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Localization in Algebraic K-Theory

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INTRODUCTION

Throughout this paper the following hypothesis will be assumed.

MAIN HYPOTHESIS. *A and B are regular rings, R is a domain of Krull dimension $d(R) \leqslant d < \infty$, $R \to A$ and $R \to B$ are monic ring homomorphisms whose images are contained in the centers of A and B respectively, and $h: A \to B$ is an epic ring homomorphism with kernel I. Furthermore, the diagram* $\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \nwarrow R \nearrow & \end{array}$ *is commutative*

The purpose of this paper is to show:

(i) The map h may induce isomorphisms $k_*(A) \overset{\approx}{\to} k_*(B)$ but fail to do so after localization at a multiplicative set in R.

(ii) If $k_*(A_{\bar{\rho}}) \overset{\approx}{\to} k_*(B_{\bar{\rho}})$ is an isomorphism for each prime ideal $\rho \subset R$, then $k_*(A) \overset{\approx}{\to} k_*(B)$ is an isomorphism.

The theorem (ii) above is proved using a Mayer–Vietoris sequence due to Gersten. Gersten also has a proof of this theorem using spectral sequences. The authors express their appreciation to R. Swan for his helpful conversations.

NOTATION AND BACKGROUND

If A is a ring, let $k_*(A)$ represent the Algebraic K-Theory of Karoubi and Villamayor (see [6] and [4]). Thus $k_0(A)$ is the classical $K_0(A)$ (see [1], [7], or [9]) and whenever A is regular, $k_n(A) = k_0(\Omega^n A)$. It is known (see [8]) that for regular A, $k_n(A) \approx K_n(A)$, where $K_n(A)$ represents the theory of Quillen. However, it is necessary to consider the K-theory of the ideal I, and k_* is defined for rings without unity. The phrase "multiplicative set of R"

means a multiplicative set which contains 1 but does not contain 0. If $S \subset R$ is a multiplicative set, then I_S denotes the ring I localized at S. If $0 \neq f \in R$, then I_f denotes I_S where S is the multiplicative set $\{1, f, f^2, \ldots\}$. Several known properties of rings and Algebraic K-Theory will be needed, and they are listed below.

Property 1. The functor k_n commutes with direct limits. Suppose $0 \neq x \in k_n(I)$. Consider all multiplicative sets $T \subset R$ such that the image of x under $k_n(I) \to k_n(I_T)$ is nonzero. Let $\{S_a\}_{a \in \alpha}$ be a maximal monotonic tower of such T, and $S = \bigcup_{a \in \alpha} S_a$. Then limit $(k_n(I_{S_a})) = k_n(I_S)$. It follows that the image of x under $k_n(I) \to k_n(I_S)$ is nonzero, but is zero for any larger multiplicative set.

As another example, suppose $S \subset R$ is a multiplicative set and $0 \neq x \in k_n(I)$ maps to zero under $k_n(I) \to k_n(I_S)$. Then there exists an $s \in S$ such that x maps to zero under $k_n(I) \to k_n(I_s)$.

Property 2. I is called K-regular provided $k_0(I) \xrightarrow{\approx} k_0(I[X])$ is an isomorphism for any set X of variables. If I is K-regular, then the loop rings $\Omega^n I$ are also K-regular. The regular rings A & B are K-regular, and from the short exact sequence of rings $0 \to I \to A \to B \to 0$ it follows that I is K-regular. Furthermore, for any multiplicative set $S \subset R$, the rings A_S, B_S, and I_S are K-regular.

Property 3. Suppose $f, g \in (R - 0)$ satisfy $(f, g) = fR + gR = R$. Then by [2], there is an exact sequence $\to k_{n+1}(I_{fg}) \to^{\partial} k_n(I) \to k_n(I_f) \oplus k_n(I_g) \to k_n(I_{fg}) \to$. Furthermore, if $0 \neq v \in R$ satisfies $(f, v) = R$, then $(f, gv) = R$ and the diagram

$$\begin{array}{ccccccc}
\longrightarrow k_{n+1}(I_{fgv}) & \xrightarrow{\partial} & k_n(I) & \longrightarrow & k_n(I_f) \oplus k_n(I_{gv}) & \longrightarrow \\
\uparrow & & \| & & \uparrow & \\
\longrightarrow k_{n+1}(I_{fg}) & \longrightarrow & k_n(I) & \longrightarrow & k_n(I_f) \oplus k_n(I_g) & \longrightarrow
\end{array}$$

is commutative.

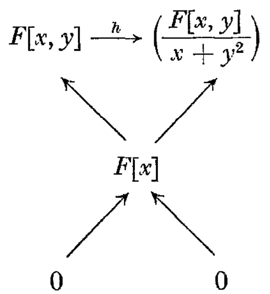
Property 4. Assume $S \subset R$ is a multiplicative set. Then the Krull dimension $d(R_S) \leqslant d(R)$. If $J(R)$ denotes the Jacobson radical of R and $S \cap J(R) \neq \phi$, then $d(R_S) < d(R)$. Assume (as always) the Main Hypothesis. Then

$$\begin{array}{ccc}
A_S & \longrightarrow & B_S \\
& \nwarrow \quad \nearrow & \\
& R_S &
\end{array}$$

also satisfies the Main Hypothesis.

An Example

This is the example promised in the Introduction. It exhibits an isomorphism in K-Theory which fails to be an isomorphism after localization. Let F be a field and $A = F[x, y]$, $R = F[x]$, $I = (x + y^2)F[x, y]$. Now $R[x] \cap I = 0$, so we have the following commutative diagram:



Part 1. For each $n \geqslant 0$, $h_* k_n(F[x, y]) \overset{\approx}{\to} k_n(F[x, y]/x + y^2)$ is an isomorphism. Let $\bar{x} = x + y^2$ and $\bar{y} = y$. Then $F[x, y]$ is also a polynomial ring over F with variables $\bar{x}$, $\bar{y}$, and h is merely the natural map $F[\bar{x}, \bar{y}] \to F[\bar{y}]$, which induces an isomorphism in K-theory.

Part 2. If $S = F[x] - 0$, then the map $k_1[(F[x, y])_S] \to k_1[(F[x, y]/x + y^2)_S]$ is not an isomorphism because it does not honor units. The ring $(F[x, y])_S$ is just $(F(x))[y]$ and its units are the nonzero elements of $F(x)$. The ring $(F[x, y]/x + y^2)_S$ is $((F(x))[y]/x + y^2)$, which is a field containing units outside of $F(x)$.

The Main Theorem

Throughout this section, the main hypothesis is assumed.

Lemma. *Suppose $n \geqslant 0$ and $k_i(I_{\bar{p}}) = 0$ for each prime ideal $\rho \subset R$ and each $i = n, n + 1, \ldots, n + d$. Then $k_n(I) = 0$. Furthermore, if $S \subset R$ is a multiplicative set, $k_n(I_S) = 0$.*

Proof. It is noted in Property 4 that the Main Hypothesis is satisfied by $I_S \to A_S \to B_S$. Note also that $k_i((I_S)_{\bar{p}}) = 0$ for each prime ideal ρ of R_S and each $i = n, n + 1, \ldots, n + d$. Thus the last statement of the lemma follows from the first part of the lemma, which we now consider.

If $d(R) = 0$, then R is a field and the hypothesis implies trivially that $k_n(I) = 0$. Suppose inductively that $d(R) \leqslant d > 0$ and that the lemma holds for Krull dimensions smaller than d.

Suppose $0 \neq x \in k_n(I)$. From Property 1 it follows that there exists a multiplicative set $S \subset R$ which is maximal with respect to the property that the image of x under $k_n(I) \to k_n(I_S)$ is nonzero. The Main Hypothesis and the hypothesis of this lemma are satisfied by the rings A_S, B_S, I_S, and R_S. Instead of considering the ring I_S, we change notation by replacing I_S with I, R_S with R, etc. This simply means that the following condition holds: There exists a nonzero element $x \in k_n(I)$ such that, if $0 \neq f \in R$ is a nonunit, the image of x under $k_n(I) \to k_n(I_f)$ is zero.

If R is a local ring with maximal ideal ρ, then $R_{\bar\rho} = R$ and the hypothesis of the theorem implies that $k_n(R) = 0$. Therefore R is not local, that is, it is false that "the sum of nonunits is a nonunit." Thus there exists nonunits f, $g \in (R - 0)$ such that $(f, g) = R$. In the sequence $k_{n+1}(I_{fg}) \to k_n(I) \to k_n(I_f) \oplus k_n(I_g)$, x lifts to an element $0 \neq x_1 \in k_{n+1}(I_{fg})$. Consider the multiplicative set $S = \{1 + fr : r \in R\}$. Now f is in the Jacobson radical of R_S, and if S_1 is any multiplicative set containing S and f, then $d(R_{S_1}) < d(R)$. By induction, $k_{n+1}((I_{fg})_S) = 0$. Thus from the last statement of Property 1, there exists $v = 1 + fr$ such that the image of x_1 under $k_{n+1}(I_{fg}) \to k_{n+1}(I_{fgv})$ is zero. Now from the commutative diagram

$$\begin{array}{ccc} k_{n+1}(I_{fgv}) & \longrightarrow & k_n(I) \\ \uparrow & & \| \\ k_{n+1}(I_{fg}) & \longrightarrow & k_n(I) \end{array}$$

it follows that $0 = x \in k_n(I)$. Q.E.D.

THEOREM. *Suppose that for each prime ideal $\rho \subset R$, $h_*: k_i(A_{\bar\rho}) \to k_i(B_{\bar\rho})$ is an isomorphism for $i = n, n+1, \ldots, n+d$, and onto for $i = n+d+1$. Then $h_*: k_n(A) \overset{\approx}{\to} k_n(B)$ is an isomorphism. Furthermore, if $S \subset R$ is a multiplicative set, then $h_*: k_n(A_S) \overset{\approx}{\to} k_n(B_S)$ is an isomorphism.*

Proof. The last statement of the theorem follows immediately from the first part of the theorem, just as in the proof of the preceeding lemma.

The long exact sequence $\to k_{i+1}(B_{\bar\rho}) \to k_i(I_{\bar\rho}) \to k_i(A_{\bar\rho}) \to$, together with the hypothesis, implies that $k_i(I_{\bar\rho}) = 0$ for $i = n, n+1, \ldots, n+d$, and any prime ideal $\rho \subset R$. Therefore, by the preceeding lemma, $k_n(I_S) = 0$ for any multiplicative set $S \subset R$. Thus $h_*: k_n(A) \to k_n(B)$ is monic, and it remains to show that it is epic.

Suppose $x \in k_n(B)$ is not in the image of h_*. Let $S \subset R$ be a multiplicative set maximal with respect to the property that x_S is not in the image of $k_n(A_S) \to k_n(B_S)$. As in the proof of the lemma, we change notation and now A_S is called A, B_S is called B, etc. This means that $x \in k_n(B)$ is not in the image of $k_n(A) \to k_n(B)$, but if $0 \neq f \in R$ is a nonunit, x_f is in the image of $k_n(A_f) \to$

$k_n(B_f)$. If R is local, the conclusion of the theorem is immediate from the hypothesis, so assume R is not local. As in the proof of the lemma, there exists nonunits $f, g \in (R - 0)$ such that $(f, g) = R$. Consider

$$\begin{array}{ccccccc}
0 & & & & & & \\
\uparrow & & & & & & \\
k_{n+1}(B_{fg}) & \longrightarrow & k_n(B) & \longrightarrow & k_n(B_f) \oplus k_n(B_g) & \longrightarrow & k_n(B_{fg}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
k_{n+1}(A_{fg}) & \longrightarrow & k_n(A) & \longrightarrow & k_n(A_f) \oplus k_n(A_g) & \longrightarrow & k_n(A_{fg}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
\end{array}$$

Since $k_n(I_S) = 0$ for any multiplicative set $S \subset R$, it follows that the left vertical arrow is epic and the three remaining are monic. The result now follows from diagram chasing. The image of x in $k_n(B_f) \oplus k_n(B_g)$ comes from some $y \in k_n(A_f) \oplus k_n(B_g)$. This y lifts to $y_1 \in k_n(A)$ and y_1 maps to some element $x_1 \in k_n(B)$. The element $x - x_1$ lifts to $x_2 \in k_{n+1}(B_{fg})$, which comes from some $y_2 \in k_{n+1}(A_{fg})$. Denote the image of y_2 in $k_n(A)$ by y_3. Then $(y_1 + y_3) \in k_n(A)$ maps to $x_1 + (x - x_1) = x \in k_n(B)$. Q.E.D.

References

1. H. Bass, "Algebraic K-Theory," Benjamin, New York, 1968.
2. S. Gersten, A Mayer–Vietoris sequence in the K-theory of localizations, to appear.
3. S. Gersten, Homotopy theory of rings, *J. Algebra* **19** (1971), 396–415.
4. S. Gersten, On Mayer–Vietoris functors and algebraic K-theory, *J. Algebra* **18** (1971), 51–88.
5. I. Kaplansky, "Commutative Rings," Allyn and Bacon, New York, 1970.
6. M. Karoubi and Villamayor, Foncteurs K^n en algèbre et en topologie, *C. R. Acad. Sci. Paris* **269** (1969), 416–419.
7. J. Milnor, "Algebraic K-Theory," Annals of Math Studies 72, Princeton University Press, Princeton, NJ, 1971.
8. D. Quillen, "Higher K-Theory for Categories With Exact Sequences," New Developments in Topology, Oxford, June 1972.
9. R. Swan, "Algebraic K-Theory," Springer Lecture Notes 76, Springer-Verlag, Berlin/New York, 1968.