\alpha \in Z^{Q^+_0} \times (Z/2)^{Q^+_0}\) is defined by

\[
\sigma_\alpha(x) = \begin{cases} 
\langle \alpha, \varepsilon_x - \varepsilon_{\tau(x)} \rangle & \text{if } x \in Q^+_0, \\
\langle \alpha, \varepsilon_x \rangle & \text{if } x \in Q^+_0. 
\end{cases}
\]

**Remark 2.3.** While \(\langle \alpha, \varepsilon_x \rangle\) is an element of \(Z/2\) when \(x \in Q^+_0\), it will be convenient for us later to think of it as an integer.

If the restriction of the polynomial function \(c^V\) to \(S\text{Rep}(Q, \beta)\) is the square of another polynomial function, we define \(p^V\) to be a square root of this function. Since \(SI(Q, \beta)\) is a domain, this is well defined up to a choice of nonzero scalar since \(x^2 = y^2\) implies that \(x = \pm y\). Since \(SG(Q, \beta)\) has no nontrivial characters, \(p^V\) is also a semi-invariant. We call this a **Pfaffian semi-invariant**. The weight of \(p^V\) is \(\frac{1}{2}\sigma_\alpha\): this makes sense for \(x \in Q^+_0\); for \(x \in Q^+_0\), we interpret \(\frac{1}{2}\sigma_\alpha(x)\), which must be an integer, to be the residue of \(\frac{1}{2}\langle \alpha, \varepsilon_x \rangle\) in \(Z/2\). Since \(c^{V\oplus V} = (c^V)^2\), one can always interpret a determinantal semi-invariant as a Pfaffian semi-invariant: \(c^V = p^V\oplus V\).

**Theorem 2.4.** Assume that the characteristic of \(K\) is different from 2. Let \(Q\) be a symmetric quiver without oriented cycles and let \(\beta\) be a symmetric dimension vector for \(Q\). The space of symmetric semi-invariants \(SSI(Q, \beta)\) is linearly spanned by the Pfaffian semi-invariants \(p^V\) such that \(\langle \dim V, \beta \rangle = 0\). The weight of \(p^V\) is \(\frac{1}{2}\sigma_{\dim V}\).
We remark that Theorem 2.4 was proven in the case of finite type and tame quivers without oriented cycles in the paper [1] along with a determination for when the determinantal semi-invariants admit square roots. The technique of proof in that paper involves extending the idea of reflection functors, but our approach will follow the ideas in [6] closely. Hence there will be three steps. The first step is to reduce to the case of a symmetric quiver that has a unique sink and source and such that the weight is 1 at the sink, and 0 elsewhere. The second step is to show how one can remove vertices of weight 0. The third and last step is to handle the case of a symmetric quiver with two vertices. The proof will be given in Section 2.4 after we state some necessary background from invariant theory.

2.3. Some results from invariant theory

2.3.1. Fundamental invariants

We recall the first fundamental theorems of invariant theory for the classical groups.

**Theorem 2.5.** Let \( V \) be a vector space of dimension \( n \) over a field \( K \).

(a) We have

\[
K[V^\oplus \mathbf{p} \oplus (V^*)^\oplus q]^{\text{GL}(V)} = K[u_{i,j}],
\]

where \( u_{i,j} (1 \leq i \leq p, 1 \leq j \leq q) \) is the function defined by \((v, f) \mapsto f_j(v_i)\).

(b) Now assume that \( V \) has a skew-symmetric nondegenerate bilinear form \((\cdot, \cdot)\). Then

\[
K[V^\oplus \mathbf{p}]^{\text{Sp}(V)} = K[u_{i,j}]
\]

where \( u_{i,j} (1 \leq i, j \leq p) \) is the function defined by \((v_1, \ldots, v_p) \mapsto (v_i, v_j)\).

(c) Now assume that the characteristic of \( K \) is different from 2 and that \( V \) has a symmetric nondegenerate bilinear form \((\cdot, \cdot)\). Then

\[
K[V^\oplus \mathbf{p}]^{\text{O}(V)} = K[u_{i,j}]
\]

where \( u_{i,j} (1 \leq i, j \leq p) \) is the function defined by \((v_1, \ldots, v_p) \mapsto (v_i, v_j)\).

**Proof.** For (a), (b), and (c), respectively, see [17, §13.6.3], [17, §13.8.5, Theorem 1], and [17, §13.8.5, Theorem 2].

In the quiver context, (a) implies the following fact. Given a representation variety of a quiver of the form \( K^p \xrightarrow{A} K^n \xrightarrow{B} K^q \), we can interpret \( A \) as \( p \) vectors of \( K^n \) and \( B \) as \( q \) covectors. Hence the \( \text{GL}(K^n) \)-invariants of \( K[A, B] \) is just \( K[AB] \), the subring generated by the entries of the product \( AB \). Both (c) and (b) have similar interpretations for symmetric representations that we will be using later.

We will also require the following extension of Theorem 2.5(a).

**Lemma 2.6.** Let \( V \) be a vector space over a field \( K \) of characteristic different from 2. Define

\[
W_{p,q,r,s} = V^\oplus \mathbf{p} \oplus (V^*)^\oplus q \oplus \left( \bigwedge^2 V^* \right)^\oplus r \oplus (\text{Sym}^2 V^*)^\oplus s.
\]
Then

\[ K[W_{p,q,r,s}]^{GL(V)} = K[u_{i,j}, \varepsilon_{i,j,k}, \eta_{i,j,k}], \]

where

- \( \varepsilon_{i,j,k} (1 \leq i < j \leq p, 1 \leq k \leq r) \) is the function defined by \((v, f, \xi, \zeta) \mapsto \xi_k(v_i, v_j)\),
- \( \eta_{i,j,k} (1 \leq i < j \leq p, 1 \leq k \leq s) \) is the function defined by \((v, f, \xi, \zeta) \mapsto \zeta_k(v_i, v_j)\), and
- \( u_{i,j} (1 \leq i < p, 1 \leq j < q) \) is the function defined by \((v, f, \xi, \zeta) \mapsto f_j(v_i)\).

**Proof.** This follows from [15, Theorem 1], but we will give a simpler proof using classical invariant theory in the case when the characteristic of \( K \) is 0.

For the definitions of full polarization and restitution, we refer to [17, §3.2.2]. First we note that the invariants of \( K[W_{p,q,r,s}] \) are graded by \( \mathbb{N}^{p+q+r+s} \). Given a multilinear homogeneous invariant, its full polarization is a multilinear function on \( W = W_{p',q',r',s'} \) for some \( p', q', r', s' \) depending on the degree of the invariant. We have an inclusion of \( GL(V) \)-representations \( W \subseteq W' = V^\otimes p' \oplus (V^*)^\otimes q' \oplus (V^* \otimes V^*)^\otimes (r'+s') \). The space of multilinear invariants of \( W' \) is \((V^\otimes p' \otimes (V^*)^\otimes (q'+2r'+2s'))^{GL(V)} \). We know from Theorem 2.5(a) that \( p' = q' + 2r' + 2s' \) and that the invariants are linear combinations of the monomials \( u_w = u_{1,w(1)} \cdots u_{p',w(p')} \) where \( w \) is in the symmetric group \( \Sigma_{p'} \). Since we are working in characteristic 0 and \( GL(V) \) is reductive, the restrictions of the \( u_w \) from \( W' \) to \( W \) generate the multilinear \( GL(V) \)-invariants. Finally, the restitution of \( u_w \) to a multihomogeneous invariant on \( W_{p,q,r,s} \) is a product of \( u_{i,j}, \varepsilon_{i,j,k}, \eta_{i,j,k} \), so our claim is proven since every multihomogeneous invariant is the restitution of some multilinear invariant.  

2.3.2. **Schur functors**

We recall some facts about the representation theory of the general linear group. For this section, we make no assumptions on the characteristic of \( K \), except for the last two results.

Let \( V \) and \( W \) be vector spaces, and let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \) be a partition of nonnegative integers. The notation \( \ell(\lambda) \) denotes the largest \( n \) such that \( \lambda_n \neq 0 \), and \( \lambda' \) denotes the conjugate partition of \( \lambda \). For \( c \in \mathbb{N} \), we define \( c\lambda = (c\lambda_1, c\lambda_2, \ldots) \).

Let \( S_\lambda(V) \) be the **Schur functor**, which is an irreducible representation of \( GL(V) \) with highest weight \( \lambda \). (This is denoted \( L_\lambda(V) \) in [21, §2.1].) Given two representations \( U \) and \( U' \), we will write \( U \sim U' \) if each representation possesses a filtration such that the associated graded representations are isomorphic. If \( K \) has characteristic 0, \( \sim \) is the same as isomorphism.

**Proposition 2.7.** We have the following decompositions:

\[
\text{Sym}^m(\text{Sym}^2 V) \sim \bigoplus_{|\lambda|=m} S_{2\lambda}(V), \tag{2.8}
\]

\[
\text{Sym}^m\left(\bigwedge^2 V\right) \sim \bigoplus_{|\lambda|=m} S_{(2\lambda)'}(V), \tag{2.9}
\]

\[
\text{Sym}^m(V \otimes W) \sim \bigoplus_{|\lambda|=m} S_\lambda(V) \otimes S_\lambda(W), \tag{2.10}
\]
where the first two identities are $GL(V)$-equivariant, and the last one is $GL(V) \times GL(W)$-equivariant.

For (2.8) and (2.9), see [17, Theorem 11.4.5] or [21, Proposition 2.3.8] and for (2.10), see [17, Chapter 9, (6.3.2)] or [21, Corollary 2.3.3].

**Corollary 2.11.** Suppose that $\dim V = 2n$ and $\lambda$ is a partition of $n$. Then $S_{\lambda}(\wedge^2 V)$ contains $SL(V)$-invariants if and only if $\lambda = (n)$.

**Proof.** By [5, Theorem 3.7], the tensor product of two Schur functors is $\sim$-equivalent to a direct sum of Schur functors, and these multiplicities are independent of characteristic. So $S_{\lambda}(\wedge^2 V)$ is a subquotient of $(\wedge^2 V)^{\otimes n}$. The $SL(V)$-invariants of this tensor product are the submodules isomorphic to $\wedge^{2n} V$. By Pieri’s rule (see [17, §9.10.2] or [21, Corollary 2.3.5]), this representation appears with multiplicity 1, and this comes from $\text{Sym}^n (\wedge^2 V)$ by (2.9). \qed

**Proposition 2.12.** Assume that the characteristic of $K$ is 0. If $\dim V = n$, then we have an isomorphism of $GL(V)$-representations $S_{\lambda}(V^*) = (\wedge^n V^*)^{\otimes \lambda_1} \otimes S_{\mu} V$ where $\mu_i = \lambda_1 - \lambda_{n+1-i}$.

See [17, §2.3.3].

**Proposition 2.13.** Assume that the characteristic of $K$ is 0. Write $\lambda' = (1^{d_1}, \ldots, (n-1)^{d_{n-1}})$ and $\mu' = (1^{e_1}, \ldots, (n-1)^{e_{n-1}})$. Then $S_{\lambda}(V) \otimes S_{\mu}(V)$ contains a nonzero $SL(V)$-invariant if and only if $d_i = e_{n-i}$ for $i = 1, \ldots, n-1$. In this case, the action of $GL(V)$ on these $SL(V)$-invariants is via the $m$th power of the determinant, where $m = d_1 + \cdots + d_{n-1} = e_1 + \cdots + e_{n-1}$.

This follows from the previous proposition and the isomorphism $S_{\lambda}(V^*) \cong S_{\lambda}(V)$ in characteristic 0 [21, Proposition 2.1.18].

2.4. Proof of Theorem 2.4

2.4.1. Reduction to a unique source and sink with nonzero weight

Let $Q$ be a symmetric quiver with a symmetric dimension vector $\beta$. It is enough to prove Theorem 2.4 for each weight space of $\text{SSI}(Q, \beta)$. Given a weight $\sigma \in \mathbb{Z}^{Q_0^+} \times (\mathbb{Z}/2)^{Q_0^+}$, we lift it to some $\tau$-invariant weight $\sigma^0 \in \mathbb{Z}^{Q_0}$.

Form a new symmetric quiver $\overline{Q}$ as follows. We add two new vertices $x_-$ and $x_+$. If $\sigma^0(x) > 0$, then we add $\sigma^0(x)$ arrows $x_- \to x$ and $\sigma^0(x)$ arrows $\tau(x) \to x_+$. If $\sigma^0(x) < 0$, then we add $-\sigma^0(x)$ arrows $x \to x_+$ and $-\sigma^0(x)$ arrows $x_- \to \tau(x)$. Let $\tau$ be the involution on $\overline{Q}_0$ that switches $x_-$ and $x_+$ and restricts to $\tau$ on $Q_0$. Define $\tau$ on $\overline{Q}$ to be the same as $\tau$ on $Q_1$ and to switch the arrows incident to $x_-$ and $x_+$ (we add arrows two groups at a time, so just fix an identification of these groups). We have $\overline{Q}_0 = \{x_-\} \cup Q_0^+$, $\overline{Q}_0^+ = Q_0^+$ and $\overline{Q}_0^+ = \{x_+\} \cup Q_0$. Also, $\overline{Q}_1^+ = Q_1^+$, while $\overline{Q}_1^\pm = Q_1^\pm$ plus the arrows incident to $x_\mp$. Set

$$\overline{\beta}(x_-) = \overline{\beta}(x_+) = \sum_{x \in Q_0^+} |\sigma^0(x)| \beta(x),$$
and $\overline{\beta}(x) = \beta(x)$ for all $x \in Q_0$. Define a symmetric weight $\sigma$ by $\sigma(x_-) = 1$, and $\sigma(x) = 0$ for all $x \in Q_0^+ \cup Q_0^-$. Also define a $\tau$-invariant weight $\sigma^\circ$ by $\sigma^\circ(x_\pm) = 1$ and $\sigma^\circ(x_\mp) = 0$ for all $x \in Q_0$.

Given a representation $W \in \text{SRep}(\overline{Q}, \overline{\beta})$, let $D(W)$ be the determinant of the matrix formed by taking the direct sum of all maps incident to $x_\mp$ (equivalently, all maps incident to $x_\pm$). Any function on $\text{SRep}(Q, \beta)$ is naturally a function on $\text{SRep}(\overline{Q}, \overline{\beta})$. We claim that $c \mapsto Dc$ gives an isomorphism $\varphi : \text{SSI}(Q, \beta) \sigma \rightarrow \text{SSI}(\overline{Q}, \overline{\beta}) \sigma$.

First, suppose that $c \in K[\text{SRep}(Q, \beta)]$ is a symmetric semi-invariant of weight $\sigma^\circ$. Pick $g \in G(\overline{Q}, \overline{\beta})$. Let $W \in \text{SRep}(\overline{Q}, \overline{\beta})$ be a symmetric representation. Then

$$(g \cdot D)(W) = \left( \det g_\pm \cdot \prod_{x \in Q_0^+ \cup Q_0^-} (\det g_x)^{-\sigma^\circ(x)} \cdot \prod_{x \in Q_0^+ \cup Q_0^-} (\det g_{\tau(x)})^{\sigma^\circ(x)} \right) D(W),$$

$$(g \cdot c)(W) = \left( \prod_{x \in Q_0^+ \cup Q_0^-} (\det g_x)^{\sigma^\circ(x)} \right) c(W).$$

Noting that $g_{\tau(x)} = (g_x^{-1})^t$ when $x \neq \tau(x)$ and $(\det g_x)^2 = 1$ when $x = \tau(x)$, we see that

$$(g \cdot Dc)(W) = (\det g_{\pm})(Dc)(W),$$

so $Dc$ is a symmetric semi-invariant of weight $\sigma$, and hence $\varphi$ is well defined. It is clear that $\varphi$ is injective, and surjectivity follows from direct calculations via (2.10).

**Proposition 2.14.** Use the notation above. Let $\overline{V}$ be a representation of $\overline{Q}$ such that $\langle \overline{V}, \beta \rangle_{\overline{Q}} = 0$. If the function $c^V$ is a square, then writing $pf^V = Dp$, we have $p = pf^V$ for some representation $V$ of $Q$.

**Proof.** Given $c^V$, we can extend it to a semi-invariant of $\text{Rep}(\overline{Q}, \overline{\beta})$ of weight $\sigma^\circ$. By [6, Proposition 2], we can write $c^V = D^- c^V D^+$ for some representation $V$ of $\overline{Q}$ which is supported on the subquiver $Q$. The restriction of both $D^+$ and $D^-$ to $\text{SRep}(\overline{Q}, \overline{\beta})$ is $D$, so restricting this identity to $\text{SRep}(\overline{Q}, \overline{\beta})$, we get $c^V = D^2 c^V$. Since $(pf^V)^2 = c^V$, we conclude that $p^2 = c^V$. \( \square \)

Therefore, to prove Theorem 2.4, we may replace $Q$ by $\overline{Q}$. So we may assume without loss of generality that our symmetric quiver has a unique source (and hence a unique sink) with weight 1, and that all other vertices have weight 0.

2.4.2. Reduction to no weight 0 vertices

To do the second step, we find a vertex $x$ such that $\sigma(x) = 0$ and delete $\{x, \tau(x)\}$ to get a new vertex set $Q'_0$. To define arrows $Q'_1$, there are two cases to consider, depending on whether or not $\tau(x) = x$. In both cases, we construct a map of the form

$$\text{res}^* : \text{SSI}(Q', \beta')_{\sigma'} \rightarrow \text{SSI}(Q, \beta)_{\sigma}. \quad (2.15)$$


Then we show that it is surjective and that the image of a Pfaffian semi-invariant is also a Pfaffian semi-invariant so that we can replace $Q$ by $Q'$.

If $x = \tau(x)$, then for any arrows $y \xrightarrow{a} x \xrightarrow{b} z$, we add an arrow $y \xrightarrow{(a,b)} z$ to $Q'_1$. We define $\tau'(a, b) = (\tau(b), \tau(a))$. For $(a, b) \in Q'_1$, we set $s'(a, b) = s(x)$. We define $\tau'$ on $Q'_0$ to be the restriction of $\tau$ from $Q_0$. Similarly, let $\sigma'$ be the restriction of $\sigma$.

Given a symmetric representation $V$ of $Q$ of dimension $\beta$, we define a symmetric representation $V' = \text{res } V$ of $Q'$ of dimension $\beta'$, where $\beta'$ is the restriction of $\beta$, by setting $V_{(a,b)} = V_b V_a$ for all of the new arrows, and by leaving everything else as is. This gives us a map on symmetric semi-invariants as in (2.15). If $s(x) = 1$, we can consider the map $\sum_{y \xrightarrow{a} x} K^{\beta(y)} \to K^{\beta(x)}$ as a choice of $p = \sum y. s_x \beta(y)$ vectors in $K^{\beta(x)}$. Note that for every $a \in Q_1$, at most one of $a$ and $\tau(a)$ appears in the sum. Hence Theorem 2.5(c) implies that (2.15) is surjective. Similarly, if $s(x) = -1$, we can use Theorem 2.5(b) to conclude that (2.15) is surjective.

If $x \neq \tau(x)$, suppose that all arrows between $x$ and $\tau(x)$ are oriented as $x \to \tau(x)$ (which we may assume without loss of generality since $Q$ has no directed cycles). For arrows $y \xrightarrow{a} x \xrightarrow{b} \tau(x) \xrightarrow{c} z$, we define an arrow $y \xrightarrow{(a,b,c)} z$ in $Q'_1$. We define $\tau'$ on these arrows by $\tau'(a, b, c) = (\tau(c), \tau(b), \tau(a))$. If this arrow is $\tau'$-invariant, we set $s(a, b, c) = s(b)$. Also, for arrows $y \xrightarrow{a} x \xrightarrow{c} z$ with $z \neq \tau(x)$, we define an arrow $y \xrightarrow{(a,c)} z$ in $Q'_1$, and we do a similar thing when $x$ is replaced by $\tau(x)$. We set $\tau'(a, c) = (\tau(c), \tau(a))$. Any other arrows not incident to $x$ or $\tau(x)$ are also added to $Q'_1$: $\tau'$ and $s'$ are defined as the restriction of $\tau$ and $s$ on these arrows.

Given a symmetric representation $W$ of $Q$ of dimension $\beta$, we define a representation $W' = \text{res } W$ of $Q'$ of dimension $\beta'$, where $\beta'$ is the restriction of $\beta$, as follows. First, set $W'(y) = W(y)$ for all $y \in Q'_0$. For any arrow of the form $y \xrightarrow{(a,b,c)} z$ we define $W'_{(a,b,c)} = W_c W_b W_a$. Similarly, for arrows of the form $y \xrightarrow{(a,c)} z$ we set $W'_{(a,c)} = W_c W_a$. For all other arrows, we define $W'$ to be the restriction of $W$. This gives a map as in (2.15), which is surjective by Theorem 2.5(a) and Lemma 2.6.

Now we need to know that under (2.15), determinantal and Pfaffian semi-invariants pull back to determinantal and Pfaffian semi-invariants, respectively.

**Proposition 2.16.** Use the notation above. The image of a determinantal semi-invariant $c^V' \in \text{SSI}(Q', \beta')_{\sigma'}$ under (2.15) is of the form $c^V$ for some representation $V$ of $Q$. Similarly, the image of a Pfaffian semi-invariant is a Pfaffian semi-invariant.

**Proof.** First we deal with determinantal semi-invariants. In the case $x = \tau(x)$, the proof of Step 2 of Theorem 1 in [6] works in our case. So we get $V$ such that the pullback of $c^V'$ is $c^V$.

Now we deal with the case $x \neq \tau(x)$. We define an intermediate quiver $\tilde{Q}$ by forgetting that $Q$ is a symmetric quiver and deleting $x$ from $Q^0$ as in Step 2 of Theorem 1 in [6]. Deleting $\tau(x)$ from $\tilde{Q}$ again as in Step 2 of Theorem 1 in [6], we get to $(Q')^0$. Hence we have maps

$$\text{Rep}(Q^0) \to \text{Rep}(\tilde{Q}) \to \text{Rep}((Q')^0).$$

Restricting the composition to $\text{SRep}(Q)$, the image is in $\text{SRep}(Q')$, and we recover our restriction map $\text{res}$. Now Step 2 of Theorem 1 in [6] implies that $\text{res}^* c^V' = c^V$ for some $V \in \text{Rep}(Q, \beta)$.

Finally, if we have a Pfaffian semi-invariant $\text{pf}^V$ (in either case), then $(\text{res}^* \text{pf}^V)^2 = c^V$ for some $V$, and hence $\text{res}^* \text{pf}^V = \text{pf}^V$. □
2.4.3. The generalized Kronecker quiver

We have to work with the quiver $\Theta_{p,r}^+$ (respectively $\Theta_{p,r}^-$) which is defined to be $p$ arrows $x_- \xrightarrow{a_i} x_+$ with $s(a_i) = 1$ (respectively $s(a_i) = -1$) and $r$ pairs of arrows $\{b_i, b'_i\}$ such that $\tau(b_i) = b'_i$ and $\tau(b'_i) = b_i$. The dimension vector is $\beta = (n, n)$, and the weight is given by $\sigma(x_-) = 1$.

This case can be handled similarly to the way it was handled in [6]. In the case of $\Theta_{p,r}^-$, with dimension vector $(n, n)$, let $V = K^n$. The representation variety is $\text{Hom}(V, V^*)^{\oplus r} \oplus (\bigwedge^n V^*)^{\oplus p}$, which splits up in a $\text{GL}(V)$-equivariant way as $(\text{Sym}^2 V^*)^{\oplus r} \oplus (\bigwedge^2 V^*)^{\oplus (p+r)}$. We are only interested in the semi-invariants of weight 1, and by (2.8), only the polynomial functions on $\bigwedge^2 V^*$ can contribute. We think of this as $(\bigwedge^2 V^*) \otimes U$ with an action of $\text{GL}(U)$ where $U \cong K^{p+r}$. (We really have $p + 2r$ arrows, so $U$ is identifying the arrows that come in pairs.) By (2.10), we have

$$K \left[ \bigwedge^2 V^* \otimes U \right] = \text{Sym} \left( \bigwedge^2 V \otimes U^* \right) / \bigoplus_{\lambda} S_{\lambda} \left( \bigwedge^2 V \right) \otimes S_{\lambda}(U^*),$$

and $S_{\lambda}(\bigwedge^2 V)$ contains an $\text{SL}(V)$-invariant of weight 1 if and only if $n$ is even and $\lambda = (n/2)$ by Corollary 2.11. Hence we have

$$\text{SSI}(\Theta_{p,r}^-, (n, n))_1 = \bigwedge^n V \otimes D^{n/2}(U^*),$$

where $D$ denotes the divided power functor. To see this, we just note that $\text{Sym}^{n/2}(\bigwedge^2 V \otimes U^*) \subset \text{Sym}^{n/2}(V \otimes V \otimes U^*)$, and the latter module contains $\bigwedge^{n/2} V \otimes \bigwedge^{n/2} V \otimes D^{n/2} U^*$ by the dual Cauchy filtration for $\bigwedge^{n/2} (V \otimes U^*)$ [21, Theorem 2.3.2(b)].

Since $D^{n/2}(U^*)$ is a highest weight module for $\text{GL}(U)$, it is enough to show that its highest weight vector is represented by a semi-invariant pf$^V$ for some representation $V$ since these kinds of semi-invariants are invariant under the action of $\text{GL}(U)$. Fix an ordering of the arrows $a_1, \ldots, a_{p+2r}$. Now define a representation $W$ of dimension $(1, p + 2r - 1)$ by $W_{a_i} = 0$ and $W_{a_i}(1) = e_{i-1}$ for all $i > 1$, where $1$ is a nonzero vector of $W_-$ and $e_1, \ldots, e_{p+2r-1}$ is a basis for $W_+$. Then $e^W = \det V_{a_1}$ (up to scalar multiple) and hence $\text{pf}^W = \text{pf} V_{a_1}$. Then $\text{pf}^W_{a_1}$ is the desired highest weight vector.

The situation of $\Theta_{p,r}^+$ is similar to that of $\Theta_{p,r}^-$. The representation variety instead decomposes as $(\text{Sym}^2 V^*)^{\oplus (p+r)} \oplus (\bigwedge^2 V^*)^{\oplus r}$, but we can proceed as above.

This concludes the proof of Theorem 2.4.

3. Semi-invariants of quivers with relations

Suppose that the characteristic is different from 2 for this section, except in the statements of Proposition 3.2 and Theorem 3.3, where the characteristic is assumed to be 0.
3.1. Quivers with relations

Let $Q$ be a symmetric quiver (without oriented cycles, as usual), and let $KQ$ be its path algebra. We let $m \subset KQ$ be the two-sided ideal generated by all arrows $a \in Q_1$. Given any two vertices $x, y \in Q_0$ we say that a linear combination of paths from $x$ to $y$ is homogeneous. A two-sided ideal $I \subset KQ$ is admissible if $I \subseteq m^2$. We will also assume that $I$ is $\tau$-invariant. We denote by $Q/I$ the quiver with relations $I$, i.e., (symmetric) representations of $Q/I$ are (symmetric) representations $V$ of $Q$ for which $IV = 0$. The requirement on admissible ideals is for convenience, since having an element of length 0 or 1 in the relations is equivalent to considering the quotient $Q'/I'$ where $Q'$ is the quiver with the corresponding vertices or edges removed, and $I'$ is the ideal $I$ with the corresponding relations removed.

3.1.1. Representations

The symmetric representation variety of $Q/I$ of dimension $\beta$ is denoted $\text{SRep}(Q/I, \beta)$. This is a closed subvariety of $\text{SRep}(Q, \beta)$ which is $\text{SG}(Q, \beta)$-invariant. Hence the surjection $K[\text{SRep}(Q, \beta)] \to K[\text{SRep}(Q/I, \beta)]$ is $\text{SG}(Q, \beta)$-equivariant. If $K$ has characteristic 0, then $\text{SG}(Q, \beta)$ is linearly reductive, so by semisimplicity, we get a surjective map of semi-invariants $\text{SSI}(Q, \beta) \to \text{SSI}(Q/I, \beta)$.

However, we are not interested in the whole variety $\text{SRep}(Q/I, \beta)$. In general, this variety is reducible, so let $\text{SRep}(Q/I, \beta)^{(1)}, \ldots, \text{SRep}(Q/I, \beta)^{(N)}$ denote its irreducible components.

For a representation $V$ of $Q/I$, let $\text{pdim} V$ denote the projective dimension of $V$ over $KQ/I$. The global dimension of $Q/I$ (abbreviated $\text{gldim } Q/I$) is defined as the largest possible projective dimension of a module of $Q/I$, and is at most $\#Q_0 - 1$ (this can be proven by induction on the number of vertices after choosing an ordering of the vertices such that arrows only go from smaller vertices to bigger ones, but we won’t use this fact). Hence we can define a modified Euler form for two representations $V$ and $W$ of $Q/I$ with dimension vectors $\alpha = \dim V$ and $\beta = \dim W$ via

$$\langle \alpha, \beta \rangle_I = \sum_{i \geq 0} (-1)^i \dim_K \text{Ext}^i_{Q/I}(V, W).$$

In fact, this definition depends only on the vectors $\alpha$ and $\beta$ [2, Proposition III.3.13]. The indecomposable projective modules of $Q/I$ are indexed by $Q_0$, and are given by $P_x = P'_x/IP'_x$ where $P'_x$ is the indecomposable projective module for $Q$ indexed by $x \in Q_0$ [2, Lemma III.2.4]. From this, we see that projective covers are well defined for representations of $Q/I$, see also [2, Theorem I.5.8].

3.1.2. Semi-invariants

We denote the ring of semi-invariants $K[\text{SRep}(Q/I, \beta)^{(j)}]^{\text{SG}(Q, \beta)}$ by $\text{SSI}(Q/I, \beta)^{(j)}$. These components need not be invariant under $G(Q, \beta)$ if $Q^+_0$ is nonempty. However, they are invariant under $\text{SG}(Q, \beta)$ since it is a connected group. We can define analogues of determinantal and Pfaffian semi-invariants for symmetric quivers with relations. When discussing semi-invariants, we will work modulo the nilpotent radical. This remark will not affect our main application in Section 4.2, but we include it here for simplicity.

Given a representation $V$ of $Q/I$, let $P_1 \to P_0 \to V \to 0$
be a minimal projective presentation of \( V \). Given any other representation \( W \), we define the map
\[
\delta^V_W : \text{Hom}_{KQ/I}(P_0, W) \to \text{Hom}_{KQ/I}(P_1, W).
\]
If it is a square matrix, we set \( \delta^V_W \) to be its determinant. In this case, we get a determinantal semi-invariant \( c^V \in \text{SSI}(Q/I, \beta)^{(j)} \). If this polynomial function is a square, we likewise define the Pfaffian semi-invariant \( \text{pf}^V \in \text{SSI}(Q/I, \beta)^{(j)} \). Similar to before, this is well defined up to a nonzero scalar since \( K[\text{SRep}(Q/I, \beta)^{(j)}] \) is a domain.

Now let \( V' \) be a representation of \( Q \) and let \( V = V'/IV' \) be the corresponding representation of \( Q/I \). The Pfaffian semi-invariant \( \text{pf}^V \) restricts to a semi-invariant in \( \text{SSI}(Q/I, \beta)^{(j)} \) for all \( j \).

In particular, its image is \( \text{pf}^V \), which can be shown using the proof of [7, Proposition 1]. So we have the following result.

**Proposition 3.2.** Suppose that the characteristic of \( K \) is 0. Let \( Q \) be a symmetric quiver without oriented cycles and \( I \subset KQ \) an admissible ideal. The Pfaffian semi-invariants \( \text{pf}^V \) linearly span \( \text{SSI}(Q/I, \beta)^{(j)} \).

For our applications, we shall only be interested in certain kinds of irreducible components. A component \( \text{Rep}(Q/I, \beta)^{(j)} \) is **faithful** if whenever \( x \in KQ \) annihilates all representations in \( \text{Rep}(Q/I, \beta)^{(j)} \), we have \( x \in I \). We make similar definitions for symmetric representation varieties.

**Theorem 3.3** (Derksen–Weyman). Suppose that the characteristic of \( K \) is 0. Let \( Q \) be an acyclic quiver and let \( I \subset KQ \) be an admissible ideal. Suppose that \( \text{Rep}(Q/I, \beta)^{(j)} \) is a faithful component of \( \text{Rep}(Q/I, \beta) \). If \( \delta^V \) is nonzero on \( \text{Rep}(Q/I, \beta) \), then \( \text{pdim} V \leq 1 \).

**Proof.** See [7, Theorem 1].}

### 3.1.3. Reminders on semicontinuity and genericity

Let \( X \) be a topological space and \( P \) be a partially ordered set. A function \( f : X \to P \) is **upper semicontinuous** if the sets \( X_{<n} = \{ x \in X \mid f(x) < n \} \) are open for all \( n \in P \). We shall mostly be interested in the case when \( P = \mathbb{N} \), so that the set where \( f \) attains its minimum is open, or the case when \( P = \mathbb{N} \cap \mathbb{Q}^0 \) for some vertex set \( Q_0 \) with the partial order \( (dx)_x \in Q_0 \leq (d'x)_x \in Q_0 \) if and only if \( dx \leq d'x \) for all \( x \in Q_0 \). We list here some functions on representation varieties that are upper semicontinuous with references for proofs. We will use these facts without explicit mention.

Let \( Q/I \) be a quiver with relations and let \( X = \text{Rep}(Q/I, \alpha) \) and \( Y = \text{Rep}(Q/I, \beta) \) for some dimension vectors \( \alpha \) and \( \beta \). The function \( \text{pdim} : X \to \mathbb{N} \) that assigns a module to its projective dimension is upper semicontinuous [16, Lemma 2.1]. By duality, the same is true for injective dimension. For any given \( i \), the function \( \text{Ext}^i : X \times Y \to \mathbb{N} \) given by \( (M, N) \mapsto \dim_K \text{Ext}^i_{Q/I}(M, N) \) is upper semicontinuous [16, Lemma 1.2]. In particular, if we fix a representation \( M \in X \), the function \( N \mapsto \dim_K \text{Ext}^i_{Q/I}(M, N) \) is upper semicontinuous, and there is a similar statement when fixing the other argument. Given two irreducible components \( C \subseteq X \) and \( C' \subseteq Y \), we will use \( \text{Ext}^i(C, C') \) to denote the minimum value of \( \text{Ext}^i \) restricted to \( C \times C' \). In this case, the minimum is attained on an open dense subset.

For representations \( M \) and \( N \), the function \( \text{Hom}_{Q/I}(M, N) \to \mathbb{N}^{Q_0} \) given by \( \varphi \mapsto \dim \ker \varphi \) is upper semicontinuous. Hence there is a unique maximal dimension vector \( \gamma \) such that the set
of linear maps $M \to N$ with rank $\gamma$ is open. We call this the generic rank for $M$ and $N$. See [19, Lemma 5.1] and its proof for details. Similarly, we can form the subbundle

$$\text{Hom}(Q/I, C, C') = \{(\varphi, M, N) \in \text{Hom}_K(K^\alpha, K^{\beta}) \times X \times Y \mid \varphi \in \text{Hom}_{Q/I}(M, N)\},$$

and the function that assigns to a triple $(\varphi, M, N)$ the dimension of $\ker \varphi$ is upper semicontinuous, so we can define the generic rank for two components $C$ and $C'$. We can also mix and match modules and irreducible components.

Finally, given a dominant morphism of two irreducible varieties $f : X \to Y$, the set $\{y \in Y \mid \dim f^{-1}(y) = \dim X - \dim Y\}$ contains a dense open subset of $Y$.

3.2. Global dimension 2

Suppose that the characteristic is arbitrary and also that $\text{gldim } Q/I \leq 2$ for this section. Let $Q_2$ be a set of homogeneous minimal relations, i.e., for all $r \in Q_2$, we have that $r$ is not contained in the ideal generated by $Q_2 \setminus \{r\}$. For $r \in Q_2$, let $tr$ and $hr$ be the beginning and ending, respectively, of the paths which are the summands in $r$. There is a canonical resolution just as in the case of a quiver $Q$:

$$0 \to \bigoplus_{r \in Q_2} V(tr) \otimes P_{hr} \to \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha} \to \bigoplus_{x \in Q_0} V(x) \otimes P_x \to V \to 0. \quad (3.4)$$

It follows that the Euler form for $Q/I$ (3.1) can be defined as

$$\langle \alpha, \beta \rangle_I = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha) + \sum_{r \in Q_2} \alpha(tr)\beta(hr).$$

**Remark 3.5.** Suppose $\dim V = \alpha$ and $\text{pdim } V \leq 1$. We claim that the weight $\sigma_\alpha$ of the semi-invariant $\hat{e}^V$ on $\text{Rep}(Q/I, \beta)$ is given by $\sigma_\alpha(x) = \langle \alpha, \varepsilon_x \rangle_I$.

First, let $0 \to P_1 \to P_0 \to V \to 0$ be a projective resolution, and choose $W \in \text{Rep}(Q/I, \beta)$. For $x \in Q_0$, we know by (3.4) that the number of times that $P_x$ appears in $P_0$ minus the number of times that $P_x$ appears in $P_1$ is $\langle \alpha, \varepsilon_x \rangle_I$. Hence when we apply $\text{Hom}(-, W)$ to the projective resolution, we can use the proof of [18, Lemma 1.4] to conclude our desired result.

**Remark 3.6.** We will use the following facts repeatedly without explicit mention. If $V$ is a module with $\text{pdim } V \leq 1$, then $\text{pdim } V' \leq 1$ for any submodule $V' \subseteq V$. To see this, first note that $\text{pdim } V \leq 1$ is equivalent to the functor $\text{Ext}^2(V, -)$ being identically 0. Using that $\text{gldim } Q/I \leq 2$, we see that $\text{Ext}^2(V, W) = 0$ implies that $\text{Ext}^2(V', W) = 0$ for all modules $W$. Dually, if $\text{idim } V \leq 1$, then $\text{idim } V/V' \leq 1$ for any submodule $V' \subseteq V$.

**Proposition 3.7.** Suppose $\text{gldim } Q/I \leq 2$, and let $M$ be a representation of $Q/I$ such that $\text{Ext}^2(M, M) = 0$. Set $\alpha = \dim M$. Then $M$ is a nonsingular point of $\text{Rep}(Q/I, \alpha)$, and

$$\dim_M \text{Rep}(Q/I, \alpha) = \sum_{a \in Q_1} \alpha(ta)\alpha(ha) - \sum_{r \in Q_2} \alpha(tr)\alpha(hr).$$

Here $\dim_X X$ means the local dimension of $X$ at $x \in X$. 


**Definition 3.8.** Set

\[ Q_0^e = \{0, 1\} \times Q_0, \]
\[ Q_1^e = \{((0, ta), (0, ha)) \mid a \in Q_1\} \cup \{((1, ta), (0, ha)) \mid a \in Q_1\} \]
\[ \cup \{((1, ta), (1, ha)) \mid a \in Q_1\}. \]

There is an abuse of notation here since two arrows might have the same head and tail, but we hope the meaning is clear. Define an ideal of relations \( I^e \) as follows. For every relation \( \sum_p \lambda_p a_{pd(p)} \cdots a_{p_2} a_{p_1} \) in \( I \), we take the homogeneous components of the relations \( \sum_p \lambda_p a'_{pd(p)} \cdots a'_{p_2} a'_{p_1} \) where \( a'_x = ((0, ta), (0, ha)) + ((1, ta), (0, ha)) + ((1, ta), (1, ha)) \). We call \( Q^e/I^e \) the extension quiver of \( Q/I \).

**Remark 3.9.** Given a representation \( V \) of \( Q^e/I^e \), there is an associated representation \( V' \) of \( Q/I \) along with a choice of submodule \( V'' \subset V' \) by setting \( V'_x = V_{(0,x)} \oplus V_{(1,x)} \) and \( V''_x = V_{(0,x)} \). Conversely, given an inclusion of \( Q/I \)-modules \( V'' \subset V' \), one can associate to it a representation \( V \) of \( Q^e/I^e \) by picking a basis for \( V'' \) and extending it to a basis for \( V' \). There is some ambiguity about this choice of basis, but it will not affect our discussions.

**Proposition 3.10.** If \( \text{gldim} \ Q/I \leq 2 \), then \( \text{gldim} \ Q^e/I^e \leq 2 \).

**Proof.** It will be enough to show that every simple representation \( S_{(n,x)} \) has projective dimension at most 2. The projective modules \( P_{(0,x)} \) are supported in \( \{0\} \times Q \), so \( \text{pd}(Q^e/I^e) S_{(0,x)} = \text{pd}(Q/I) S_x \leq 2 \) by assumption. Otherwise, we claim that

\[ 0 \to \bigoplus_{r \in Q_2, tr = x} P_{(0,hr)} \oplus P_{(1,hr)} \xrightarrow{d_2} \bigoplus_{a \in Q_1, ta = x} P_{(0,ha)} \oplus P_{(1,ha)} \xrightarrow{d_1} P_{(1,x)} \xrightarrow{d_0} S_{(1,x)} \to 0 \]

is a projective resolution. It is clear that \( d_0 \) is surjective and that image \( d_1 = \ker d_0 \). An element in \( \ker d_1 \) is a linear combination of paths \( p_{(n,a)} \) starting at \( (n, ha) \) for various \( n \in \{0, 1\} \) and \( a \in Q_1 \), with \( ta = x \), all of which end at a common vertex \((m, y)\), such that appending \( a \) to the beginning of \( p_{(n,a)} \) and taking the sum gives 0, i.e., is a relation between \((1, x)\) and \((m, y)\) in \( I^e \). Hence image \( d_2 = \ker d_1 \).

Finally, we show that \( d_2 \) is injective. An element in \( \ker d_2 \) is a linear combination of paths \( p_{(n,r)} \) starting at \( (n, hr) \) for various \( n \in \{0, 1\} \) and \( r \in Q_2 \) with \( tr = x \) and \( r = \sum_p \lambda_p a_{pd(p)} \cdots a_{p_2} a_{p_1} \), all of which end at a common vertex \((m, y)\), such that appending \( \sum_p \lambda_p a_{pd(p)} \cdots a_{p_2} \) to the beginning of each \( p_{(n,r)} \) and taking the sum gives 0. If this element is nonzero, this will imply that there is a minimal relation in \( I^e \) such that the beginning of one of its paths coincides with the ending of one of the paths of \( r \). This contradicts the fact that \( \text{gldim} \ Q/I \leq 2 \) and our definition of \( I^e \). \( \square \)

**Proposition 3.11.** Let \( N \subset M \) be representations of \( Q/I \) so that \( M' = (N \subset M) \) is naturally a representation of \( Q^e/I^e \). If \( \text{Ext}^2(M/N, M/N) = 0 \), then \( \text{Ext}^2_{Q^e/I^e}(M', M') = 0 \).
Proof. Let \( P_\bullet \to M' \to 0 \) be the canonical resolution (3.4) of \( M' \). Write \( P_i^{(n)} \) for the summands of \( P_i \) starting at vertices of \( \{n\} \times Q_0 \). Since \( P_3 = 0 \), it is enough to show that \( \text{Hom}(P_1, M') \to \text{Hom}(P_2, M') \) is surjective. This homomorphism can be broken up into two pieces. The first piece is \( \text{Hom}(P_1^{(0)}, M') \to \text{Hom}(P_2^{(0)}, M') \), which is the same thing as \( \text{Hom}(P_1^{(0)}, N) \to \text{Hom}(P_2^{(0)}, N) \), whose cokernel can be identified with \( \text{Ext}^2_{Q/I}(M, N) = 0 \). Hence this part is surjective, and the second piece is \( \text{Hom}(P_1^{(1)}, M') \to \text{Hom}(P_2^{(1)}, M') \), which is the same as \( \text{Hom}(P_1^{(1)}, M/N) \to \text{Hom}(P_2^{(1)}, M/N) \). But this map is also surjective since the cokernel of this map can be identified with \( \text{Ext}^2_{Q/I}(M/N, M/N) = 0 \). 

Example 3.12. Let \( Q/I \) be the quiver \( \begin{array}{c} 1 \rightarrow \beta \\ \downarrow \alpha \rightarrow 2 \end{array} \), with the relation \( \beta \alpha = 0 \). Let \( M \) be the representation \( K \rightarrow 0 \rightarrow K \) and let \( N \) be the subrepresentation \( 0 \rightarrow 0 \rightarrow K \). The projective resolution for \( M' = (N \subset M) \) is \( Q^e/I^e \) is

\[
0 \to P_{(1,3)} \oplus P_{(0,3)} \to P_{(0,2)} \oplus P_{(1,2)} \to P_{(1,1)} \oplus P_{(0,3)} \to M' \to 0.
\]

The first map in the above proof becomes

\[
0 = \text{Hom}(P_1^{(0)}, N) \to \text{Hom}(P_2^{(0)}, N) = K
\]

which we naturally think of as \( \text{Ext}^2_{Q/I}(M, N) = K \) so that \( \text{Ext}^2(M', M') = K \). In this example, it is easy to produce a nonzero element of \( \text{Ext}^2(M', M') \):

\[
0 \to K 0 0 0 K 0 0 \oplus 0 K 0 0 0 K 0 0 0 0 K 0 0 0 K 0 0 0 \to 0,
\]

where we have drawn \( Q^e/I^e \) as

\[
\begin{array}{c}
(1, 1) \rightarrow (1, 2) \rightarrow (1, 3) \\
(0, 1) \rightarrow (0, 2) \rightarrow (0, 3),
\end{array}
\]

and the arrows in each direct summand are nonzero whenever possible. If we instead took \( N = (K \to 0 \to 0) \), then \( \text{Ext}^2(M', M') = 0 \).

3.3. Quiver Grassmannians

Suppose that the characteristic is arbitrary except in Proposition 3.15 where the characteristic is different from 2. We continue to use the notation of the previous section. Given two nonnegative integers \( k \leq n \), \( \text{Gr}(k, n) \) denotes the grassmannian, a variety parametrizing the \( k \)-dimensional subspaces of a fixed \( n \)-dimensional vector space. It is a projective variety of dimension \( k(n-k) \). Given a quiver with relations \( Q/I \), a module \( W \), and a dimension vector \( \gamma \), we let \( \text{Gr}(\gamma, W) \) be the quiver grassmannian, which is the variety of submodules of \( W \) of dimension \( \gamma \). This is a projective variety and the calculation of its dimension in some special cases will be the subject of our attention for this section. We will only be using topological properties of this variety, so its scheme structure will not play a role in this paper.
Let $\beta$ be a dimension vector and let $\text{Rep}(Q/I, \beta)'$ be the closure inside $\text{Rep}(Q/I, \beta)$ of points corresponding to representations with injective dimension at most 1, and let $W$ be such a representation. Fix a dimension vector $\gamma \leq \beta$, and let $\text{Gr}(\gamma, W)'$ denote the subvariety of the quiver Grassmannian consisting of submodules $U \subseteq W$ such that $\dim U = \gamma$ and $\text{idim } U \leq 1$.

**Theorem 3.13.** With the notation above, suppose that $\text{Gr}(\gamma, W)'$ is nonempty for a general $W$ with $\text{idim } W \leq 1$. Then there is a nonempty open set $\Omega \subseteq \text{Rep}(Q/I, \beta)'$ such that $\dim \text{Gr}(\gamma, W)' = \langle \gamma, \beta - \gamma \rangle_I$ for $W \in \Omega$.

**Proof.** Let $\text{Gr}(\gamma, \beta) = \prod_{x \in Q_0} \text{Gr}(\gamma(x), \beta(x))$ denote a product of ordinary Grassmannians and let $\text{Rep}(Q/I, \gamma \subset \beta)$ be the closure inside of $\text{Rep}(Q/I, \beta)' \times \text{Gr}(\gamma, \beta)$ of the set of points $(W, V)$ such that $\dim W \leq 1$ and such that the subspaces determined by $V$ form a submodule of $W$ with $\dim V \leq 1$. Consider the projection $\pi_2 : \text{Rep}(Q/I, \gamma \subset \beta) \to \text{Gr}(\gamma, \beta)$. The fiber over a point $V$ consists of all representation structures on $K\beta$ such that $V$ forms a submodule with $\text{idim } V \leq 1$.

To better describe this fiber, first choose a splitting $K\beta = K\gamma \oplus K\beta - \gamma$ of vector spaces. Define a dimension vector $\alpha$ of $Q^e/I^e$ by $\alpha(0, x) = \gamma(x)$ and $\alpha(1, x) = (\beta - \gamma)(x)$.

Let $Z \subseteq \text{Rep}(Q^e/I^e, \alpha)$ be the subvariety consisting of modules $M$ such that $\text{idim } M \leq 1$ when thought of as a module over $Q/I$. There is a map $p : Z \to \text{Rep}(Q/I, \gamma)$ which sends $W$ to the restriction of $W$ to the subquiver $\{0\} \times Q_0$. The subvariety $\text{Rep}(Q/I, \gamma)'$ of $\text{Rep}(Q/I, \gamma)$ consisting of representations with injective dimension at most 1 is open (and nonempty by our assumptions), so $Z' = p^{-1}(\text{Rep}(Q/I, \gamma)')$ can be identified with $\pi_2^{-1}(V)$. We know that $\text{gldim } Q^e/I^e \leq 2$ by Proposition 3.10. So by Proposition 3.11, the local dimension at every point of $Z'$ is given by Proposition 3.7. Hence we have

$$\dim \pi_2^{-1}(V) = \sum_{x \in Q_0} \alpha(x)^2 - \langle \alpha, \alpha \rangle_{I^e}$$

$$= \sum_{x \in Q_0} (\beta - \gamma)(x)\beta(x) + \sum_{x \in Q_0} \gamma(x)^2 - \langle \beta - \gamma, \beta \rangle_I - \langle \gamma, \gamma \rangle_I,$$

and thus

$$\dim \text{Rep}(Q/I, \gamma \subset \beta) = \dim \text{Gr}(\gamma, \beta) + \dim \pi_2^{-1}(V)$$

$$= \sum_{x \in Q_0} \gamma(x)(\beta - \gamma)(x) + \dim \pi_2^{-1}(V)$$

$$= \sum_{x \in Q_0} \beta(x)^2 - \langle \beta - \gamma, \beta \rangle_I - \langle \gamma, \gamma \rangle_I.$$

By Proposition 3.7, $\text{Rep}(Q/I, \beta)'$ is equidimensional of dimension $\sum_{x \in Q_0} \beta(x)^2 - \langle \beta, \beta \rangle_I$. For $W \in \text{Rep}(Q/I, \beta)'$, we have $\pi_1^{-1}(W) = \text{Gr}(\gamma, W)'$. Hence there is an open set $\Omega$ such that for $W \in \Omega$, one has

$$\dim \text{Gr}(\gamma, W)' = \dim \text{Rep}(Q/I, \gamma \subset \beta) - \dim \text{Rep}(Q/I, \beta)'$$

$$= -\langle \beta - \gamma, \beta \rangle_I - \langle \gamma, \gamma \rangle_I + \langle \beta, \beta \rangle_I = \langle \gamma, \beta - \gamma \rangle_I.$$  \qed
Remark 3.14. We can also consider the dual situation of a representation $W$ with $\text{pdim} \ W \leq 1$ and its quiver Grassmannian $\textbf{Gr}(W, \gamma)'$ of $\gamma$-dimensional quotients $U$ with $\text{pdim} \ U \leq 1$. Following the above proof will also show that $\dim \textbf{Gr}(W, \gamma)' = \langle \beta - \gamma, \gamma \rangle_1$ for generic $W$, or we can apply the duality functor $\text{Hom}_K(-, K)$.

We need to calculate the dimension of quiver Grassmannians when $W$ is a symmetric representation of a triple flag quiver (see Section 4.2).

Proposition 3.15. Suppose that the characteristic is different from 2. Let $\mathcal{Q}$ be an $r$-tuple flag quiver and $\beta^\delta$ denote the corresponding dimension vector. Then for general symmetric $W$ where the appropriate maps are injective and surjective, and $\gamma \leq \beta^\delta$ such that $\gamma(u) \geq \gamma(X_n) + \gamma(\tau(X_n))$ for all $X \in \{x, y, z\}$, we have $\dim \textbf{Gr}(\gamma, W)' = \langle \gamma, \beta - \gamma \rangle_1$.

Proof. Let $Q/I$ be the following symmetric quiver with relations:

$$K^n \xrightarrow{\alpha} K^{2n+\delta} \xrightarrow{\tau(a)} K^n$$

where $\tau(a) = 0$ and $K^{2n+\delta}$ has a bilinear form. Let $\beta$ be this dimension vector and pick $\gamma \leq \beta$ such that $\gamma \geq \gamma_1 + \gamma_3$ and $\textbf{Gr}(\gamma, W)'$ is nonempty for general symmetric $W$. We use the setup from the proof of Theorem 3.13:

$$\text{SRep}(Q/I, \beta) \xleftarrow{\pi_1} \text{SRep}(Q/I, \gamma \subset \beta) \xrightarrow{\pi_2} \text{Gr}(\gamma, \beta).$$

Then $\pi_2$ is surjective by assumption on $\gamma$, so

$$\dim \textbf{Gr}(\gamma, W)' = \dim \textbf{Gr}(\gamma, \beta) + \dim \pi_2^{-1} - \dim \text{SRep}(Q/I, \beta),$$

where $\dim \pi_2^{-1}$ is the dimension of a general fiber of $\pi_2$.

We can show directly that $\dim \textbf{Gr}(\gamma, W)' = \langle \gamma, \beta - \gamma \rangle_1$ when $W$ is injective and $W_{\tau(a)}$ is surjective. To get a $\gamma$-dimensional submodule $U$, we first choose a $\gamma_1$-dimensional subspace in $K^n$, which gives $\gamma_1(n - \gamma_1)$ dimensions of choices. This determines its image in $K^{2n+\delta}$. Set $d = \gamma_2 - \gamma_1 - \gamma_3$. We need $\dim \ker U_{\tau(a)} = \gamma_2 - \gamma_3$ and $U_a(K^{\gamma_1}) \subset \ker U_{\tau(a)}$, so to pick the rest of the kernel, we pick a $d$-dimensional subspace of $\ker W_{\tau(a)}/U_a(K^{\gamma_1})$. Since $\dim \ker W_{\tau(a)} = n + \delta$, this gives us $d(n + \delta - \gamma_1 - d)$ dimensions of choice.

Finally, we choose a $\gamma_3$-dimensional subspace in $K^{2n+\delta}/(K^{\gamma_1} + K^d)$ disjoint from the image of $W_a(K^n)$. A generic choice of subspace works, so we get $\gamma_3(2n + \delta - \gamma_2)$ dimensions of choice. This determines the $\gamma_3$-dimensional subspace in the last $K^n$. Thus,

$$\dim \textbf{Gr}(\gamma, W)' = \gamma_1(n - \gamma_1) + d(n + \delta + \gamma_3 - \gamma_2) + \gamma_3(2n + \delta - \gamma_2)$$

$$= \gamma_1(n - \gamma_1) + (\gamma_2 - \gamma_1)(2n + \delta - \gamma_2) + d(\gamma_3 - n)$$

$$= \gamma_1(n - \gamma_1) + \gamma_2(2n + \delta - \gamma_2) + \gamma_3(n - \gamma_3)$$

$$- \gamma_1(2n + \delta - \gamma_2) - \gamma_2(n - \gamma_3) + \gamma_1(n - \gamma_3)$$

$$= \langle \gamma, \beta - \gamma \rangle_1.$$
From this, it is a quick check to show that \( \dim \text{Gr}(\gamma, W)' = \langle \gamma, \beta - \gamma \rangle I \) for the symmetric quiver with relations

\[
K^1 \to K^2 \to \cdots \to K^n \stackrel{a}{\to} K^{2n+\delta} \stackrel{r(a)}{\to} K^n \to \cdots \to K^1.
\]

We get the formula

\[
\dim \pi_2^{-1} = -\sum_{a \in Q_1} \gamma(ta)(\beta - \gamma)(ha) + \sum_{r \in Q_2} \gamma(tr)(\beta - \gamma)(hr) + \dim \text{SRep}(Q/I, \beta).
\]

This formula is “additive” with respect to the number \( r \) of arms (the formula only involves arrows and relations, and not vertices), and the general fiber \( \pi_2^{-1} \) when there are \( r \) arms can be thought of as the direct product of \( r \) copies of the general fiber when there is 1 arm. So we can work as in the proof of Theorem 3.13 to conclude that \( \dim \text{Gr}(\gamma, W)' = \langle \gamma, \beta - \gamma \rangle I \) for \( r \) arms.

3.4. Calculating \( \text{Ext}^1_{Q/I} \)

Suppose that the characteristic is arbitrary in this section. Recall from Section 3.3 that \( \text{Gr}(\gamma, W)' \) denotes the variety of submodules \( U \) of \( W \) with \( \dim U = \gamma \) and \( \text{idim} U \leq 1 \).

**Lemma 3.16.** Suppose \( \text{gldim } Q/I \leq 2 \), and let \( \alpha \) be a dimension vector and let \( W \) be a representation of dimension \( \beta \). Assume that the following conditions hold.

(a) There is an irreducible component \( C \subseteq \text{Rep}(Q/I, \alpha) \) such that a general representation \( V \) satisfies \( \text{idim} V \leq 1 \) and \( \text{pdim} V \leq 1 \). Let \( \gamma \) be the generic rank of a homomorphism between a representation of \( C \) and \( W \).

(b) \( \text{idim} W \leq 1 \) and \( \dim \text{Gr}(\gamma, W)' = \langle \gamma, \beta - \gamma \rangle I \).

Then \( \dim \text{Ext}^1(C, W) = -\langle \alpha - \gamma, \beta - \gamma \rangle I \).

**Proof.** The proof proceeds like the proof of [19, Theorem 5.2]. Define

\[
\text{Hom}(K^\alpha, \gamma, W) = \{(\varphi, U) \in \text{Hom}_K(K^\alpha, W) \times \text{Gr}(\gamma, W)' \mid \varphi(K^\alpha) = U\}.
\]

For \( U \in \text{Gr}(\gamma, W)' \), the fiber of \( U \) in \( \text{Hom}(K^\alpha, \gamma, W) \) can be identified with an open subset of \( \text{Hom}_K(K^\alpha, U) \), so has dimension \( \sum_{x \in Q_0} \alpha(x)\gamma(x) \). So by (b), we have

\[
\dim \text{Hom}(K^\alpha, \gamma, W) = \langle \gamma, \beta - \gamma \rangle I + \sum_{x \in Q_0} \alpha(x)\gamma(x).
\]

Now define

\[
\text{Hom}(Q, \alpha, \gamma, W) = \{(M, \varphi, U) \in C \times \text{Hom}(K^\alpha, \gamma, W) \mid \varphi \in \text{Hom}_{Q/I}(M, U)\},
\]

and let \( \pi_1 \) and \( \pi_2 \) be the projections of \( \text{Hom}(Q, \alpha, \gamma, W) \) to \( C \) and \( \text{Hom}(K^\alpha, \gamma, W) \), respectively. A point in \( \pi_2^{-1}(\varphi, U) \) is a collection of linear maps \( \{M_a : K^\alpha(ta) \to K^\alpha(ha)\}_{a \in Q_1} \) that lift the maps in \( U \).
We can rephrase these lifts as certain representations of $Qe/Ie$ of dimension $\alpha e$ where $\alpha e(0, x) = (\alpha - \gamma)(x)$ and $\alpha e(1, x) = \gamma(x)$ for $x \in Q_0$. A general representation $M \in \text{Rep}(Qe/Ie, \alpha e)$ satisfies $\text{Ext}^2(M, M) = 0$ by (a) and Proposition 3.11. So by Proposition 3.7, 

$$\dim \text{Rep}(Qe/Ie, \alpha e) = \sum_{x \in Q_0} \alpha(x)(\alpha - \gamma)(x) - \langle \alpha, \alpha - \gamma \rangle I + \sum_{x \in Q_0} \gamma(x)^2 - \langle \gamma, \gamma \rangle I.$$ 

There is a map $q : \text{Rep}(Qe/Ie, \alpha e) \to \text{Rep}(Q/I, \gamma)$ which sends a representation to its quotient which is supported on $[1] \times Q_0$. Let $N \in \text{Rep}(Q/I, \gamma)$ be a representation which appears as the image of a map $V \to W$ for general $V \in \mathcal{C}$. Let $C' \subseteq \text{Rep}(Q/I, \gamma)$ be the irreducible component containing $N$. Then the general rank of a homomorphism between $C'$ and $\text{Rep}(Q/I, \beta)$ is $\gamma$, so we may rechoose the above $N$ so that $\dim q^{-1}(N) = \dim \text{Rep}(Qe/Ie, \alpha e) - \dim C'$. This also shows that the image of $\pi_2$ contains a nonempty open set. Also, $\text{idim} N \leq 1$ since it is a quotient of $V$, and we can identify $q^{-1}(N)$ with $\pi_2^{-1}(\varphi, U)$, so 

$$\dim \text{Hom}(Q, \alpha, \gamma, W) = \dim \text{Hom}(K^\alpha, \gamma, W) + \dim \pi_2^{-1}(\varphi, U)$$
$$= \dim \text{Hom}(K^\alpha, \gamma, W) + \dim \text{Rep}(Qe/Ie, \alpha e) - \dim C'$$
$$= \langle \gamma, \beta - \gamma \rangle I + \sum_{x \in Q_0} \alpha(x)\gamma(x)$$
$$+ \sum_{a \in Q_1} \alpha(ta)(\alpha - \gamma)(ha) - \sum_{r \in Q_2} \alpha(tr)(\alpha - \gamma)(hr) \quad (3.17)$$

On the other hand, there is an open subset $\Omega \subseteq \mathcal{C}$ that is contained in the image of $\pi_1$ by definition of $\gamma$. So $\dim \pi_1^{-1}(\Omega) = \dim \text{Hom}(Q, \alpha, \gamma, W)$. Furthermore, for general $M \in \Omega$ such that $\dim \text{Hom}_{Q/I}(M, W) = \text{Hom}(C, W)$, we have that the fiber $\pi_1^{-1}(M)$ is a dense open subset of $\text{Hom}_{Q/I}(M, W)$. So we get 

$$\dim \text{Hom}(Q, \alpha, \gamma, W) = \dim \text{Rep}(Q/I, \alpha) + \dim \pi_1^{-1}(M)$$
$$= \sum_{a \in Q_1} \alpha(ta)\alpha(ha) - \sum_{r \in Q_2} \alpha(tr)\alpha(hr) + \text{Hom}(C, W). \quad (3.18)$$

Putting together (3.17) and (3.18) we get 

$$\langle \gamma, \beta - \gamma \rangle I + \langle \alpha, \gamma \rangle I = \text{Hom}(C, W).$$

Now $\text{Ext}^2(C, W) = 0$ by (b), so we can rewrite this equality as 

$$\langle \alpha, \beta \rangle I + \text{Ext}^1(C, W) = \langle \gamma, \beta - \gamma \rangle I + \langle \alpha, \gamma \rangle I,$$

from which we conclude that $\text{Ext}^1(C, W) = -\langle \alpha - \gamma, \beta - \gamma \rangle I$. \hfill $\square$

**Theorem 3.19.** Let $Q/I$ be a quiver with relations and let $\alpha$ and $\beta$ be dimension vectors. Assume that the conditions of Lemma 3.16 hold and use the same notation. Then for generic $V \in \mathcal{C}$, we have $\dim \text{Ext}^1_{Q/I}(V, W) = \max_{\beta'} -\langle \alpha, \beta' \rangle I$ where the maximum is over all dimension vectors $\beta'$ of factor modules of $W$. 
**Proof.** Pick $V \in C$ so that a generic map $V \to W$ has rank $\gamma$. We always have the inequality $\dim \text{Ext}^1(V, W) \geq \max_{\gamma'} -\langle \alpha, \beta' \rangle_I$, so the content of the result is that there exists a factor module $W'$ of $W$ such that $\dim \text{Ext}^1(V, W') = -\langle \alpha, \dim W' \rangle_I$.

If $\gamma = 0$, then $\text{Hom}(V, W) = 0$ and hence $\text{Ext}^1(V, W) = -\langle \alpha, \beta \rangle_I$ since $\text{Ext}^2(V, W) = 0$ by Lemma 3.16(b). Otherwise if $\gamma \neq 0$, pick a map of rank $\gamma$, and let $V'$ be the kernel and $W'$ be the cokernel. Let $C_{V'}$ and $C_{W'}$ be the irreducible components containing $V'$ and $W'$, respectively. Now $\dim W' \leq 1$ since $\dim W \leq 1$, so $\text{Ext}^2(V', W') = 0$, which means that $\dim \text{Ext}^1(V', W') \geq -\langle \alpha - \gamma, \beta - \gamma \rangle_I$. We also have surjections

$$\text{Ext}^1(V, W) \to \text{Ext}^1(V, W') \to \text{Ext}^1(V', W'),$$

the first because $\text{pdim} V \leq 1$, and the second because $\text{idim} W' \leq 1$. By Lemma 3.16, $\dim \text{Ext}^1(V, W) = -\langle \alpha - \gamma, \beta - \gamma \rangle_I$, which forces $\dim \text{Ext}^1(V', W') = -\langle \alpha - \gamma, \beta - \gamma \rangle_I$, which is its minimal possible value, so $\text{Ext}^1(C_{V'}, C_{W'}) = -\langle \alpha - \gamma, \beta - \gamma \rangle_I = \text{Ext}^1(C, W)$. This also implies that the above surjections are isomorphisms so that $\text{Ext}^1(C, C_{W'}) \cong \text{Ext}^1(C, W)$. But we know that $\text{Ext}^1(C_{V'}, C_{W'}) \cong \text{Ext}^1(C, C_{W'})$ since $\text{idim} W' \leq 1$. In particular, $\text{Ext}^1(C, C_{W'}) = \text{Ext}^1(C, W)$.

Now a general representation $C_{W'}$ satisfies the assumptions of Lemma 3.16: the fact that $\dim C_{W'} \leq 1$ we have already mentioned; we can apply Theorem 3.13 because if $\gamma'$ is the generic rank of a map $\varphi : V'' \to W'$, where $V'' \in C$ is generic, then $\text{Gr}(\gamma', W')$ is nonempty because $\varphi(V'')$ is a quotient of $V''$ and hence $\text{idim}(\varphi(V'')) \leq 1$. Therefore, by induction on $\text{idim} W$, we conclude that for general $X \in C_{W'}$, $\dim \text{Ext}^1(V, X) = -\langle \alpha, \beta'' \rangle_I$ where $\beta'' = \dim W''$ for some factor module $W''$ of $X$. Since a general representation in this component has a factor module of dimension $\beta''$, the same is true for $W'$ [19, Lemma 3.1], and hence is also true for $W$. □

4. Orthogonal and symplectic Littlewood–Richardson coefficients

4.1. Representation theory of the classical groups

Suppose that the characteristic is 0. Let $E$ be a $(2n + \delta)$-dimensional vector space over $K$ (where $\delta \in \{0, 1\}$) and let $\omega$ be a nondegenerate symplectic or symmetric bilinear form on $E$. Let $G$ be the subgroup of $\text{SL}(E)$ which preserves $\omega$. In order to be precise let us just list the cases:

1. Case $B_n$: We have $\delta = 1$, $\omega$ is symmetric, $G = \text{SO}(E) \cong \text{SO}(2n + 1)$.
2. Case $C_n$: We have $\delta = 0$, $\omega$ is skew-symmetric, $G = \text{Sp}(E) \cong \text{Sp}(2n)$.
3. Case $D_n$: We have $\delta = 0$, $\omega$ is symmetric, $G = \text{SO}(E) \cong \text{SO}(2n)$.

We identify the weight lattice of $G$ with $Z^n = \mathbb{Z}^n(\varepsilon_1, \ldots, \varepsilon_n)$ equipped with the standard dot product. Since we have assumed that $K$ is algebraically closed, we can find a basis $e_1, \ldots, e_{2n + \delta}$ for $E$ such that $1 = \omega(e_i, e_{2n+\delta+1-i}) = \pm \omega(e_{2n+\delta+1-i}, e_i)$ (the sign depending on whether $\omega$ is symmetric or skew-symmetric) for $i = 1, \ldots, n + \delta$, and all other pairings are 0. Representing elements of $G$ as matrices with respect to this ordered basis, we can take our maximal torus $T$ to be the subgroup of diagonal matrices, and our Borel subgroup $B$ to be the subgroup of upper triangular matrices. We identify $(\lambda_1, \ldots, \lambda_n) \in Z^n$ with the character $\lambda : T \to K^*$ given by

$$\text{diag}(d_1, \ldots, d_{2n+\delta}) \mapsto d_1^{\lambda_1} d_2^{\lambda_2} \cdots d_n^{\lambda_n}.$$
Table 1
Roots and weights.

<table>
<thead>
<tr>
<th></th>
<th>$SO(2n + 1)$</th>
<th>$Sp(2n)$</th>
<th>$SO(2n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple roots</td>
<td>$\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots$</td>
<td>$\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots$</td>
<td>$\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_{n-1} - \varepsilon_n, \varepsilon_n$</td>
<td>$\varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$</td>
<td>$\varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n$</td>
</tr>
<tr>
<td>Dominant weights</td>
<td>$\lambda_1 \geq \cdots \geq \lambda_n \geq 0$</td>
<td>$\lambda_1 \geq \cdots \geq \lambda_n \geq 0$</td>
<td>$\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq</td>
</tr>
</tbody>
</table>

To be completely explicit, we list the simple roots and the conditions for a weight to be dominant under this identification in Table 1.

We review the relevant details of Weyl’s construction for these representations in characteristic 0. Given $1 \leq i < j \leq d$, we have a contraction map

$$\Psi_{i < j} : E^{\otimes d} \to E^{\otimes d-2},$$

$$v_1 \otimes \cdots \otimes v_d \mapsto \omega(v_i, v_j)v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes v_d,$$

and we define

$$E^{(d)} = \bigcap_{1 \leq i < j \leq d} \ker \Psi_{i < j}.$$

The irreducible polynomial representation of $GL(E)$ of highest weight $\lambda$ can be constructed using a Young symmetrizer acting on $E^{\otimes d}$ where $d = |\lambda|$. The details of this construction won’t be needed, but details can be found in [17, §9.7] and [21, §2.2]. Let $S_\lambda(E)$ denote such a realization. This notation is compatible with its use in Section 2.3. Also, recall that $\ell(\lambda)$ is the number of nonzero parts of a partition $\lambda$. Finally, we define

$$S_{[\lambda]}(E) = E^{(d)} \cap S_\lambda(E).$$

**Proposition 4.1.** The intersection $S_{[\lambda]}(E)$ is nonzero if and only if $\ell(\lambda) \leq n$. For $\ell(\lambda) \leq n$, we have the following cases.

(a) Case $B_n$: $S_{[\lambda]}(E)$ is an irreducible representation of $SO(E)$ with highest weight $\lambda$.

(b) Case $C_n$: $S_{[\lambda]}(E)$ is an irreducible representation of $Sp(E)$ with highest weight $\lambda$.

(c) Case $D_n$: $S_{[\lambda]}(E)$ is an irreducible representation of $O(E)$. If $\lambda_n = 0$, then $S_{[\lambda]}(E)$ is an irreducible representation of $SO(E)$ with highest weight $\lambda$. Otherwise, if $\lambda_n > 0$, then $S_{[\lambda]}(E)$ is the direct sum of two irreducible representations of $SO(E)$, one of highest weight $(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n)$, and one of highest weight $(\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n)$.

**Proof.** See [17, §§11.6.3, 11.6.6].

**Remark 4.2.** In types $B_n$ and $C_n$, all representations are self-dual. The same is true for type $D_n$ when $n$ is even. When $n$ is odd, the dual of a representation of type $D_n$ with highest weight $(\beta_1, \ldots, \beta_{n-1}, -\beta_n)$ has highest weight $(\beta_1, \ldots, \beta_{n-1}, -\beta_n)$. This follows from [17, Proposition 10.5.3]. In particular, we see from Proposition 4.1 that $S_{[\lambda]}(E)^* \cong S_{[\lambda]}(E)$ in all cases.
4.2. Flag quivers

Suppose that the characteristic is different from 2. Pick positive integers \( n \) and \( r \). We construct symmetric quivers \( Q_{r,n}^+ \) and \( Q_{r,n}^- \) as follows. The vertices consist of \( x_i^j \) and \( \tau(x_i^j) \) where \( j = 1, \ldots, r \) and \( i = 1, \ldots, n \) along with an additional vertex \( u = \tau(u) \). For notation we use \( x_i^{j+1} \) to denote \( u \) for any \( j = 1, \ldots, r \). The action of the involution \( \tau \) is suggested by the notation and \( s(u) \) is given by the superscript of \( \tau(x_i^j) \). For each \( j = 1, \ldots, r \) and each \( i = 1, \ldots, n \), we have an arrow \( x_i^j \xrightarrow{a_i^j} x_{i+1}^j \) and \( \tau(x_{i+1}^j) \xrightarrow{\tau(a_i^j)} \tau(x_i^j) \). Again, the action of \( \tau \) is suggested by the notation.

Furthermore, we impose the relations \( \tau(a_i^j)a_n^j = 0 \) for \( j = 1, \ldots, r \). We have drawn a diagram in the case \( r = 3 \):

\[
\begin{array}{ccccccccc}
  x_1^1 & \xrightarrow{a_1^1} & x_2^1 & \cdots & x_n^1 \\
  x_1^2 & \xrightarrow{a_1^2} & x_2^2 & \cdots & x_n^2 & \xrightarrow{u = \tau(u)} & x_1^2 \\
  x_1^3 & \xrightarrow{a_1^3} & x_2^3 & \cdots & x_n^3 & \xrightarrow{\tau(x_n^3)} & x_1^3 \\

  \tau(x_n^1) & \xrightarrow{\tau(a_{n-1}^1)} & \cdots & \tau(x_2^1) & \xrightarrow{\tau(a_1^1)} & x_1^1 \\
  \tau(x_n^2) & \xrightarrow{\tau(a_{n-1}^2)} & \cdots & \tau(x_2^2) & \xrightarrow{\tau(a_1^2)} & x_1^2 \\
  \tau(x_n^3) & \xrightarrow{\tau(a_{n-1}^3)} & \cdots & \tau(x_2^3) & \xrightarrow{\tau(a_1^3)} & x_1^3 \\
\end{array}
\]

We call \( Q_{r,n}^+ \) the **orthogonal flag quiver** and \( Q_{r,n}^- \) the **symplectic flag quiver**. We define a dimension vector \( \beta^\delta \) by \( \beta^\delta(x_i^j) = i \) for \( j = 1, \ldots, r \) and \( i = 1, \ldots, n \), and define \( \beta^\delta(u) = 2n + \delta \).

Since \( \delta = 0 \) in the symplectic case, we will usually write \( \beta = \beta_0 \) in this case. We will use \( Q_{r,n} \) to denote the underlying quiver with relations.

**Proposition 4.3.** The quiver \( Q_{r,n} \) has global dimension 2.

**Proof.** It is enough to show that every simple module \( S_x \) has a projective resolution of length at most 2. We write down the resolutions and leave the verification to the reader. For \( u \), we have

\[
0 \rightarrow \bigoplus_{j=1}^r P_{\tau(x_i^j)} \rightarrow P_u \rightarrow S_u \rightarrow 0.
\]

For each \( j = 1, \ldots, r \) and \( i = 1, \ldots, n \) we have (with the convention that \( P_{\tau(x_0^j)} = 0 \)):

\[
0 \rightarrow P_{\tau(x_{i-1}^j)} \rightarrow P_{\tau(x_i^j)} \rightarrow S_{\tau(x_i^j)} \rightarrow 0,
\]

\[
0 \rightarrow P_{x_{i+1}^j} \rightarrow P_{x_i^j} \rightarrow S_{x_i^j} \rightarrow 0 \quad (1 \leq i \leq n - 1),
\]

\[
0 \rightarrow P_{\tau(x_n^j)} \rightarrow P_u \rightarrow P_{x_n^j} \rightarrow S_{x_n^j} \rightarrow 0.
\]

From now on, we replace all instances of orthogonal groups by special orthogonal groups in the definition of semi-invariants. In effect, we are ignoring the action of \( O(V)/SO(V) \) on semi-
invariants. Hence we do not have to worry about irreducible components of $\text{SRep}(Q/I, \beta)$ not being closed under the action of $G(Q, \beta)$.

A symmetric representation $V \in \text{SRep}(Q^+, \beta)$ is given by the data of $(2n + \delta)$-dimensional vector space $V(u)$ equipped with a nondegenerate symmetric form, in addition to arbitrary vector spaces $V(x^{\dagger}_i)$ of dimension $i$ and arbitrary linear maps $V_{a^{\dagger}_j}$ for $j = 1, \ldots, r$ and $i = 1, \ldots, n$. The relation $\tau(a^{\dagger}_j)a^{\dagger}_j = 0$ is equivalent to saying that the image of $V(x^{\dagger}_i)$ under $V_{a^{\dagger}_j}$ is an isotropic subspace. This can be seen by picking a hyperbolic basis for $V(u)$. There is a similar interpretation for a symmetric representation in $\text{SRep}(Q^-, \beta)$.

Let $Z' \subset \text{Hom}(V(x^{\dagger}_i), V(u))$ be the subvariety of maps whose image is an isotropic subspace. Let $Y$ be the Grassmannian of $n$-dimensional isotropic subspaces of $V(u)$. Then $Y$ is equipped with a trivial vector bundle $V(u) \times Y$ and a tautological subbundle $R \subset V(u) \times Y$ given by $\{(x, W) \mid x \in W\}$. We can also form the vector bundle $Z = \text{Hom}(V(x^{\dagger}_i), R) = V(x^{\dagger}_i)^* \otimes R$. Then $Z$ consists of pairs $(\varphi, W)$ where $\varphi : V(x^{\dagger}_i) \to W$, so there is a natural projection $\pi : Z \to \text{Hom}(V(x^{\dagger}_i), V(u))$ whose image is $Z'$.

**Proposition 4.4.** The map $\pi : Z \to Z'$ is a projective birational morphism.

**Proof.** Since $Y$ is a projective variety, the projection $V(u) \times Y \to V(u)$ is projective, so the same is true for the restriction $\pi$. The set of injective maps in $Z'$ is open and dense, and there is a uniquely defined inverse on this open set. □

**Corollary 4.5.** The open subset of $Z'$ consisting of injective maps is nonsingular.

In the symplectic case and the odd orthogonal case, $Y$ is an irreducible variety since it has a transitive action of $\text{Sp}(V(u))$ and $\text{SO}(V(u))$, respectively. So $Z'$ is also irreducible in these cases. In the even orthogonal case, $\text{SO}(V(u))$ does not act transitively on $Y$. However, $\text{O}(V(u))$ does act transitively, so $Y$ has two connected components. To describe them, fix a maximal isotropic subspace $W \subset V(u)$. Then one component consists of subspaces whose intersection with $W$ has even dimension, and the other component consists of subspaces whose intersection with $W$ has odd dimension. In particular, the subvariety $Z' \subset \text{Hom}(V(x^{\dagger}_i), V(u))$ of isotropic maps has two irreducible components. Each component has a dense open subset consisting of injective maps $\varphi$ and the components are distinguished by whether $\dim(\varphi(V(X_n)) \cap W)$ is even or odd. Their intersection consists of the non-injective maps.

**Proposition 4.6.** The varieties $\text{SRep}(Q^+, \beta^1)$ and $\text{SRep}(Q^-, \beta)$ are irreducible. The variety $\text{SRep}(Q^+, \beta^0)$ has $2^r$ irreducible components. Each of the components is faithful (see Section 3.1.2).

**Proof.** The varieties $\text{Hom}(V(x^{\dagger}_j), V(x^{\dagger}_{j+1}))$ are irreducible for $j = 1, \ldots, r$ and $i = 1, \ldots, n - 1$, so the statements about the irreducible components follow from the preceding discussion. The faithfulness of the components in $\text{SRep}(Q^+, \beta^0)$ follows from their explicit description in the preceding discussion. □

**Corollary 4.7.** The restriction of $\pi$ for each irreducible component $X$ of $\text{SRep}(Q^+, \beta^0)$ is a rational desingularization. In particular, they are normal varieties.

This follows from the results of [21, Chapter 5]. We won’t need it, so we omit the details.
Proposition 4.8. Let $V$ be a representation of $\mathcal{D}_n^\pm$. Then $\text{pdim} V \leq 1$ if and only if the maps $V(x^n_j) \to V(u)$ are injective for $j = 1, \ldots, r$. Dually, $\text{idim} V \leq 1$ if and only if the maps $W(u) \to W(\tau(x^n_j))$ are surjective for $j = 1, \ldots, r$.

Proof. Throughout the proof, let $Q/I$ denote either the quiver with relations $\mathcal{D}_{r,n}^+$ or $\mathcal{D}_{r,n}^-$. First we prove that if the maps $V(x^n_j) \to V(u)$ are injective, then $\text{pdim} V \leq 1$. Let $V'$ be the submodule generated by the $V(x^n_j)$ for $j = 1, \ldots, r$. Then $\text{pdim} V/V' \leq 1$ since $V/V'$ is supported on a quiver without relations. So $\text{pdim} V \leq 1$ if we can prove that $\text{pdim} V' \leq 1$. Furthermore, the submodule $V''$ of $V'$ generated by the $V' \cdot (\tau(x^n))$ is supported in a quiver without relations, so it is enough to show that $\text{pdim} W \leq 1$ where $W = V''/V'''$.

Set $P_0 = \bigoplus_{j=1}^r P_{x^n,\lambda}(x^n_j)$. To say that the kernel $P_1$ of the surjection $P_0 \to W \to 0$ is projective, it is enough to say that the submodule of $P_0$ generated by $P_1(u)$ is projective, which in our situation amounts to saying that the maps $P_1(\tau(x^n_j))$ are injective. Pick a basis $v_1, \ldots, v_N$ for $P_1(u)$. Each basis vector $v_i$ can be written as $v_i = v_i^1 + \cdots + v_i^r$ where $v_i^j \in P_{x^n,\lambda}(x^n_j)(u)$, and its image in $P_1(\tau(x^n_j))$ is $v_i - v_i^j$. Suppose that the vectors $v_1 - v_1^j, \ldots, v_N - v_N^j$ are linearly independent. Since $v_1, \ldots, v_N$ are linearly independent, this means that some nonzero linear combination of the $v_i$ is in $P_{x^n,\lambda}(x^n_j)(u)$. Hence the map $P_{x^n,\lambda}(x^n_j)(u) \to W(u)$ coming from $P_0 \to W$ has a nonzero kernel, which contradicts that $W(x^n_j) \to W(u)$ is injective. Therefore the images of $v_1, \ldots, v_N$ under each $P_1(\tau(x^n_j))$ are linearly independent, so we are done.

On the other hand, if the map $V(x^n_j) \to V(u)$ is not injective, then $P_0$ will contain a direct summand $P_{x^n,\lambda}(x^n_j)$ such that $P_{x^n,\lambda}(x^n_j)(u)$ will be in the kernel of $P_0 \to V \to 0$. This requires that $P_1$ contain a summand $P_{x^n,\lambda}(x^n_j)$, and the restriction of the map $P_1 \to P_0$ to this summand will not be injective.

The dual statement about injective dimension is proved in a similar manner, or can be obtained by applying the duality functor $\text{Hom}_K(-, K)$. □

4.3. Back to semi-invariants

Suppose that the characteristic is 0. Now we calculate the space of semi-invariants of $S\text{Rep}(\mathcal{D}_{r,n}^\pm, \beta^\delta)$. First, we need a better understanding of what the coordinate ring of $S\text{Rep}(\mathcal{D}_{r,n}^\pm, \beta^\delta)$ is. Since $S\text{Rep}(\mathcal{D}_{r,n}^\pm, \beta^\delta)$ is an $r$-fold product of $S\text{Rep}(\mathcal{D}_{1,n}^\pm, \beta^\delta)$, it is enough to calculate the coordinate ring in the case $r = 1$. Let $R$ denote this variety in the case $r = 1$ and set $x_i = x_i^n$ and let $Z \subset \text{Hom}(V(x_n), V(u))$ denote the subvariety of maps whose image is an isotropic subspace. Then $R$ is a product of an affine space with $Z$, so we just need to describe the coordinate ring of $Z$.

By the Cauchy identity (2.10), the polynomial functions on $\text{Hom}(V(x_n), V(u))$ are given by

$$\text{Sym}(V(x_n) \otimes V(u)^*) = \bigoplus_{\lambda} S_{\lambda}(V(x_n)) \otimes S_{\lambda}(V(u)^*).$$

Imposing the relation that the image of the map must be an isotropic subspace means that after applying the Schur functor $S_{\lambda}$, the image of $S_{\lambda}(V(x_n)) \to S_{\lambda}(V(u))$ must be in the kernel of
each contraction map as described in Section 4.1. So the coordinate ring of $Z$ is a quotient of

$$
\bigoplus_{\lambda} S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*).
$$

In fact, we have equality. To see this, note that $S_{\lambda}(V(x_n)) \otimes S_{\lambda}(V(u)^*)$ is nonzero on some map $\varphi$. We can conjugate $\varphi$ by $\text{GL}(V(u))$ to $g\varphi$ so that the image of $g\varphi$ is an isotropic subspace. Hence $S_{\lambda}(V(x_n)) \otimes S_{\lambda}(V(u)^*)$ will also be nonzero on $g\varphi$, and $S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*)$ has to be nonzero on $g\varphi$. Furthermore, letting $G$ be either $\text{GL}(V(x_n)) \times \text{O}(V(u))$ or $\text{GL}(V(x_n)) \times \text{Sp}(V(u))$ depending on which case we are in, $S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*)$ is an irreducible $G$-module and $Z$ is a $G$-invariant subvariety of $\text{Hom}(V(x_n), V(u))$, so no function of $S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*)$ can be identically zero on $Z$.

By Proposition 4.6, the variety $R$ is irreducible in types $B$ and $C$, but has 2 components $R^+$ and $R^-$ in type $D$. We want to describe the coordinate rings for $\text{SO}(V(u))$-module, but that when $\lambda_n > 0$, it splits into the direct sum of two representations when considered as an $\text{SO}(V(u))$-module. Call these two summands $S_{[\lambda]}(V(u))$. For notational purposes, the symbols $[\lambda]\pm$ mean $[\lambda]$ in the case that $\lambda_n = 0$. Fix an element $g \in \text{O}(V(u)) \setminus \text{SO}(V(u))$. Conjugation by $g$ is an outer automorphism of $\text{SO}(V(u))$ that transforms $S_{[\lambda]}(V(u))$ into $S_{[\lambda]}(V(u))$ and vice versa. Furthermore, while $R^+$ and $R^-$ are preserved by $\text{SO}(V(u))$, the element $g$ swaps the two of them and preserves $R^0$. The points in $R^0$ consist of non-injective maps $V(x_n) \to V(u)$. So when $\lambda_n > 0$, the functions in $S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*)$ vanish on $R^0$.

We claim that the “positive” and the “negative” representations of $\text{SO}(V(u))$ form the coordinate rings of the two separate components of $R$. If not, suppose that both $S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*)$ and $S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*)$ are nonzero on the component $R^+$ where $\lambda_n > 0$ and $\lambda_n > 0$ and choose highest weight vectors $f_{\lambda}$ and $f_{\mu}$ in both. Since $R^+$ is irreducible, $f_{\lambda} \neq f_{\mu}$ and has weight $(\lambda + \mu, [\lambda]^+ + [\mu]^+)$ when $n$ is even, and has weight $(\lambda + \mu, [\lambda]^+ + [\mu]^+)$ when $n$ is odd (Remark 4.2). But there are no representations in $K[Z]$ with this highest weight, which is a contradiction. Therefore, setting

$$
A = \bigoplus_{\lambda, \lambda = 1}^{n-1} S_{\lambda}(V(x_i)) \otimes S_{\lambda}(V(x_{i+1}))^*,
$$

we have

$$
K[R^+] = A \otimes \bigoplus_{\lambda} S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*),
$$

$$
K[R^-] = A \otimes \bigoplus_{\lambda} S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*),
$$

$$
K[R^0] = A \otimes \bigoplus_{\lambda, \lambda = 0} S_{\lambda}(V(x_n)) \otimes S_{[\lambda]}(V(u)^*).
$$

There is no inherent way of distinguishing the two components or the representations $S_{[\lambda]}(V(u)^*)$, so the formulas above should be taken as sign conventions.
Proposition 4.9. The coordinate ring of \( S\text{Re}p(\mathcal{D}_{r,n}^{\pm}, \beta^\delta) \) is

\[
\bigoplus_{\lambda} \left( \bigotimes_{1 \leq j \leq r} S_{\lambda(x_j^i)}(V(x_j^i)) \otimes S_{\lambda(x_j^{i+1})}(V(x_j^{i+1})^*) \otimes \bigotimes_{1 \leq j \leq r} S_{\lambda(x_j^r)}(V(x_j^r)) \otimes S_{\lambda(x_n^1)}(V(u)^*) \right),
\]

where \( \lambda \) ranges over all partition-valued functions of the set \( \{x_j^i\}_{1 \leq i \leq n} \). The coordinate ring of an irreducible component in the type D case is given by a choice of pluses and minuses \( \varepsilon : \{x_j^i\} \rightarrow \{+, -, \} \) to add to the \( \lambda(x_n^1) \). We use the notation \( S\text{Re}p(\mathcal{D}_{r,n}^+, \beta^\delta) \) to denote the corresponding component.

Given this description, we can now calculate the space of semi-invariants in terms of representations of the corresponding classical group. This is essentially the same as [6, §3] but we have tried to provide more details.

Lemma 4.10. If the direct summand corresponding to the function \( \lambda \) of Proposition 4.9 contains a nonzero \( SG(\mathcal{D}_{r,n}^{\pm}, \beta^\delta) \)-invariant, then there exist numbers \( \sigma(x_j^i) \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, r \) such that

\[
\lambda(x_j^i)' = (1^{\sigma(x_j^1)}, 2^{\sigma(x_j^2)}, \ldots, i^{\sigma(x_j^i)}).
\]

Furthermore, \( \sigma \) is the symmetric weight of the corresponding semi-invariant.

Proof. We prove this by induction on \( i \). For \( i = 1 \), we have that \( S_{\lambda(x_j^i)}(V(x_j^i)) \) is a nonzero \( SL(V(x_j^i)) \)-invariant if and only if \( \lambda(x_j^i)' = (1^{\sigma(x_j^1)}) \) for some \( \sigma(x_j^1) \geq 0 \) since \( \dim V(x_j^1) = 1 \).

For general \( i \), write \( \mu' = (1^{\sigma(x_j^1)}, 2^{\sigma(x_j^2)}, \ldots, (i - 1)^{\sigma(x_j^{i-1})}) \). Then

\[
S_{\lambda(x_j^{i-1})}(V(x_j^{i-1})^*) \otimes S_{\lambda(x_j^i)}(V(x_j^i)) \cong \left( \bigwedge^i V(x_j^i) \right) \otimes S_{\mu'}(V(x_j^i)) \otimes S_{\lambda(x_j^i)}(V(x_j^i))
\]

as \( GL(V(x_j^i)) \)-representations, where \( \Sigma = \sigma(x_j^1) + \cdots + \sigma(x_j^{i-1}) \) by Proposition 2.12. We see that if there is a nonzero \( SL(V(x_j^i)) \)-invariant, then \( k \) must appear as a part of \( \lambda(x_j^i)' \) with multiplicity \( \sigma(x_j^k) \) for \( k = 1, \ldots, i - 1 \), and \( i \) can appear with arbitrary multiplicity \( \sigma(x_j^i) \) by Proposition 2.13. Hence \( \lambda(x_j^i) \) is also in the form (4.11), and the action of \( GL(V(x_j^i)) \) on the \( SL(V(x_j^i)) \)-invariants is via the \( \sigma(x_j^i) \)th power of the determinant again by Proposition 2.13.

Proof of Theorem 1.1. Let \( Q/I \) denote either \( \mathcal{D}_{r,n}^+ \) or \( \mathcal{D}_{r,n}^- \) depending on which group we are interested in. If \( (V_{N\lambda} \otimes \cdots \otimes V_{N\lambda'})^G \neq 0 \), then by Lemma 4.10, there is a corresponding nonzero weight space \( SSI(Q/I, \beta^\delta)^{\varepsilon}_{N\sigma} \) for some irreducible component of \( S\text{Re}p(Q/I, \beta^\delta) \) (in type B or C, there is only one component, in which case the superscript \( \varepsilon \) means nothing). Since this space is spanned by Pfaffian semi-invariants by Proposition 3.2, there must be a nonzero determinantal semi-invariant in \( SSI(Q/I, \beta^\delta)^{\varepsilon}_{2N\sigma} \). There is not a unique solution \( \alpha \) for \( 2N\sigma = \sigma_N \alpha \) since \( \sigma(u) \) is not defined, so the variable \( \alpha(u) \) is not yet determined, and the variables \( \alpha(\tau(X_j)) \)
are \( \alpha(u) \) plus some constant. When we calculate \( \langle \alpha, \beta^\delta \rangle_I \), the coefficient of \( \alpha(u) \) is \( 2n + \delta \) (independent of what \( r \) is), so using the fact that \( \langle \alpha, \beta^\delta \rangle_I = 0 \), we can solve for \( \alpha(u) \) uniquely.

So there is a representation \( V \in \text{Rep}(Q/I, N\alpha) \) such that \( \tilde{c}^V \) is nonzero on \( \text{SRep}(Q/I, \beta^\delta) \). This implies that \( \tilde{c}^V \) is nonzero on \( \text{Rep}(Q/I, \beta^\delta) \), so we can conclude that \( \text{pdim} V \leq 1 \) by Theorem 3.3. Since \( \text{pdim} V \leq 1 \), Remark 3.5 gives

\[
0 \geq -\sigma^\circ(\tau(x^j_n)) = \langle \alpha, \varepsilon_{\tau(x^j_n)} \rangle_I = \alpha(\tau(x^j_n)) - \alpha(u) + \alpha(x^j_n),
\]

so \( \alpha(u) \geq \alpha(x^j_n) + \alpha(\tau(x^j_n)) \) for \( j = 1, \ldots, r \). Since \( \text{pdim} V \leq 1 \) and \( \tilde{c}^V \neq 0 \), we have that \( \text{Ext}^1(V, -) \) vanishes generically on \( \text{SRep}(Q/I, \beta^\delta) \). Let \( W \) be a general element of \( \text{SRep}(Q/I, \beta^\delta) \). This implies that \( \langle N\alpha, \beta' \rangle_I \geq 0 \) for all dimension vectors \( \beta' \) of factor modules of \( W \). To see this, let \( W' \subseteq W \) be a submodule of dimension \( \beta^\delta - \beta' \). Since \( \text{pdim} V \leq 1 \), we have \( \text{Ext}^2(V, W') = 0 \), so \( \text{Ext}^1(V, W/W') = 0 \) since \( \text{Ext}^1(V, W) = 0 \). This gives \( \langle N\alpha, \beta' \rangle_I = \text{idim}(V, W/W') \geq 0 \), which is the desired inequality.

So \( \langle \alpha, \beta' \rangle_I \geq 0 \) for all dimension vectors \( \beta' \) of factor modules of \( W \). By Proposition 4.8, \( \text{idim} W \leq 1 \) and \( \text{pdim} W \leq 1 \), and the same is true for a general representation of dimension \( \alpha \) since we have shown that \( \alpha(u) \geq \alpha(x^j_n) + \alpha(\tau(x^j_n)) \) for \( j = 1, \ldots, r \). Also, if \( \gamma \) is the generic rank of a map between a general representation of dimension \( \alpha \) and \( W \), then \( \gamma(u) \geq \gamma(x^j_n) + \gamma(\tau(x^j_n)) \) for \( j = 1, \ldots, r \): the image \( U \) of such a generic map satisfies \( \text{idim} U \leq 1 \) (being a quotient of a module with injective dimension at most 1) and \( \text{pdim} U \leq 1 \) (being a submodule of \( W \)), so \( \dim \text{Gr}(\gamma, W) = \langle \gamma, \beta - \gamma \rangle_I \) by Proposition 3.15. So the hypotheses of Lemma 3.16 are satisfied and Theorem 3.19 gives \( V' \in \text{Rep}(Q/I, \alpha) \) such that \( \text{Ext}^1_{Q/I}(V', W) = 0 \). Hence \( \text{Ext}^1_{Q/I}(V', -) \) vanishes generically on \( \text{SRep}(Q/I, \beta^\delta) \). Finally, this means that

\[
(V_{2^a} \otimes \cdots \otimes V_{2^r})^G = \text{SSI}(Q/I, \beta^\delta)_{2^a} \neq 0.
\]

**Example 4.12.** Let \( n = 2 \) and \( r = 3 \) so that

\[
\begin{array}{ccc}
1 & 2 & 2 \\
\beta^\delta = 1 & 2 & 4 + \delta \\
2 & 2 & 1
\end{array}
\]

The product \( V_{(2)} \otimes V_{(2)} \otimes V_{(4)} \) has a \( G \)-invariant, and from Lemma 4.10, this gives a symmetric weight \( \sigma \). The corresponding non-symmetric weight is

\[
\begin{array}{cccc}
1 & 0 & 0 & -1 \\
\sigma^\circ = 1 & 0 & ? & 0 & -1 \\
2 & 0 & 0 & -2
\end{array}
\]

Note that we have to halve the weights when we think of \( \sigma \) as a non-symmetric weight \( \sigma^\circ \). The question mark means that there is nothing assigned to the symmetric weight at that vertex, and hence the non-symmetric value cannot be determined yet. Using the fact that \( \langle \alpha, \varepsilon_x \rangle_I = \sigma^\circ_x \) gives us that

\[
\begin{array}{cccc}
1 & 1 & a - 1 & a - 2 \\
\alpha = 1 & 1 & a & a - 1 & a - 2 \\
2 & 2 & a - 2 & a - 4
\end{array}
\]

for some value of \( a \). Now \( 0 = \langle \alpha, \beta \rangle_I = (a - 4)(4 + \delta) \) means that \( a = 4 \).
The product $V(1) \otimes V(1) \otimes V(2)$ also has a $G$-invariant, but the corresponding symmetric weight $\sigma'$ is not divisible by 2, which means that we cannot express $\sigma'$ as a non-symmetric weight. This means that the $\sigma'$-weight space is spanned by Pfaffian semi-invariants but does not contain any determinantal semi-invariants.

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References