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Original Article

Generalized rough sets based on neighborhood systems and topological spaces

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Abstract Rough sets theory is an important method for dealing with uncertainty, fuzziness and undefined objects. In this paper, we introduce a new approach for generalized rough sets based on the neighborhood systems induced by an arbitrary binary relation. Four pairs of the dual approximation operators are generated from the core of neighborhood systems. Relationship among different approximation operators are presented. We generate different topological spaces by using the core of these neighborhood systems. Relationship among different generated topologies are discussed.

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1. Introduction

There are many mathematical tools to deal with inexact or uncertain knowledge in information systems such probability theory, fuzzy sets [1] and rough sets [2]. Rough sets was proposed by Pawlak [3] as an useful tool to deal with uncertainty and incomplete information. Since then rough sets and its applications have attracted the interest of researchers in many fields [4–17]. The indiscernibility relation is the starting point of Pawlak rough set which was first described by equivalence relation. However, the requirement of equivalence relation such as the indiscernibility impose restrictions and limitations in many applications. In the light of this, equivalence relation has been extended to some other relations such as similarity relation [18], tolerance relation [19], fuzzy relations [20], arbitrary relation [17,21–23] and coverings of the universal sets [24–31].

Topology is regarded as an important and significant branch of mathematics. In recent years many researchers have used topological approaches in the study of rough sets and its applications. The combination of topological spaces and rough sets and the properties of topological rough spaces are discussed by Wu et al. [32]. Lin [12,13,33] used neighborhood systems and topological concept in the study of approximations. Also, neighborhood systems can be induced by the binary relations. The equivalence class of each element in the equivalence relation

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can be viewed as a neighborhood of this element [34,35]. Yao [23,36] introduced the successor elements of any element in an arbitrary binary relation as its right neighborhood. A concept of neighborhood assignment of general topology is considered by Hung [8]. Zuoming et al. [30] used the same concept, is called "core of neighborhoods", and defined two classes of new rough sets based on neighborhood systems in terms of cores.

In this paper, we introduce a new approach for generalization rough sets based on an arbitrary binary relation via the concept of the core of neighborhoods. Four classes of new rough sets are defined. The properties of new rough sets are established and compared with the properties of other approaches. We discuss the relationship among the four approximation. We claim that our approach is an extension of the classical rough sets. We generate four different topologies in terms of cores. Relationship among four different topologies are discussed. Our paper is considered an important evidence for the relationship between topology and rough set theory.

2. Preliminaries

Definition 2.1. [2] Let U be a non empty set, is called the universe of discourse, and R be an equivalence relation on U. Then, the pair $K = (U, R)$ is called an approximation space. For any subset $X \subseteq U$, $\underline{\Re}(X)$, $\overline{\Re}(X)$ are called the lower and upper approximations, respectively, and are defined as follows:

$$\underline{\Re}(X) = \{x \in U : [x]_R \subseteq X\}, \overline{\Re}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$$

where $[x]_R$ is the equivalence class of x with respect to R.

Proposition 2.1. [2] Let $K = (U, \Re)$ be an approximation space. Then, the following properties hold, for $X, Y \subseteq U$:

(1L) $\underline{\Re}(U) = U$;
(1H) $\overline{\Re}(U) = U$;
(2L) $\underline{\Re}(\emptyset) = \emptyset$;
(2H) $\overline{\Re}(\emptyset) = \emptyset$;
(3L) $\underline{\Re}(X) \subseteq X$;.
(3H) $X \subseteq \overline{\Re}(X)$.
(4L) $\underline{\Re}(X \cap Y) = \underline{\Re}(X) \cap \underline{\Re}(Y)$;
(4H) $\overline{\Re}(X \cup Y) = \overline{\Re}(X) \cup \overline{\Re}(Y)$;
(5) $\underline{\Re}(-X) = -\overline{\Re}(X)$, where $(-X)$ is the complement of X;
(6L) $\underline{\Re}(\underline{\Re}(X)) = \underline{\Re}(X)$;
(6H) $\overline{\Re}(\overline{\Re}(X)) = \overline{\Re}(X)$;
(7L) $X \subseteq Y \Rightarrow \underline{\Re}(X) \subseteq \underline{\Re}(Y)$;
(7H) $X \subseteq Y \Rightarrow \overline{\Re}(X) \subseteq \overline{\Re}(Y)$;
(8L) $\underline{\Re}(-\underline{\Re}(X)) = -\underline{\Re}(X)$;
(8H) $\overline{\Re}(-\overline{\Re}(X)) = -\overline{\Re}(X)$;
(9L) $\underline{\Re}(X) \cup \underline{\Re}(Y) \subseteq \underline{\Re}(X \cup Y)$;
(9H) $\overline{\Re}(X \cap Y) \subseteq \overline{\Re}(X) \cap \overline{\Re}(Y)$;

Definition 2.2. [36] Let R be a binary relation on the universe U and $x, y \in U$. If $(x, y) \in R$, then we say that y is related to x by R and the class $RN(x) = \{y \in U : xRy\}(LN(x) = \{y \in U : yRx\})$ is called the right neighbored (the left neighbored) of x induced by R, respectively.

Definition 2.3. [8,30] Let R be a binary relation on the universe U and $x, y \in U$. Then, the set $\{y \in U : N(y) = N(x)\}$ is called the core of neighborhood of x induced by R and is denoted by $CN(x)$.

Definition 2.4. [37] Let U be a non empty set, τ be a family of subsets of U and the following properties hold:

(i) $U, \emptyset \in \tau$;
(ii) τ is closed under an arbitrary union;
(iii) τ is closed under finite intersection.

Then, τ is called a topology on U and the pair (U, τ) is called a topological space. The elements of U are called points of the space. The subsets of U belonging to τ are called open sets and the complement of the open subsets are called closed sets.

3. Generalized rough sets based on neighborhood systems

In this section, we introduce a study of rough sets based on the core of neighborhood systems induced by an arbitrary binary relation. We define four different pairs of dual approximation operators. Also, we compare between our approach and some others approaches.

Definition 3.1. Let U be a non empty set, R be an arbitrary binary relation on U. Then, we can define four types of the core of neighborhood systems induced by R as follows:

(i) The core of right neighborhood($CN_r(x)$): $CN_r(x) = \{y \in U : RN(x) = RN(y)\}$.
(ii) The core of left neighborhood($CN_l(x)$): $CN_l(x) = \{y \in U : LN(x) = LN(y)\}$.
(iii) The core of union neighborhood($CN_u(x)$): $CN_u(x) = CN_r(x) \cup CN_l(x)$.
(iv) The core of intersection neighborhood($CN_i(x)$): $CN_i(x) = CN_r(x) \cap CN_l(x)$.

Definition 3.2. Let U be a non empty set, R be an arbitrary binary relation on U and $CN_j(x)$ be the core of neighborhood systems where $j \in \{r, l, u, i\}$ and $x \in U$. Then (U, R, CN_j) is called an approximation space based on neighborhood induced by the binary relation R (briefly called CN_j-approximation space).

Remark 3.1. Let (U, R, CN_j) be a CN_j-approximation space. If R is an equivalence relation, then the right and left neighborhoods are identical for each element of U. Therefore, $CN_j(x) = [x]_R$ for all $j \in \{r, l, u, i\}$, where $[x]_R$ is the equivalence class of $x \in U$ induced by R. Consequently, our approach is considered a generalization to Pawlak's approximation space.

Lemma 3.1. *Let (U, R, CN_j) be a CN_j-approximation space. Then:*

(i) *$x \in CN_j(x)$ for all $x \in U$ and $j \in \{r, l, u, i\}$.*
(ii) *if $y \in CN_j(x)$. Then $CN_j(x) = CN_j(y)$, for all $x, y \in U$ and $j \in \{r, l, u, i\}$.*

Proof. The proof is obvious from Definition 3.1. □

Definition 3.3. Let (U, R, CN_j) be a CN_j-approximation space and $X \subseteq U$. For each $j \in \{r, l, u, i\}$ and $x \in U$, we define the CN_j-lower approximation and the CN_j-upper approximation of X respectively, as follows:

(i) $\underline{\aleph}_j(X) = \bigcup\{CN_j(x) : CN_j(x) \subseteq X\}$.
(ii) $\overline{\aleph}_j(X) = \bigcup\{CN_j(x) : CN_j(x) \cap X \neq \emptyset\}$.

Definition 3.4. Let (U, R, CN_j) be a CN_j-approximation space, $X \subseteq U$. Then, the subset X is called CN_j-exact set if $\underline{\aleph}_j(X) = \overline{\aleph}_j(X) = X$ for all $j \in \{r, l, u, i\}$. Otherwise, the subset X is called CN_j-rough.

Definition 3.5. Let (U, R, CN_j) be a CN_j-approximation space and $X \subseteq U$. For all $j \in \{r, l, u, i\}$, the CN_j-boundary region, the CN_j-positive region and the CN_j-negative region can be defined as follows, respectively:

(i) $BND_j(X) = \overline{\aleph}_j(X) - \underline{\aleph}_j(X)$.
(ii) $POS_j(X) = \underline{\aleph}_j(X)$.
(iii) $NEG_j(X) = U - \overline{\aleph}_j(X)$

Definition 3.6. (the accuracy measure)

Let (U, R, CN_j) be a CN_j-approximation space, $X \subseteq U$. Then, CN_j-accuracy of the approximations of the subset X is defined as follows:

$$\delta_j(X) = \frac{|\underline{\aleph}_j(X)|}{|\overline{\aleph}_j(X)|}$$

where $|\overline{\aleph}_j(X)| \neq \emptyset$, $|X|$ is the cardinality of X and for all $j \in \{r, l, u, i\}$

Remark 3.2. From the definition of the accuracy measure, we deduce that:

(i) $0 \leq \delta_j(X) \leq 1, \forall X \subseteq U$.
(ii) If $\delta_j(X) = 1$, then the subset X is CN_j-exact and $BND_j(X) = 0$. Otherwise, X is CN_j-rough set.

Proposition 3.1. Let (U, R, CN_j) be a CN_j-approximation space and $X, Y \subseteq U$. Then, CN_j-lower and upper approximations have the following properties:

(1L) $\underline{\aleph}_j(U) = U$;
(1H) $\overline{\aleph}_j(U) = U$;
(2L) $\underline{\aleph}_j(\emptyset) = \emptyset$;
(2H) $\overline{\aleph}_j(\emptyset) = \emptyset$;
(3L) $\underline{\aleph}_j(X) \subseteq X$;
(3H) $X \subseteq \overline{\aleph}_j(X)$.
(4L) $\underline{\aleph}_j(X \cap Y) = \underline{\aleph}_j(X) \cap \underline{\aleph}_j(Y)$;
(4H) $\overline{\aleph}_j(X \cup Y) = \overline{\aleph}_j(X) \cup \overline{\aleph}_j(Y)$;
(5) $\underline{\aleph}_j(-X) = -\overline{\aleph}_j(X)$, where $(-X)$ is the complement of X;
(6L) $\underline{\aleph}_j(\underline{\aleph}_j(X)) = \underline{\aleph}_j(X)$;
(6H) $\overline{\aleph}_j(\overline{\aleph}_j(X)) = \overline{\aleph}_j(X)$;
(7L) $X \subseteq Y \Rightarrow \underline{\aleph}_j(X) \subseteq \underline{\aleph}_j(Y)$;
(7H) $X \subseteq Y \Rightarrow \overline{\aleph}_j(X) \subseteq \overline{\aleph}_j(Y)$;
(8L) $\underline{\aleph}_j(-\underline{\aleph}_j(X)) = -\underline{\aleph}_j(X)$;
(8H) $\overline{\aleph}_j(-\overline{\aleph}_j(X)) = -\overline{\aleph}_j(X)$;
(9L) $\underline{\aleph}_j(X) \cup \underline{\aleph}_j(Y) \subseteq \underline{\aleph}_j(X \cup Y)$;
(9H) $\overline{\aleph}_j(X \cap Y) \subseteq \overline{\aleph}_j(X) \cap \overline{\aleph}_j(Y)$;

Proof. The proofs of (1*L*), (1*H*), (2*L*), (2*H*), (3*L*), (3*H*), (6*L*), (6*H*), (9*L*) and (9*H*) are obvious. For each $x \in U$, we will prove:

(4L) Let $y \in \underline{\aleph}_j(X \cap Y)$. Then $y \in \bigcup\{CN_j(x) : CN_j(x) \subseteq (X \cap Y)\}$. Therefore, there exists at least CN_{j0} such that $y \in CN_{j0} \subseteq (X \cap Y) \Rightarrow y \in CN_{j0} \subseteq X$ and $y \in CN_{j0} \subseteq Y$. Hence, $y \in \bigcup\{CN_j(x) : CN_j(x) \subseteq X\}$ and $y \in \bigcup\{CN_j(x) : CN_j(x) \subseteq Y\} \Rightarrow y \in (\underline{\aleph}_j(X) \cap \underline{\aleph}_j(Y))$. Thus, $\underline{\aleph}_j(X \cap Y) \subseteq (\underline{\aleph}_j(X) \cap \underline{\aleph}_j(Y))$. Conversely, we can prove $(\underline{\aleph}_j(X) \cap \underline{\aleph}_j(Y)) \subseteq \underline{\aleph}_j(X \cap Y)$. Then, $\underline{\aleph}_j(X \cap Y) = \underline{\aleph}_j(X) \cap \underline{\aleph}_j(Y)$. Similarly, the proof of (4*H*);

(5) Since $-\overline{\aleph}_j(X) = -\bigcup\{CN_j(x) : CN_j(x) \cap X \neq \emptyset\} = -\bigcup\{CN_j(x) : CN_j(x) \cap X \neq \emptyset\} = \bigcup\{CN_j(x) : CN_j(x) \cap X = \emptyset\} = \bigcup\{CN_j(x) : CN_j(x) \subseteq (-X)\} = \underline{\aleph}_j(-X)$.

(7L) Let $y \in \underline{\aleph}_j(X) \Rightarrow y \in \bigcup\{CN_j(x) : CN_j(x) \subseteq (X)\}$. But $X \subseteq Y \Rightarrow y \in \bigcup\{CN_j(x) : CN_j(x) \subseteq (Y)\} \Rightarrow y \in \underline{\aleph}_j(Y)$. Then, $\underline{\aleph}_j(X) \subseteq \underline{\aleph}_j(Y)$. Similarly, the proof of (7*H*).

(8L) Since $\underline{\aleph}_j(-\underline{\aleph}_j(X)) \subseteq -\underline{\aleph}_j(X)$. Conversely, let $y \in (-\underline{\aleph}_j(X)) \Rightarrow y \in (-\bigcup\{CN_j(x) : CN_j(x) \subseteq (X)\}) \Rightarrow y \in (\bigcup\{CN_j(x) : CN_j(x) \cap X = \emptyset\}) \Rightarrow y \in (\bigcup\{CN_j(x) : CN_j(x) \cap \underline{\aleph}_j(X) = \emptyset\}) \Rightarrow y \in (\bigcup\{CN_j(x) : CN_j(x) \subseteq -\underline{\aleph}_j(X)\})$. This implies that, $y \in \underline{\aleph}_j(-\underline{\aleph}_j(X))$. Hence $-\underline{\aleph}_j(X) \subseteq \underline{\aleph}_j(-\underline{\aleph}_j(X))$. Similarly, the proof of (8*H*).

□

Table 1 Comparison among different approaches's properties of rough sets.

Pawlak's model	*Yao's approach* [36]	*Yu* et al. [30]	*Our approach*
1*L*	×	×	×
1*H*		×	×
2*L*		×	×
2*H*		×	×
3*L*		×	×
3*H*		×	×
4*L*	×		×
4*H*	×	×	×
5	×		×
6*L*		×	×
6*H*			×
7*L*	×	×	×
7*H*	×	×	×
8*L*			×
8*H*			×
9*L*	×	×	×
9*H*	×	×	×

Remark 3.3. We notice from Proposition 3.1 that our approach satisfies the same properties as Pawlak's rough sets model. Since R is an arbitrary relation in our approach. Therefore, we think that our approach is an ideal generalization of rough sets. Although there are many generalizations of rough set theory, but many of them did not satisfy all the properties of rough sets. In Table 1, a comparison between our approach and other approaches of rough sets approximations where × shows that the property hold.

Corollary 3.1. *$\underline{\aleph}_j(X) \cup \underline{\aleph}_j(Y)$ and $\overline{\aleph}_j(X \cap Y)$ are proper subsets of $\underline{\aleph}_j(X \cup Y)$ and $\overline{\aleph}_j(X) \cap \overline{\aleph}_j(Y)$, respectively and the equality does not hold generally. The following example illustrates this corollary.*

Example 3.1. Let $U = \{a, b, c, d\}$ be a non empty set and $R = \{(a, a), (a, c), (b, b), (b, d), (c, c), (c, a), (d, c),\}$ be an arbitrary relation. Then $RN(a) = \{a, c\}, RN(b) = \{b, d\}, RN(c) = \{a, c\}, RN(d) = \{c\}$ and $LN(a) = \{a, c\}, LN(b) = \{b, d\}, LN(c) = \{a, c\}, LN(d) = \{D\}$. Therefore, $CN_r(a) = \{a, c\}, CN_r(b) = \{b\}, CN_r(c) = \{a, c\}, CN_r(d) = \{c\}$, $CN_l(a) = \{a, c\}, CN_l(b) = \{b, d\}, CN_l(c) = \{a, c\}, CN_l(d) = \{b, d\}$. From Table 2, if $X = \{a\}, Y = \{c, d\}$ and $X \cup Y = \{a, c, d\}$. Then $\underline{\aleph}_r(X) \cup \underline{\aleph}_r(Y) = \{d\}$ and $\underline{\aleph}_r(X \cup Y) = \{a, c, d\}$. Hence, $\underline{\aleph}_r(X) \cup \underline{\aleph}_r(Y) \neq \underline{\aleph}_r(X \cup Y)$. Also, if $Z = \{a, b\}$, $E = \{c, d\}$

Table 2 Comparison between Yao's approach and our approach.

	Yao's method [36]		Our method			
The subset	$\underline{\mathfrak{R}}_r(X)$	$\overline{\mathfrak{R}}_r(X)$	$\underline{\aleph}_r(X)$	$\overline{\aleph}_r(X)$	$\underline{\aleph}_u(X)$	$\overline{\aleph}_u(X)$
$\{a\}$	$\emptyset$	$\{a, c\}$	$\emptyset$	$\{a, c\}$	$\emptyset$	$\{a, c\}$
$\{b\}$	$\emptyset$	$\{b\}$	$\{b\}$	$\{b\}$	$\emptyset$	$\{b, d\}$
$\{d\}$	$\emptyset$	$\{b\}$	$\{d\}$	$\{d\}$	$\emptyset$	$\{b, d\}$
$\{a, b\}$	$\emptyset$	U	$\{b\}$	$\{a, b, c\}$	$\emptyset$	U
$\{a, b, c\}$	$\{a, c, d\}$	U	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, c\}$	U
$\{a, b, d\}$	$\{b\}$	$\{a, b, c\}$	$\{b, d\}$	U	$\{b, d\}$	U
$\{a, c, d\}$	$\{a, c, d\}$	U	$\{a, c, d\}$	$\{a, c, d\}$	$\{a, c\}$	U

and $Z \cap E = \emptyset$. Then $\overline{\aleph}_u(Z) \cap \overline{\aleph}_u(E) = U$ and $\overline{\aleph}_u(Z \cap E) = \emptyset$. Hence, $\overline{\aleph}_u(Z) \cap \overline{\aleph}_u(E) \neq \overline{\aleph}_u(Z \cap E)$.

Remark 3.4. We notice from Table 2 that many of subsets become CN_j-exact by using our approach. On the other hand, Yao's method does not satisfy the basic properties of rough sets for some subsets of U.

3.1. Relationships among different types of $\mathrm{CN_j}$*-approximations operators*

In this section, we introduce a comparison among different types of CN_j-approximations operators. Also, a comparison among different types of accuracy of CN_j-approximations is introduced.

Proposition 3.2. *Let (U, R, CN_j) be a CN_j-approximation space and $X \subseteq U$. Then, the following properties hold:*

(i) $\underline{\aleph}_u(X) \subseteq \underline{\aleph}_r(X) \subseteq \underline{\aleph}_i(X)$;
(ii) $\underline{\aleph}_u(X) \subseteq \underline{\aleph}_l(X) \subseteq \underline{\aleph}_i(X)$;
(iii) $\overline{\aleph}_i(X) \subseteq \overline{\aleph}_r(X) \subseteq \overline{\aleph}_u(X)$;
(iv) $\overline{\aleph}_i(X) \subseteq \overline{\aleph}_l(X) \subseteq \overline{\aleph}_u(X)$.

Proof. We will prove parts (*i*) and (*iii*). The proofs of other parts are similar:$\forall x \in U$

(i) Let $y \in \underline{\aleph}_u(X)$. Then $y \in \bigcup\{CN_u(x) : CN_u(x) \subseteq X\}$. But, $CN_u(x) = CN_r(x) \cup CN_l(x) \Rightarrow y \in \bigcup\{CN_r(x) : CN_r(x) \subseteq X\}$. Hence, $y \in \underline{\aleph}_r(X)$. Therefore, $\underline{\aleph}_u(X) \subseteq \underline{\aleph}_r(X)$.
Now, Let $y \in \underline{\aleph}_r(X)$. Then $y \in \bigcup\{CN_r(x) : CN_r(x) \subseteq X\}$. But, $CN_i(x) = CN_r(x) \cap CN_l(x) \Rightarrow y \in \bigcup\{CN_i(x) : CN_i(x) \subseteq X\}$. Hence, $y \in \underline{\aleph}_i(X)$. Therefore, $\underline{\aleph}_r(X) \subseteq \underline{\aleph}_i(X)$.
(iii) Let $y \in \overline{\aleph}_i(X)$. Then $y \in \bigcup\{CN_i(x) : CN_i(x) \cap X \neq \emptyset\}$. But, $CN_i(x) = CN_r(x) \cap CN_l(x) \Rightarrow y \in \bigcup\{CN_r(x) : CN_r(x) \cap X \neq \emptyset\}$. Hence, $y \in \overline{\aleph}_r(X)$. Therefore, $\overline{\aleph}_i(X) \subseteq \underline{\aleph}_r(X)$.
Now, Let $y \in \overline{\aleph}_r(X)$. Then $y \in \bigcup\{CN_r(x) : CN_r(x) \cap X \neq \emptyset\}$. But, $CN_u(x) = CN_r(x) \cup CN_l(x) \Rightarrow y \in \bigcup\{CN_u(x) : CN_u(x) \cap X \neq \emptyset\}$. Hence, $y \in \overline{\aleph}_u(X)$. Therefore, $\overline{\aleph}_r(X) \subseteq \overline{\aleph}_u(X)$.

□

Remark 3.5. The equality relation does not hold in Proposition 3.2., in general. The following example illustrates this remark.

Example 3.2. Continued from Example 3.1, we find that $\underline{\aleph}_u\{a, b, c\} = \{a, c\} \neq \{a, b, c\} = \underline{\aleph}_r\{a, b, c\}$ and $\underline{\aleph}_l\{a, b\} = \emptyset \neq \{b\} = \underline{\aleph}_i\{a, b\}$. Also, we find that, $\overline{\aleph}_r\{b\} = \{b\} \neq \{b, d\} = \overline{\aleph}_u\{b\}$ and $\overline{\aleph}_i\{c, d\} = \{a, c, d\} \neq U = \overline{\aleph}_l\{c, d\}$.

Corollary 3.2. *Let (U, R, CN_j) be a CN_j-approximation space and $X \subseteq U$. Then, the following properties hold:*

(i) $BN_i \subseteq BN_r \subseteq BN_u$
(ii) $BN_i \subseteq BN_l \subseteq BN_u$

Corollary 3.3. *Let (U, R, CN_j) be a CN_j-approximation space and $X \subseteq U$. Then, the following properties hold:*

(i) $\delta_u(X) \leq \delta_r(X) \leq \delta_i(X)$
(ii) $\delta_u(X) \leq \delta_l(X) \leq \delta_i(X)$

Proposition 3.3. *Let (U, R, CN_j) be a CN_j-approximation space and $X \subseteq U$. Then, the following properties hold:*

(i) *If X is CN_u-exact. Then X is CN_r-exact, which implies that X is CN_i-exact.*
(ii) *If X is CN_u-exact. Then X is CN_l-exact, which implies that X is CN_i-exact.*

Proof. We will prove part (*i*). The proof of part (*ii*) is similar:

(i) By Corollary 3.2, let X be CN_u-exact. Then $BN_u = \emptyset \Rightarrow BN_r = \emptyset$. Hence, X is CN_r-exact. Now, X is CN_r-exact. Then, $BN_r = \emptyset \Rightarrow BN_i = \emptyset$. Therefore, X is CN_i-exact

□

Remark 3.6. From Example 3.1, we can see that the converse of Corollary 3.2 and Corollary 3.3 are not true generally. Also, $\underline{\aleph}_i(X)$ and $\overline{\aleph}_i(X)$ are the more accurate approximation operators in the approximation space (U, R, CN_j).

4. Topological spaces induced by the core of neighborhoods

Topology is a significant and interesting topic in pure mathematics. There are many methods for generating topological spaces such as interior and closure operators. Topology induced by binary relations has attracted the interest of many researchers. In this section, some types of topologies are generated from the core of neighborhoods which are induced from the binary relation.

Proposition 4.1. *Let (U, R, CN_j) be a CN_j-approximation space. Then the families of topologies which can be generated from the core of neighborhoods induced by the binary relation R can be defined as: $\forall j \in \{r, l, u, i\}$*

$$\tau_j = \{G \subseteq U : CN_j(x) \subseteq G, x \in G\}$$

Proof.

(i) $U, \emptyset \in \tau_j$, obviously.
(ii) Let $G_1, G_2, G_3, ..., G_i, ... \in \tau_j$, $i \in I$ and $x \in \bigcup G_i$. Then, there exists at least G_{i0} such that $x \in G_{i0} \in \bigcup G_i$ and $G_{i0} \in \tau$. This implies that, $CN_j(x) \subseteq G_{i0}$. Therefore, $CN_j(x) \subseteq \bigcup G_i$ and $\bigcup G_i \in \tau_j$.
(iii) Let $G_1, G_2 \in \tau_j$ and $x \in G_1 \cap G_2$. Then, $x \in G_1$, $x \in G_2 \Rightarrow CN_j(x) \subseteq G_1$ and $CN_j(x) \subseteq G_2$. Therefore, $CN_j(x) \subseteq G_1 \cap G_2$. Hence, $G_1 \cap G_2 \in \tau_j$

□

Example 4.1. Let $U = \{a, b, c, d\}$ be a non empty set and $R = \{(a, a), (a, b), (b, b), (b, a), (c, c), (c, d), (d, b), \}$

be an arbitrary relation. Then, $RN(a) = \{a, b\}, RN(b) = \{a, b\}, RN(c) = \{c, d\}, RN(d) = \{b\}$ and $LN(a) = \{a, b\}, LN(b) = \{a, b, d\}, LN(c) = \{c\}, LN(d) = \{c\}$. Therefore, $CN_r(a) = \{a, b\}, CN_r(b) = \{a, b\}, CN_r(c) = \{c\}, CN_r(d) = \{d\}$, $CN_l(a) = \{a\}, CN_l(b) = \{b\}, CN_l(c) = \{c, d\}, CN_l(d) = \{c, d\}$, $CN_u(a) = \{a, b\}, CN_u(b) = \{a, b\}, CN_u(c) = \{c, d\}, CN_u(d) = \{c, d\}$ and $CN_i(a) = \{a\}, CN_i(b) = \{b\}, CN_i(c) = \{c\}, CN_i(d) = \{d\}$. Hence $\tau_r = \{U, \emptyset, \{a, b\}, \{c\}, \{d\}, \{a, b, c\}, \{a, b, d\}, \{c, d\}\}$, $\tau_l = \{U, \emptyset, \{a\}, \{b\}, \{c, d\}, \{a, b\}, \{a, c, d\}, \{b, c, d\}\}$, $\tau_u = \{U, \emptyset, \{a, b\}, \{c, d\}\}$ and $\tau_i = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, , \{b, c, d\}, \{a, c, d\}, \{b, c, d\}\}$

Proposition 4.2. *Let (U, R, CN_j) be a CN_j-approximation space and τ_j are topologies induced by CN_j generated by the binary relation R. Then, the following properties hold:*

(i) $\tau_u \subseteq \tau_r \subseteq \tau_i$;
(ii) $\tau_u \subseteq \tau_l \subseteq \tau_i$;
(iii) $\tau_u \subseteq \tau_i$.

Proof. We will prove parts (i) and (iii). The proof of (ii) parts is similar:

(i) Let $G \in \tau_u$. Then, $CN_u(x) \subseteq G, x \in G \Rightarrow CN_r(x) \subseteq G, x \in G$. Therefore, $G \in \tau_r$ and $\tau_u \subseteq \tau_r$. Also, Let $G \in \tau_r$. Then, $CN_r(x) \subseteq G, x \in G \Rightarrow CN_i(x) \subseteq G, x \in G$. Therefore, $G \in \tau_i$ and $\tau_r \subseteq \tau_i$.
(iii) Let $G \in \tau_u$. Then, $CN_u(x) \subseteq G, x \in G \Rightarrow CN_i(x) \subseteq G, x \in G$. Therefore, $G \in \tau_i$ and $\tau_u \subseteq \tau_i$.

□

Remark 4.1. In Proposition 4.2, the equality relation does not hold generally. From Example 4.1, $\tau_u \neq \tau_r$, $\tau_u \neq \tau_l$ and $\tau_u \neq \tau_i$. Also, $\tau_u \neq \tau_i \neq \tau_l$

Corollary 4.1. *Let (U, R, CN_j) be a CN_j-approximation space and τ_j are topologies induced by CN_j generated by the binary relation R and R be a symmetric relation. Then, $\tau_r = \tau_l = \tau_u = \tau_i$.*

Proof. Let R be a symmetric relation. Then $RN(x) = LN(x), \forall x \in U \Rightarrow CN_r(x) = CN_l(x) = CN_u(x) = CN_i(x)$. Hence and by Proposition 4.1, $\tau_r = \tau_l = \tau_u = \tau_i$. □

5. Conclusion

In this paper, four new types of rough sets are introduced. We generalized Pawlak's rough set using an arbitrary general relation. The concept of core of neighborhoods induced by an arbitrary binary relation are used to define new approximations. We established the properties of new approximation spaces. The relationship among four approximation operators are discussed. Four topological spaces are generated via core of neighborhoods induced from an arbitrary relation. The relationships among the four topologies are established. We think that our approach is an important meeting point between general topology and rough set theory. In future, we will discuss more applications of topological concepts in rough sets theory.

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References

[1] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.
[2] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning About Date, Kluwer Academic Publishers, Boston, 1991.
[3] Z. Pawlak, Rough Sets, Int. J. Inf. Comput. Sci. 11 (1982) 341–356.
[4] G. Cattaneo, Abstract approximation spaces for rough theories, in: L. Polkowski, A. Skoeron (Eds.), Rough Sets in Knowledge Discovery, Methodology and Applications, vol. 1, Physica-Verlag, Heidelberg, 1998, pp. 59–98.
[5] E.K.V. Douwen, W.F. Pfeffer, Some properties of Sorgenfrey line and related spaces, Pac. J. Math. 81 (1979) 371–377.
[6] G. Gruenhage, A survey on D-spaces, Contemp. Math. 533 (2011) 13–28.
[7] T. Herawan, M.M. Deries, Rough set theorey for topological spaces in information systems, in: 2009 Third Asia International Conference on Modeling and Simulation, pp. 107–112.
[8] H. Hung, Symmetric and tufted assignments of neighborhoods and metrization, Topol. Appl. 155 (2008) 2137–2142.
[9] M. Kondo, On the structure of generalized rough sets, Inf. Sci. 176 (2006) 586–800.
[10] E. Lashin, A. Kozae, A. Khadra, T. Medhat, Rough set theory for topological spaces, Int. J. Approx. Reason. 40 (2005) 35–43.
[11] G. Liu, W. Zhu, The algebraic structures of generalized rough set theory, Inf. Sci. 178 (2008) 4105–4113.
[12] T. Lin, Neighborhood systems and relational datebase, in: Proceedings of CSCÆ88, 1988.
[13] T. Lin, Neighborhood systems-application to qualitative fuzzy and rough sets, in: P.P. Wang (Ed.), Advances in Machine Intelligence and Soft Computing, Department of Electrical Engineering, Duke University, Durham, NC, USA, 1997, pp. 132–155.
[14] J. Dai, Q. Xu, Approximations and uncertainty measures in incomplete information systems, Inf. Sci. 198 (2012) 62–80.
[15] J. Jarvinen, On the structure of rough approximations, Fundam. Inform. 53 (2002) 135–153.
[16] G.L. Liu, Y. Sai, A comparison of two types of rough sets induced by coverings.
[17] Y.Y. Yao, Constructive and algebraic methods of theory of rough sets, Inf. Sci. 109 (1998) 21–47.
[18] R. Slowinski, D. Vanderpooten, A generalized definition of rough approximations based on similarity, IEEE Trans. Knowl. Date Eng. 12 (2000) 331–336.
[19] Z. Pawlak, A. Skowron, Rough sets and boolean reasoning, Inf. Sci. 177 (2007) 41–73.
[20] G.L. Liu, Using one axiom to characterize rough set and fuzzy rough set approximations, Inf. Sci. 223 (2013) 285–296.
[21] T. Yang, Q. Li, Reduction about approximation spaces of covering generalized rough sets, Int. J. Approx. Reason. 51 (3) (2010) 335–345.
[22] Y.Y. Yao, On generalizing pawlak approximation operators, Lecture Notes in Artif. Intell. 424 (1998a) 298–307.
[23] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, Inf. Sci. 111 (1998b) 239–259.
[24] W. Zhu, Generalized rough sets based on relations, Inf. Sci. 177 (22) (2007) 4997–5011.
[25] Z. Bonikowski, E. Bryniarski, U. Wybraniec-Skardowska, Extensions and intentions in the rough set theory, Inf. Sci. 107 (1998) 149–167.
[26] E. Bryniarski, A calculus of rough sets of the first order, Bull. Pol. Acad. Sci. 36 (16) (1989) 71–77.
[27] J.A. Pomykala, Approximation operations in approximation space, Bull. Pol. Acad. Sci. 35 (910) (1987) 653–662.
[28] W. Zakowski, Approximations in the space (u, π), Demonstr. Math. 16 (1983) 761–769.
[29] W. Zhu, F.Y. Wang, Reduction and axiomization of covering generalized rough sets, Inf. Sci. 152 (2003) 217–230.
[30] Z.Y. Xiaole, Z. Yun, A study of rough sets based on 1-neighborhood systems, Inf. Sci. 248 (2013) 103–113.

[31] G. Liu, Special types of coverings and axiomatization of rough sets based on partial orders, Knowl. Based Syst. 85 (2015) 316–321. Int. J. Approx. Reason.50(3)(2009), 521–528
[32] Q.E. Wu, T. Wang, Y.X. Huang, J.S. Li, Topology theory on rough sets, IEEE Trans. Syst. Man Cybern. Part B: Cybern. 38 (1) (2008) 68–77.
[33] T. Lin, Q. Liu, K. Huang, W. Chen, Rough sets, neighborhood systems and approximation, in: Z. Ras, M. Zemakova, M. Emrichm (Eds.), Methodologies for Intelligent systems, Proceedings of the Fifth International Symposium on Methodologies of Intelligent Systems, Knoxville, Tennessee, North-Holland, New York,, vol. 2527, 1990, pp. 130–141.
[34] E. Orlowska, Semantics analysis of inductive reasoning, Theor. Comput. Sci. 43 (1986) 81–89.
[35] A. Wasilewska, Conditional knowledge representation systems model for an implementation, Bull. Pol. Acad. Sci.: Math. 37 (1987) 63–69.
[36] Y. Yao, T. Lin, Generalization of rough sets using modal logic, Intell. Autom. Soft Comput. Int. J. 2 (1996) 103–120.
[37] J.L. Kelly, General topology, Graduate Texts in Mathematics, vol. 27, Springer-Verlag, 1955.