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Cozero complemented spaces; when the space of minimal prime ideals of a $C(X)$ is compact

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Abstract

If X is a Tychonoff space, $C(X)$ its ring of real-valued continuous functions, and $f \in C(X)$, then the cozeroset of f is $\operatorname{coz}(f) = \{x \in X\colon f(x) \neq 0\}$. If, for every cozeroset V of X, there is a disjoint cozeroset V' such that $V \cup V'$ is dense in X, then X is said to be *cozero complemented*. It has long been known that X is cozero complemented iff the space $\operatorname{Min} C(X)$ of minimal prime ideals of $C(X)$ (in the hull-kernel or Zariski topology) is compact iff the classical ring of fractions of $C(X)$ is von Neumann regular. While many characterizations of cozero complemented spaces are known, they seem not to be adequate to answer some natural questions about them raised by R. Levy and J. Shapiro in an unpublished preprint. These questions concern the relationship between a space being cozero complemented and certain kinds of subspaces having this property, and between a product of two spaces being cozero complemented and the factor spaces being cozero complemented. Also, some conditions are given that guarantee that a space that is locally cozero complemented has this property globally. In this paper partial answers are given to these questions. Sample results: If X is weakly Lindelöf and dense in T, then X is cozero complemented iff T is cozero complemented; if $X \times Y$ is weakly Lindelöf and cozero complemented, then X and Y are cozero complemented, but if D is an uncountable discrete space, then $\beta D \times \beta D$ is not cozero complemented even though βD is cozero complemented. If X is locally cozero complemented and either weakly Lindelöf or paracompact, then X is cozero complemented.

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1. Introductory remarks

A Tychonoff topological space is called *cozero complemented* if for each cozeroset V of X there is a (not necessarily unique) cozeroset W such that $V \cap W = \emptyset$ and $V \cup W$ is dense in X. This paper is devoted to the study of such spaces.

The motivation for this study arises in part because of the fact that X is cozero complemented if and only if the space $\operatorname{Min} C(X)$ of minimal prime ideals (with the hull-kernel topology) of the ring $C(X)$ (of real valued continuous functions with domain X) is compact. We give a brief discussion of the more general algebraic setting below.

There is a large mathematical literature devoted to the study of the space of proper prime ideals of a commutative ring A endowed with the hull-kernel topology (called the Zariski topology by many commutative algebraists). We will be concerned exclusively with the subspace $\operatorname{Min} A$ of minimal prime ideals of A, and then only if $A = C(X)$. The pioneering papers of this latter sort are [13] and [16]. Many of the results we use or prove are true in more generality, but will be stated only in this special case. In case $A = C(X)$, it is shown in [13] that $\operatorname{Min} C(X)$ is always a 0-dimensional Hausdorff space that is countably compact but not always compact. Moreover, if $\operatorname{Min} C(X)$ is compact, then it is basically disconnected (i.e., the closure of each cozeroset of $\operatorname{Min} C(X)$ is compact). No attempt will be made to survey the many studies of the topology of $\operatorname{Min} C(X)$. Instead we address the question of how to determine from the topological properties of X when X is cozero complemented. There is no loss of generality in confining our study to Tychonoff spaces. The terminology used with rare exception is that of [9,11,13]. and unless the contrary is mentioned explicitly the word “space” will be used to abbreviate “Tychonoff space”, Characterizations have been given of those X such that $\operatorname{Min} C(X)$ is compact in terms of properties of cozerosets of functions in $C(X)$, but their usefulness is limited by the fact that unlike open sets, the family of cozerosets is closed under formation of countable, but not arbitrary, unions.

This paper is inspired in large part by a preprint written by Levy and Shapiro [19] in which they study such spaces, and give some interesting examples. They also raise some questions that will be answered (in part) below. Among these questions are:

(1) If X is cozero complemented, and T is a subspace of X, under what conditions is T cozero complemented?
(2) Let $X = Y \times Z$. What is the relation between X being cozero complemented and Y and Z being cozero complemented?

Next, we make more formal the definitions and terminology used both above and in what follows.

Throughout X will denote a Tychonoff space, $C(X)$ the algebra of real-valued continuous functions on X, and A a commutative ring with identity element 1. If $S \subset A$, we let $S^d = \{a \in A\colon\ aS = \{0\}\}$, (called the *annihilator* of S) and if s is a singleton, then $s^d = \{s\}^d$). It is well known and easily seen that $\{a^d\colon\ a \in C(X)\}$ is closed under countable intersection. Let $T(A) = \{\frac{a}{d}\colon\ a \in A, d$ not a zero divisor$\}$ denote the classical ring of fractions of A. As a special case of results in [12], it is known that for $A = C(X)$, then

$T(A)$ is a von Neumann regular ring if and only if $\operatorname{Min} A$ is compact (if and only if X is cozero complemented).

The lemma that follows is well known and is the major tool for recognizing when a prime ideal is minimal.

1.1. Lemma. *A prime ideal P of A is minimal if and only if for each $p \in P$, there is a $q \notin P$ such that pq is nilpotent. Thus a prime ideal P of $C(X)$ is minimal if and only if $p \in P$ implies $p^d \subsetneq P$. In particular every element of a minimal prime ideal is a zero divisor.*

As usual, if $f \in C(X)$, its *zeroset* $f^{\leftarrow}(0)$ is denoted by $Z(f)$, its *cozeroset* $X \backslash Z(f)$ by $\operatorname{coz}(f)$, $\mathcal{Z}(X) = \{Z(f)\colon f \in C(X)\}$, and $\mathcal{C}oz(X) = \{\operatorname{coz}(f)\colon f \in C(X)\}$.

1.2. Definition. A space X is said to be *cozero complemented* if for each $f \in C(X)$, there is a $g \in C(X)$ such that the union of their cozerosets is dense and intersection of their cozerosets is empty. In this case $\operatorname{coz}(f)$ and $\operatorname{coz}(g)$ are called *complementary* cozerosets, and $\operatorname{cl}_X[\operatorname{coz}(f)] = \operatorname{cl}_X[\operatorname{int}_X Z(g)]$.

It is easy to see that any space in which the closure of a cozeroset is a zeroset must be cozero complemented.

The following equivalences are known, but do not all appear in any one paper, are sometimes disguised by being stated in different terminology, and are often stated in greater generality.

1.3. Theorem. *If X is a Tychonoff space, the following assertions are equivalent*:

(a) $\operatorname{Min} C(X)$ *is compact.*
(b) X *is cozero complemented.*
(c) *For each $f \in C(X)$, there is an $f' \in C(X)$ such that $f^{dd} = (f')^d$.*
(d) *For each $f \in C(X)$, there is an $f' \in C(X)$ such that* $\operatorname{cl}[\operatorname{int} Z(f)] = \operatorname{cl}[\operatorname{coz}(f')]$.
(e) $T[C(X)]$ *is a von Neumann regular ring.*
(f) *For each $f \in C(X)$, there is a nonzero divisor $d \in C(X)$ such that $fd = f^2$.*
(g) *For each $f \in C(X)$, there is an $f' \in C(X)$ such that $ff' = 0$ and $|f| + |f'|$ is a nonzero divisor.*
(h) *If every element of a prime ideal P of $C(X)$ is a zero divisor, then P is minimal.*

As usual, $C^*(X)$ will denote the subring of bounded functions in $C(X)$, and βX denotes the Stone–Čech compactification of X. Recall from [11] that $C^*(X)$ and $C(\beta X)$ are isomorphic. Recall also that a space X is said to be *extremally* (respectively *basically*) *disconnected* if the closure of every open (respectively cozero) set is open. A subspace Y of a space X such that the map $Z \to Z \cap Y$ is a surjection of $\mathcal{Z}(X)$ onto $\mathcal{Z}(Y)$ is said to be *z-embedded* in X.

Parts (a), (b), and (c) of the next result appear in [13], and part (d) is shown in [21]. By [24, 10.7], every cozeroset and every Lindelöf subspace of a space are z-embedded in it.

1.4. Theorem.

(a) $\mathrm{Min}(C(X))$ *is compact if and only if* $\mathrm{Min}(C^*(X))$ *is compact. Thus X is cozero complemented if and only if βX is cozero complemented.*
(b) $\mathrm{Min}\,C(X)$ *is a countably compact zero-dimensional space that need not be compact. Indeed, no point of* $\mathrm{Min}\,C(\beta\omega\backslash\omega)$ *has a compact neighborhood.*
(c) *If* $\mathrm{Min}(C(X))$ *is compact, then* $\mathrm{Min}(C(X))$ *is basically disconnected.*
(d) *If Y is dense and z-embedded in X, then* $\mathrm{Min}(C(Y))$ *and* $\mathrm{Min}(C(X))$ *are homeomorphic, and so Y is cozero complemented if and only if X is cozero complemented. In particular, a dense cozeroset Y of a space X is cozero complemented if and only if X is cozero complemented.*

Next we give some examples of spaces that are cozero complemented as well as some that are not. The following lemma is useful for this purpose. Its last part generalizes Lemma 3.1 of [19].

1.5. Lemma.

(a) *If $V \in \mathcal{C}oz(X)$ and $W \in \mathcal{C}oz(V)$, then $W \in \mathcal{C}oz(X)$.*
(b) *Cozero subspaces of cozero complemented spaces are cozero complemented.*
(c) *A free union of cozero complemented spaces is cozero complemented.*
(d) *If L is a Lindelöf space that contains an uncountable dense set D of isolated points, and if D contains a countable subset C for which $L\backslash D \subset \mathrm{cl}(C)$, then L is not cozero complemented.*

Proof. (a) This is [5, 1.2].

(b) Let $V \in \mathcal{C}oz(X)$ and $W \in \mathcal{C}oz(V)$. By (a), $W \in \mathcal{C}oz(X)$, so by hypothesis, there is a $U \in \mathcal{C}oz(X)$ that is a cozero complement of W in X. As V is open in X, it follows easily that $V \cap U$ is a cozero complement of W in V. The result follows.

(c) This is straightforward.

(d) Because C is a countable union of clopen sets, it is a cozeroset of L. If V were a cozero complement of C in L, then $V = D\backslash C$ since $L\backslash D \subset \mathrm{cl}(C)$. Because cozerosets are F_σ-sets and V is uncountable, there would be an uncountable closed subset A of L contained in V. Then A is Lindelöf since L is. But A is uncountable and discrete, which is a contradiction. So C has no cozero complement, which concludes the proof of (d). $\square$

1.6. Examples of cozero complemented spaces. (a) Recall that a space in which every closed set is a zeroset is said to be *perfectly normal* and that every metrizable space has this property. Clearly, every perfectly normal space is cozero complemented.

(b) Recall from [3] that a space in which every regular closed set is a zeroset is called an *Oz-space*. In this paper, R. Blair showed that a space is an Oz-space if and only if each of its open sets is z-embedded. It follows that every Oz-space is cozero complemented.

(c) By (b) and the remarks preceding Theorem 1.4, every basically disconnected space is cozero complemented. Note also that every extremally disconnected space is both basically disconnected and an Oz-space.

(d) Recall that a space in which every collection of pairwise disjoint open sets is countable is called a *ccc-space* or is said to have *countable cellularity* or is said to satisfy the *countable chain condition*. Any separable space is a ccc-space, but the cube $[0,1]^{\aleph}$ and the generalized Cantor space $\{0,1\}^{\aleph}$ are ccc for any cardinal $\aleph$, and they both fail to be separable if the exponent $\aleph$ exceeds 2^{ω}. In fact, continuous images of dense subspaces of ccc-spaces are ccc. See 2N(6) and 3RST of [22].

If X is a ccc-space and $C \in \mathcal{C}oz(X)$, then by Zorn's lemma, $X \setminus \mathrm{cl}_X C$ contains as a dense subset the union C' of a (necessarily countable) maximal collection of pairwise disjoint cozerosets. Clearly C' is a cozero complement of C, and we conclude that every ccc-space is cozero complemented.

In [14], Hager and Martinez study a class of spaces called *fraction-dense* which could be defined as those spaces such that $\mathrm{Min}\, C(X)$ is compact and extremally disconnected. These authors characterize such spaces as being those such that each regular closed set is the closure of a cozeroset. It follows from this that every fraction-dense space is cozero complemented, and that every Oz-space is fraction-dense.

(e) A subset S of a space X is called a *P-set* (respectively *almost P-set*) of X if, whenever $S \subset Z(f)$ for some $f \in C(X)$, it follows that $S \subset \mathrm{int}_X Z(f)$ (respectively $S \subset \mathrm{cl}_X \mathrm{int}_X Z(f)$). If $p \in X$ and $\{p\}$ is a P-set (respectively almost P-set), then p is called a *P-point* (respectively *almost P-point*). If every point of X is a P-point (respectively almost P-point), then X is called a *P-space* (respectively *almost P-space*). It is well known and easy to see every cozeroset of a P-space is clopen, so every P-space is cozero complemented. Note that an almost P-point of a cozero complemented space X is a P-point because by 1.3(g), M_p is a minimal prime ideal. For X compact, this is noted in [14].

(f) Every ordinal (space) δ (with the interval topology) is cozero complemented. For if not, let δ_0 denote the smallest ordinal that is not cozero complemented. Clearly, δ_0 must be an uncountable limit ordinal. There are two cases.

Suppose first that δ_0 has countable cofinality, and let $(\alpha_n)_{n<\omega}$ be an (increasing) countable cofinal subset of δ_0 such that $\alpha_0 = 0$ and α_n is not a limit ordinal if $n > 0$. Then

$$\delta_0 = \bigcup_{n<\omega} [\alpha_n + 1, \alpha_{n+1}],$$

where we use standard interval notation. Each summand is a clopen subset of δ_0 and hence of the ordinal space α_{n+1}. By the minimality of δ_0, each $\alpha_{n+1} + 1$ is cozero complemented as is its clopen subspace $[\alpha_n + 1, \alpha_{n+1})$. Thus δ_0 is a free union of cozero complemented spaces and hence is cozero complemented by Lemma 1.5(a), (b), which is a contradiction.

Now suppose that δ_0 has uncountable cofinality. Arguing almost exactly as in [11], we see that if $f \in C(\delta_0)$, then there exists $\alpha < \delta_0$ such that f is constant on $[\alpha, \delta_0)$. Thus if $V \in \mathcal{C}oz(\delta_0)$, then there exists $\alpha_V < \delta_0$ such that either $V \subset [0, \alpha_V)$ or else $[\alpha_V, \delta_0) \subset V$. By the minimality of δ_0, the ordinal space α_V is cozero complemented, so $V \cap [0, \alpha_V)$ has a cozero complement U in α_V. One easily checks that either $U \cup [\alpha_V, \delta_0)$ (in the first case) or U (in the second case) is a cozero complement of V in δ_0. Thus δ_0 is again cozero complemented, in contradiction to its choice. The result follows.

1.7. Examples of spaces that are not cozero complemented. (a) As noted in 1.6(e), no space with an almost P-point that is not a P-point can be cozero complemented; e.g., the one-point compactification of an uncountable discrete space is not cozero complemented. It is shown in [10] that if X is locally compact and realcompact, then $\beta X\backslash X$ is an infinite compact almost P-space; such a subspace cannot be cozero complemented because not all of its points are P-points.

(b) The Alexandroff double X of a compact metric space M without isolated points is compact first countable space and a compactification of a discrete space of power 2^{ω} that satisfies the hypotheses of 1.5(d). As a set, X is the union of two copies $M(0)$ and $M(1)$ of M, where each point of $M(1)$ isolated, the relative topology of $M(0)$ is that of M, and a neighborhood of a point of $M(0)$ is the union of the corresponding neighborhood in M and the deleted neighborhood in $M(1)$. Then X becomes a compact (Hausdorff) first countable space in which a copy of a countable dense subspace C of M contained in $M(1)$ is a cozeroset of X that is not complemented. See [23] or [9, 3.1.26] for details about Alexandroff doubles. Hence by 1.5(d), X is an example of a compact first countable space that is not cozero complemented.

(c) Let $\alpha D = D \cup \{p\}$ denote the one-point compactification of an uncountable discrete space D, and let $q \in \beta\omega\backslash\omega$, and consider the space Y obtained from the free union of αD and $\beta\omega$ by attaching p and q and imposing the quotient topology. The point obtained by identifying p and q is an almost P-point of Y that is not a P-point. So Y is not cozero complemented by 1.6(e).

(d) The *Michael line* $M = (\mathbb{R}, \mu)$ is the real line with the topology generated from the usual topology by making each member of the set P of irrational numbers an isolated point. See [9, 5.1.22 and 5.1.32]. Let C denote a countable subset of P that is dense in the usual topology of $\mathbb{R}$. Then $C \in \mathcal{C}oz(M)$, and $Q = \mathbb{R}\backslash P \subset \mathrm{cl}_M C$. Clearly, any cozero complement W of C in M would have to be $M\backslash(C \cup Q)$. As cozerosets are F_σ-sets, it follows that

$$M\backslash(C \cup Q) = \bigcup_{n<\omega} T_n,$$

where each T_n is closed in M. If $Q \cap \mathrm{cl}_{\mathbb{R}} T_n = \emptyset$ for each n, then

$$\big\{\{x\}\colon x \in C \cup Q\big\} \cup \{\mathrm{cl}_{\mathbb{R}} T_n\colon n < \omega\}$$

is a countable family of closed nowhere dense subsets of $\mathbb{R}$ whose union is $\mathbb{R}$, which contradicts the Baire Category Theorem. Hence there is a $k < \omega$ for which $Q \cap \mathrm{cl}_{\mathbb{R}} T_k \neq \emptyset$. But

$$\emptyset \neq Q \cap \mathrm{cl}_{\mathbb{R}} T_k = Q \cap \mathrm{cl}_M T_k = Q \cap T_k,$$

which contradicts the choice of T_k. Thus $M\backslash(C \cup Q) \notin \mathcal{C}oz(M)$ and so M is not cozero complemented. Thus the Michael line is an example of a first countable hereditarily paracompact space that is not cozero complemented.

We close this section by outlining the contents of the rest of this paper.

Section 2 is devoted to studying when a subspace of a cozero complemented space is cozero complemented.

Section 3 is concerned with the relationship between the product of two spaces being cozero complemented and the individual factors being cozero complemented, and to a lesser extent, what kinds of continuous mappings preserve being cozero complemented either directly or inversely. We know only about rather special mappings that do the latter.

In Section 4, spaces such that each of their extensions is cozero complemented are studied. Our nicest result of this kind is that all cozero complemented weakly Lindelöf spaces have this property (see 2.9) while spaces that are not pseudocompact have this property only if they are weakly Lindelöf (see 4.6). Some instructive examples of spaces with cozerosets that fail to be cozero complemented in any extension are also given.

In Section 5, we study when being locally cozero complemented is enough to guarantee that this property holds globally. This is the case for spaces that are either weakly Lindelöf or paracompact, but there is an example of a space that is both locally weakly Lindelöf and locally cozero complemented without being cozero complemented.

While we answer some of the questions posed by Levy and Shapiro, many remain open and we raise some of our own in a brief final Section 6.

2. Subspaces of cozero complemented spaces

In [19] it is asked whether every (a) dense (b) open or (c) dense and open subspace of a cozero complemented space must be cozero complemented.

In this section we continue our study of which kinds of subspaces of cozero complemented spaces must be cozero complemented. In 1.5(b), we showed that cozerosets of cozero complemented spaces are cozero complemented, and in 3.11 below, we answer (c), and hence (a) and (b) in the negative.

If X is z-embedded in T, then the zerosets of X are completely determined by those of T. Because every element of a minimal prime ideal is a zero divisor, information about zerosets of elements of minimal prime ideals of $C(X)$ depends only on zerosets of elements of $\mathcal{Z}(T)$ with nonempty interior. Zerosets of this kind were studied in [15] from which we recall the following definitions.

2.1. Definitions. Suppose X and T are (Tychonoff) spaces

(a) Let $\mathcal{Z}^{\#}(T) = \{\mathrm{cl}(\mathrm{int}\, Z(f))\colon\ f \in C(T)\}$.
(b) A subspace X of a space T such that for each $f \in C(X)$ there is a $g \in C(T)$ such that $\mathrm{cl}_X(\mathrm{int}_X\, Z(f)) = X \cap\ \mathrm{cl}_T(\mathrm{int}_T\, Z(g))$ is said to be *$\mathcal{Z}^{\#}$-embedded in Y*.

Remark. In [15, 3.1], the concept of $\mathcal{Z}^{\#}$-embedding is defined only for dense subspaces, a restriction we do not impose in what follows.

Frequent use will be made of the next well-known lemma which is recorded in [11, 0.12].

2.2. Lemma. *If X is dense in T and V is open in T, then* $\mathrm{cl}_T(V \cap X) = \mathrm{cl}_T\, V$.

Recall that if every open cover of a space X contains a countable subfamily whose union is dense in X, then X is called a weakly *Lindelöf space*. Every Lindelöf space and every ccc-space is weakly Lindelöf, while an uncountable discrete space is not weakly Lindelöf. It is shown in 1.2 of [8] that regular closed subsets and cozerosets of weakly Lindelöf spaces are weakly Lindelöf. The next three lemmas and the theorem that follows them illustrates the importance of $\mathcal{Z}^{\#}$-embedding for determining which subspaces of a cozero complemented space are cozero complemented.

2.3. Lemma. *If X is a subspace of T that is either dense or open, then the following are equivalent*:

(a) *X is $\mathcal{Z}^{\#}$-embedded in T.*
(b) *If $C \in \mathcal{C}oz(X)$, there is a $V \in \mathcal{C}oz(T)$ such that* $\mathrm{cl}_X C = X \cap \mathrm{cl}_T V$.

Proof. (a) implies (b). By (a), there is a $V \in \mathcal{C}oz(T)$ such that

$$\mathrm{cl}_X \mathrm{int}_X(X \backslash C) = X \cap \mathrm{cl}_T \mathrm{int}_T(T \backslash V).$$

Taking complements in X yields:

$$\mathrm{int}_X \mathrm{cl}_X C = X \cap \mathrm{int}_T \mathrm{cl}_T V.$$

Because $C \subset \mathrm{int}_X \mathrm{cl}_X C \subset \mathrm{cl}_X C$, we have:

$$\mathrm{cl}_T C = \mathrm{cl}_T(\mathrm{int}_X \mathrm{cl}_X C) = \mathrm{cl}_T(X \cap \mathrm{int}_T \mathrm{cl}_T V). \tag{i}$$

If X is dense in T, we have

$$\mathrm{cl}_T(X \cap \mathrm{int}_T \mathrm{cl}_T V) = \mathrm{cl}_T\big(\mathrm{int}_T(\mathrm{cl}_T V)\big) = \mathrm{cl}_T V.$$

So (i) becomes

$$\mathrm{cl}_T C = \mathrm{cl}_T V.$$

Intersecting with X yields

$$\mathrm{cl}_X C = X \cap \mathrm{cl}_T V,$$

and (b) holds in this case.

Next assume X is open in T, and intersect each side of (i) with X to obtain:

$$\mathrm{cl}_X C = \mathrm{cl}_X(X \cap \mathrm{int}_T \mathrm{cl}_T V). \tag{ii}$$

Now suppose $p \in \mathrm{cl}_X(X \cap \mathrm{int}_T \mathrm{cl}_T V)$, and let U denote an open neighborhood of p in T. Then

$$(U \cap X) \cap \mathrm{int}_T \mathrm{cl}_T V \neq \emptyset.$$

Because $U \cap X$ is open in T, it follows that $U \cap X \cap V \neq \emptyset$. Thus $U \cap V \neq \emptyset$ and so $p \in X \cap \mathrm{cl}_T V$. Thus:

$$\mathrm{cl}_X(X \cap \mathrm{int}_T \mathrm{cl}_T V) \subset X \cap \mathrm{cl}_T V. \tag{iii}$$

Conversely, suppose $p \in X \cap \mathrm{cl}_T V$, and let U denote an open neighborhood of p in T. Then $U \cap X$ and hence $U \cap X \cap V$ are nonempty. But $X \cap V \subset \mathrm{int}_T \mathrm{cl}_T V$ because $X \cap V$ is open in T, so $(U \cap X) \cap \mathrm{cl}_X(X \cap \mathrm{int}_T \mathrm{cl}_T V) \neq \emptyset$. Thus, $p \in \mathrm{cl}_X(X \cap \mathrm{int}_T \mathrm{cl}_T V)$, so:

$$X \cap \mathrm{cl}_T V \subset \mathrm{cl}_X(X \cap \mathrm{int}_T \mathrm{cl}_T V). \tag{iv}$$

Combining (ii), (iii) and (iv) yields that (b) holds in this case as well.

(b) implies (a). Suppose (b) holds. Taking complements in X yields

$$\mathrm{int}_X(X\backslash C) = X\backslash \mathrm{cl}_X C = X \cap (T\backslash \mathrm{cl}_T V) = X \cap \mathrm{int}_T(X\backslash V).$$

Thus

$$\mathrm{cl}_X \mathrm{int}_X(X\backslash C) = \mathrm{cl}_X\big(X \cap \mathrm{int}_T(T\backslash V)\big). \tag{v}$$

If we assume that X is dense in T, then by 2.2, this latter is equal to

$$X \cap \mathrm{cl}_T \mathrm{int}_T(T\backslash V),$$

so (a) holds in this case.

Next we assume that X is open in T. By (v), we obtain:

$$\mathrm{cl}_X \mathrm{int}_X(X\backslash C) = X \cap \mathrm{cl}_T\big(X \cap \mathrm{int}_T(T\backslash V)\big) \subset X \cap \mathrm{cl}_T \mathrm{int}_T(T\backslash V). \tag{vi}$$

Now assume next that $p \in X \cap \mathrm{cl}_T \mathrm{int}_T(T \setminus V)$ and let $p \in U$, where U is open in T. Then $U \cap X$ is open in T and is a neighborhood of p. Thus $U \cap X \cap \mathrm{int}_T(T\backslash V) \neq \emptyset$, and so $p \in X \cap \mathrm{cl}_T(X \cap \mathrm{int}_T(T\backslash V))$. Thus by (vi), $p \in \mathrm{cl}_X \mathrm{int}_X(X \setminus C)$ and the reverse of the inclusion in (vi) holds.

Combining this with (v) completes the proof that (b) implies (a) if X is open in T. □

2.4. Lemma. *If X is a weakly Lindelöf subspace that is either dense or open in a space T, then X is $\mathcal{Z}^\#$-embedded in T.*

Proof. The proof when X is dense in T appears in 3.7(b) of [15]. If X is open in T and $C \in \mathcal{C}oz(X)$, then C is open in T and hence is the union of a collection $\mathcal{A}$ of cozerosets of T. By 1.2 of [8], since X is weakly Lindelöf, so is its cozeroset C. So there is a countable subcollection of $\mathcal{A}$ whose union V is dense in C. Then $V \in \mathcal{C}oz(T)$ and $\mathrm{cl}_T V = \mathrm{cl}_T C$. Thus $\mathrm{cl}_X C = X \cap \mathrm{cl}_T V$, so by 2.3, X is $\mathcal{Z}^\#$-embedded in T. □

2.5. Lemma. *Suppose X is either dense or open as well as being $\mathcal{Z}^\#$-embedded in a space T. If T is cozero complemented, then so is X.*

Proof. If $C \in \mathcal{C}oz(X)$, then by 2.4 there is a $V \in \mathcal{C}oz(T)$ such that $\mathrm{cl}_X C = X \cap \mathrm{cl}_T V$. By assumption, there is a cozero complement W of V in T. Then $W \cap X \in \mathcal{C}oz(X)$ and because $W \cap \mathrm{cl}_T V = \emptyset$, we have $C \cap (W \cap X) = \emptyset$.

Suppose $p \in X$ and U is a neighborhood of p in T. Because $W \cup V$ is dense in T, it follows that $U \cap (W \cup V) \neq \emptyset$.

Suppose first that X is dense in T. Then

$$U \cap (W \cup V) \cap X \neq \emptyset, \quad \text{so}$$
$$(U \cap X) \cap \big[(W \cap X) \cup (V \cap X)\big] \neq \emptyset.$$

Since $\mathrm{cl}_X C = X \cap \mathrm{cl}_T V$, it follows that

$$(U \cap X) \cap \big[(W \cap X) \cup C\big] \neq \emptyset$$

and so $W \cap X$ is a cozero complement of C in X.

Suppose next that X is open in T. Then $U \cap X$ is open in T, and since $W \cup V$ is dense in T, we conclude again that $(U \cap X) \cap (W \cup V) \neq \emptyset$. The rest of the argument proceeds as above. □

2.6. Theorem. *If $S \cap W$ is a weakly Lindelöf space, where S is a dense subspace and W is an open subspace of a cozero complemented space T, then $S \cap W$ is cozero complemented.*

Proof. Clearly $S \cap W$ is dense in W, so since $S \cap W$ is weakly Lindelöf, so is W. Because W is open in T, it is $\mathcal{Z}^{\#}$-embedded in T by 2.4. Thus W is cozero complemented. But $S \cap W$ is dense in W, so by 2.4, $S \cap W$ is $\mathcal{Z}^{\#}$-embedded in W. Then by 2.5, the conclusion holds. □

2.7. Remarks. (a) Not every weakly Lindelöf subspace of a cozero complemented space need be cozero complemented. For example, the compact extremally disconnected space $\beta\omega$ is cozero complemented by 1.6(c), but by 1.7(b), its compact subspace $\beta\omega \setminus \omega$ is not cozero complemented.

(b) In 3.12 an example will be given of a dense open subspace of a cozero complemented space that is not cozero complemented. Thus, "weakly Lindelöf" cannot be dropped from the hypothesis of 2.6.

The result that follows is proved in 1.3 of [19] and 7.6 of [21] under the stronger hypothesis that X is z-embedded and dense in T. Theorem 2.8 follows also from results in [7] from which can be inferred that if X is $\mathcal{Z}^{\#}$-embedded and dense in T, then $C(X)$ is rigidly embedded (see [7] for a definition) in $C(T)$, and from this latter, it follows that $\operatorname{Min} C(T)$ and $\operatorname{Min} C(X)$ are homeomorphic. Clearly, any space containing a dense Lindelöf subspace is weakly Lindelöf.

2.8. Theorem. *Suppose X is $\mathcal{Z}^{\#}$-embedded and dense in T. Then*:

(a) *X is cozero complemented if and only if T is cozero complemented.*
(b) *If T is cozero complemented, then* $\operatorname{Min} C(T)$ *and* $\operatorname{Min} C(X)$ *are homeomorphic.*

Proof. (a) If T is cozero complemented, then so is X by 1.5. Suppose X is cozero complemented and $C \in \mathcal{C}oz(T)$. Then $C \cap X \in \mathcal{C}oz(X)$, so there is $W \in \mathcal{C}oz(X)$ such that $W \cap (C \cap X) = \emptyset$ and $W \cup (C \cap X)$ is dense in X. By 2.3, there is a $V \in \mathcal{C}oz(T)$ such that $\operatorname{cl}_X W = X \cap \operatorname{cl}_T V$. We will show that V is a cozero complement of C in T.

To see this, suppose first that $V \cap C \neq \emptyset$. Since X is dense in T, $C \cap (V \cap X) \neq \emptyset$, so $C \cap \operatorname{cl}_T (V \cap X) \neq \emptyset$. But by 2.2, $\operatorname{cl}_T(V \cap X) = \operatorname{cl}_T W$, so $C \cap \operatorname{cl}_T W$ and hence $C \cap W$ are nonempty, as is $W \cap (C \cap X)$, contrary to the choice of W. Thus, $V \cap C = \emptyset$.

It remains to show that $V \cup C$ is dense in T. To see this, we will show that $\operatorname{cl}_T(V \cup C)$ contains the dense subspace X of T. With this end in mind, recall first that by 2.2, $\operatorname{cl}_X(C \cap X) = X \cap \operatorname{cl}_T(C \cap X) = X \cap \operatorname{cl}_T C$, so

$$\begin{aligned} X \cap \operatorname{cl}_T(V \cup C) &= (X \cap \operatorname{cl}_T V) \cup (X \cap \operatorname{cl}_T C) \\ &= \operatorname{cl}_X W \cup \operatorname{cl}_X(C \cap X) = \operatorname{cl}_X\bigl[W \cup (C \cap X)\bigr] = X. \end{aligned}$$

Thus T is cozero complemented.

(b) Recall from [13] that for any commutative ring A, the hull-kernel topology on the space $\operatorname{Min} A$ of minimal prime ideal of A has as a base for its closed sets $\{h(a)\colon a \in A\}$, where $h(a) = \{P \in \operatorname{Min} A\colon a \in A\}$, and that $\operatorname{Min} A$ is always a zero-dimensional Hausdorff space. If X is $\mathcal{Z}^{\#}$-embedded densely in T, let $\varphi : C(T) \to C(X)$ be given by $\varphi(f) = f|_X$, and for each $P \in \operatorname{Min} C(T)$, let $\Phi[P] = \{\varphi(f)\colon f \in C(T)\}$. Then $\Phi[P]$ is a prime ideal and since every element of P is a divisor of 0, the same is true in $\Phi[P]$. Since each of these spaces of minimal prime ideals is compact, and since X is dense in T, it follows from 1.3(g) that Φ is a one–one map of $\operatorname{Min} C(T)$ onto $\operatorname{Min} C(X)$. It follows also that for each $f \in C(T)$, $\Phi[h(f)] = h(f|_X)$, so both Φ and its inverse are closed maps. Thus Φ is the desired homeomorphism. □

Combining 2.4 and 2.8, we obtain:

2.9. Corollary. *If X is weakly Lindelöf and dense in T, then*

(a) *X is cozero complemented if and only if T is cozero complemented.*
(b) *If T is cozero complemented, then* $\operatorname{Min} C(T)$ *and* $\operatorname{Min} C(X)$ *are homeomorphic.*

2.10. Remarks.

(a) Clearly any space containing a dense Lindelöf space is weakly Lindelöf. If the converse of the latter held, then 2.9 would say nothing beyond the previously known result for Lindelöf subspaces. The converse does not hold; see, for example, 2.2 of [4] where a first countable ccc-space X is given whose only Lindelöf subspaces are nowhere dense. (Because X is ccc, it is weakly Lindelöf; see 3P(3) of [22].)
(b) Corollary 2.9 relates to 7.6 of [21], where it is shown that if X is dense and z-embedded in T, then X and T have homeomorphic spaces of minimal prime ideals.

3. Products and mappings of cozero complemented spaces

The purpose of this section is to investigate the preservation of cozero complementation under the formation of products and various sorts of continuous images.

3.1. Questions.

(a) If X and Y are cozero complemented, must $X \times Y$ be cozero complemented?
(b) If $X \times Y$ is cozero complemented, must X and Y be cozero complemented?

3.2. Questions. Suppose $f : X \to Y$ is a continuous surjection.

(a) If X is cozero complemented, what conditions on f guarantee that Y is cozero complemented?

(b) If Y is cozero complemented, what conditions on f guarantee that X is cozero complemented?

These questions are related by the projection maps of a product space onto its factors and complicated by fact that cozerosets are not preserved by projection maps. While we have partial answers to all of them, they have no clear pattern. In fact, 3.1(a) has a negative answer even if $X = Y$ is compact. While the answer to 3.1(b) is affirmative in many special cases, the general situation remains open.

We begin with two easy cases. The projection map of $X \times Y$ onto X is denoted by π_X.

3.3. Lemma.

(a) *If $X \times Y$ is cozero complemented and Y has an isolated point, then X is cozero complemented.*
(b) *If Y has a countable dense set D of isolated points, then $X \times Y$ is cozero complemented if and only if X is cozero complemented.*

Proof. (a) follows from 1.5.

(b) The necessity is immediate from (a). Conversely, suppose X is cozero complemented, $V \in \mathcal{C}oz(X \times Y)$, and $d \in D$. Then

$$\pi_X\big[V \cap \big(X \times \{d\}\big)\big] \in \mathcal{C}oz(X).$$

Since X is cozero complemented, this latter set has a cozero complement U_d in X, and it is easy to see that $\bigcup\{U_d \times \{d\}\colon d \in D\}$ is a cozero complement of V in $X \times Y$. □

It is sometimes possible to describe the cozerosets of a product of two spaces in terms of the cozerosets of each factor, and we invoke material from [6] for this purpose. The product of a cozeroset of X and a cozeroset of Y is called a *cozero-rectangle of $X \times Y$*. If these cozerosets are clopen, then the resulting cozero-rectangle is called a *clopen-rectangle*. A countable union of cozero-rectangles. is called a *σ-rectangle*.

The next result is proved in 1.1 and 1.4 of [6].

3.4. Lemma (Blair–Hager). *If $X \times Y$ is a Lindelöf space, then each cozeroset of $X \times Y$ is a σ-rectangle.*

3.5. Definition. For an uncountable discrete space D, let $L = D \cup \{p\}$. Imposing on L the topology generated by the singletons of D and the sets containing p together with a co-countable subset of D yields a regular Lindelöf P-space called *the one-point Lindelöfication of D*.

Clearly L is cozero complemented. (See 1.6(e).)

3.6. Theorem. *Suppose Y is a Lindelöf space and L is the one-point Lindelöfication of an uncountable discrete space D. Then Y is cozero complemented if and only if $L \times Y$ is cozero complemented.*

Proof. It follows from 3.3(a) that if $L \times Y$ is cozero complemented, then so is Y. Conversely suppose Y is cozero complemented. It is well known and easy to see that the product of two Lindelöf spaces, one of which is a P-space, is Lindelöf. So, if $C \in \mathcal{C}oz(L \times Y)$, then by Lemma 3.4, there are, for each $i < \omega$, cozerosets V_i of L and E_i of Y such that $C = \bigcup_{i<\omega} V_i \times E_i$. Because L is a P-space, V_i and $L \backslash V_i$ are clopen. Moreover, V_i is countable or co-countable according as $p \in V_i$ or not. Let $I = \{i < \omega\colon |V_i| \leqslant \omega\}$, let $J = \omega \backslash I$, and let

$$M = \left(\bigcup_{i \in I} V_i\right) \cup \left(L \backslash \bigcap_{i \in J} V_i\right).$$

Then the countable subset M of D is clopen in L and

$$C = \left[C \cap (M \times Y)\right] \cup \left[C \cap \left((L \backslash M) \times Y\right)\right].$$

As M is discrete, by 1.5(c), $M \times Y$ is cozero complemented since Y is. Hence there is a cozero complement S of $C \cap (M \times Y)$ in $M \times Y$. Because $M \times Y$ is clopen in $L \times Y$, $S \in \mathcal{C}oz(L \times Y)$.

If $i \in J$, then

$$(V_i \times E_i) \cap \left((L \backslash M) \times Y\right) = (L \backslash M) \times E_i$$

because $L \backslash M \subset V_i$. So

$$C \cap \left[(L \backslash M) \times Y\right] = \bigcup_{i \in J} (L \backslash M) \times E_i = (L \backslash M) \times \bigcup_{i \in J} E_i.$$

Now $\bigcup_{i \in J} E_i \in \mathcal{C}oz(Y)$, so as Y is cozero complemented, there is a $T \in \mathcal{C}oz(Y)$ such that $T \cup \bigcup_{i \in J} E_i$ is dense in Y, but $T \cap \bigcup_{i \in J} E_i = \emptyset$. Thus,

$$(L \backslash M) \times T \in \mathcal{C}oz(L \times Y).$$

It is now easy to see that $S \cup ((L \backslash M) \times T)$ is a cozero complement of C in $L \times Y$. □

Next, we consider product spaces where one factor is βD for some discrete space D.

Recall that a space in which every point has a separable neighborhood is said to be *locally separable.*

3.7. Theorem. *If M is a locally separable metric space* (*in particular if M is locally compact and metric*) *and D is a discrete space, then $M \times \beta D$ is cozero complemented.*

Proof. It is shown in [1] and noted in 4.4F(c) of [9] that every locally separable metric space is a free union of separable metric spaces. Because by 1.5(c) free unions of cozero complemented are cozero complemented, it is enough to prove 3.7 in case M is separable, in which case M is a dense subspace of a compact metric space K. Clearly $M \times \beta D$ is a dense Lindelöf subspace of $K \times \beta D$, so by 2.9(a) above, it suffices to show that $K \times \beta D$ is cozero complemented. Because K is compact with a countable base and D is discrete, we conclude that $K \times D$ is z-embedded in $K \times \beta D$ from Theorem 3.2 of [6]. So, by 2.8(a) we conclude that $K \times \beta D$ is cozero complemented, which completes the proof. □

The space U discussed in 3.11 shows that the hypothesis "metric" cannot be omitted in 3.7.

In this last argument, essential use was made of the local separability of M, but one might ask:

3.8. Question. If M is a metric space and D is discrete, must $M \times \beta D$ be cozero complemented?

In the remainder of this section, $D = D(\kappa)$ will denote a discrete space of cardinality κ, and $\mathbb{R}$ will denote the real line.

Before presenting a negative answer to 3.1(a), we state a lemma that follows easily from Lemma 3.4 once one observes that: (i) If D is infinite, then a rectangle $S \times T$ in $\beta D \times \beta D$ is clopen if it is of the form $\mathrm{cl}_{\beta D} A \times \mathrm{cl}_{\beta D} B$, where A and B are subsets of D, and (ii) $(\beta D \times \beta D)\backslash(D \times D)$ has empty interior in $\beta D \times \beta D$.

3.9. Lemma. *If V and V' are cozerosets of $\beta D \times \beta D$, they are complementary if and only if $V \cap (D \times D)$ and $V' \cap (D \times D)$ are disjoint σ-rectangles whose union is $D \times D$.*

3.10. Theorem. *$\beta D(\kappa) \times \beta D(\kappa)$ is cozero complemented if and only $\kappa \leqslant \omega$.*

Proof. If $\kappa \leqslant \omega$, then $\beta D(\kappa) \times \beta D(\kappa)$ is separable and hence is cozero complemented by 1.6(d).

Let Δ denote the diagonal of $D(\mathbf{c})\times D(\mathbf{c})$ and let $V = D(\mathbf{c})\times D(\mathbf{c})\backslash\Delta$. By identifying the set $D(\mathbf{c})$ with the space $\mathbb{R}$ with its usual topology and $D(\mathbf{c})\times D(\mathbf{c})$ with $\mathbb{R}\times\mathbb{R}$, one sees how to write V as a countable union $\bigcup_{n<\omega} A_n \times B_n$ of clopen rectangles with rational endpoints. Thus, $\bigcup_{n<\omega}(\mathrm{cl}_{\beta D} A_n) \times (\mathrm{cl}_{\beta D} B_n)$ is a cozeroset U of $\beta D(\mathbf{c})\times\beta D(\mathbf{c})$ that meets $D(\mathbf{c})\times D(\mathbf{c})$ in V. If W were a cozero complement of U in $\beta D \times \beta D$, then by 3.4 W would be a σ-rectangle of $\beta D \times \beta D$. Thus $W \cap (D \times D)$ would be a σ-rectangle of $D \times D$. Now clearly $W \cap (D \times D) = \Delta$. But Δ is not a σ-rectangle since any clopen rectangle of $D(\mathbf{c})\times D(\mathbf{c})$ contained in Δ is a singleton. So, by Lemma 3.4, $\beta D(\mathbf{c})\times \beta D(\mathbf{c})$ is not cozero complemented.

Suppose $\omega < \kappa \leqslant \mathbf{c}$, regard $D(\kappa)$ as a subspace of $D(\mathbf{c})$, and define Δ and V as above. Because V is a σ-rectangle in $D(\mathbf{c})\times D(\mathbf{c})$, $V \cap (D(\kappa) \times D(\kappa))$ is a σ-rectangle in $D(\kappa) \times D(\kappa)$ whose complement in $D(\kappa) \times D(\kappa)$ is not a σ-rectangle in $D(\kappa) \times D(\kappa)$. Hence the conclusion follows in this case as well.

Finally, if $\kappa > c$, then $\beta D(c) \times \beta D(c)$ is a clopen subspace of $\beta D(\kappa) \times \beta\ D(\kappa)$. So, if the former were cozero complemented, so would be the latter, which it is not. $\square$

Observe that $\beta\omega\times\ \beta\omega$ is cozero complemented; this follows from its separability as well as from the last result. But, by 5.8 of [3], this space is not Oz. So, a separable fraction dense space need not be Oz.

The authors wish to acknowledge a useful conversation with Kenneth Kunen concerning the proof of 3.10.

The next example answers a question of Levy and Shapiro posed in [19].

3.11. Example. A compact cozero complemented space with a dense open subspace that is not cozero complemented.

Let $L = D \cup \{p\}$ be as in 3.5, let $K = \beta L$, and let $U = K \setminus \{p\}$. Observe that U is pseudocompact because each of its countable sets has compact closure. So, $\beta D \times U$ is pseudocompact. Thus by a well-known theorem of Glicksberg (see 4AG in [22]) $\beta(\beta D \times U) = \beta D \times \beta D$, whence by Theorems 3.10 and 1.4(a), $\beta D \times U$ is not cozero complemented. By Theorem 3.6, $\beta D \times L$ is Lindelöf and cozero complemented. Hence by 2.9, $\beta D \times K$ is cozero complemented. Thus, $\beta D \times K$ is the desired compact cozero complemented space whose dense open subspace $\beta D \times U$ is not cozero complemented.

Note that by 3.3(b), $\beta D \times U$ is locally cozero complemented.

The argument just given depends essentially on the pseudocompactness of the space U. This inspires:

3.12. Question. Must every realcompact dense open subspace of a compact cozero complemented space be cozero complemented?

We return to the question posed in 3.1(b).

3.13. Theorem. *If $X \times Y$ is cozero complemented and Y is separable, then X is cozero complemented.*

Proof. It follows from 1.6(d) that the hypothesis implies that Y is cozero complemented.

If $V \in \mathcal{C}oz(X)$, then $V \times Y \in \mathcal{C}oz(X \times Y)$, so by assumption there is a $W \in \mathcal{C}oz(X \times Y)$ such that $W \cup (V \times Y)$ is dense in $X \times Y$ and $W \cap (V \times Y) = \emptyset$. Thus $\pi_X[W] \cap V = \emptyset$ and $\pi_X[W] \cup V$ is dense in X.

By assumption, there is a sequence $\{y_n\}_{n<\omega}$ that is dense in Y. For each $n < \omega$, $W \cap (X \times \{y_n\}) \in \mathcal{C}oz(X \times \{y_n\})$ and hence there exists a $U_n \in \mathcal{C}oz(X)$ such that

$$W \cap \bigl(X \times \{y_n\}\bigr) = U_n \times \{y_n\}.$$

If $U = \bigcup_{n<\omega} U_n$, then $U \in \mathcal{C}oz(X)$ and since each U_n is contained in $\pi_X[W]$, $U \cap V = \emptyset$.

If $U \cup V$ failed to be dense in X, it would be disjoint from some nonempty open subset S of X. Now the nonempty set $S \times Y \subset (X \setminus V) \times Y$, and because $W \cup (V \times Y)$ is dense in $X \times Y$, it follows that $(S \times Y) \cap W \neq \emptyset$. Thus $\pi_Y[(S \times Y) \cap W]$ is a nonempty open subset of Y, which must contain some element y_k of the dense set $\{y_n\}_{n<\omega}$. It follows that there is an $s \in S$ such that $(s, y_k) \in W$, so $S \cap U \neq \emptyset$, contrary to our choice of S. Thus $U \cup V$ is dense in X and U is a cozero complement of V. □

The preceding theorem prompts us to ask:

3.14. Question. Must the product of a cozero complemented space X and a separable space Y be cozero complemented?

In the special case where X is also σ-compact, if T is a countable dense subspace of Y, $X \times T$ is a dense Lindelöf subspace of both $X \times Y$ and $X \times \beta Y$, so that $X \times Y$ is weakly

Lindelöf. It follows from 2.9 that $X \times Y$ is cozero complemented if and only if $X \times \beta Y$ is cozero complemented. So in trying to settle Question 3.14 in this special case, we may assume that Y is compact as well as separable.

Next, we consider the questions raised in 3.2, not only for their own sake, but as a tool for answering those raised in 3.1(b). Some of the difficulties involved are illustrated by the following

3.15. Example. Suppose D is an uncountable discrete space, K is the compact space described in Example 3.12, and consider the sequence of spaces and mappings illustrated by

$$\beta(D \times D) \to \beta D \times \beta D \to \beta D \times K \to (D \times D)^*,$$

where $(D \times D)^*$ denotes the one-point compactification of the discrete space $D \times D$. Each map in the above sequence is a perfect irreducible surjection that fixes $D \times D$ pointwise. By 1.6 and the remarks preceding it, together with the observation made in 3.10, the first and third spaces in this sequence are cozero complemented. By 3.11 and an easy exercise, the second and fourth are not. This shows that neither the property of being cozero complemented nor its negation is preserved under this natural class of mappings.

3.16. Definition. Suppose $f : X \to Y$ is a continuous surjection.

(a) If $f[\mathcal{Z}(X)] \subset \mathcal{Z}(Y)$ then f is said to be *zeroset preserving*.
(b) If $f[\mathcal{C}oz(X)] \subset \mathcal{C}oz(Y)$ then f is said to be *cozeroset preserving*.

Clearly cozeroset preserving maps are open. However zeroset preserving maps and closed continuous surjections are independent concepts, and not every open continuous surjection is cozeroset preserving as is shown next.

3.17. Examples. If X is a space and p is a point such that $\{p\}$ is not a G_δ-set, then $\{p\} \notin \mathcal{Z}(X)$. If $Y = X \oplus \{a\}$ is the free union of X and a one-point set, define a map $f : Y \to X$ such that $f(a) = p$ and $f|X$ is the identity map. Then f is a closed continuous surjection that is not zeroset preserving.

An example of a zeroset preserving continuous surjection that is not closed is given in [24, 15.17(2)].

Finally, let $L = D \cup \{p\}$ be as in 3.5, and let $X = D \oplus L$ denote the free union of D and L. Define $f : X \to L$ by letting its restriction to its first summand D shift it to its copy in L, and by letting its restriction to L be the identity map. It is clear that f is a continuous open surjection that is not cozeroset preserving because the image of its first summand is not a cozeroset of L.

The following definition is given in [20].

3.18. Definition. A space is called *δ-normally separated* if closed sets and disjoint zerosets are completely separated.

This concept was introduced in [25] where it was called *property Z*. Every normal space is δ-normally separated, but the converse fails. For example, every P-space and every countably compact space is δ-normally separated, but there are nonnormal P-spaces and nonnormal countably compact spaces. For other examples of δ-normally separated spaces, see 1R and 5Q in [22].

3.19. Lemma. *If V is an open subset of a δ-normally separated space X that is a countable union of zerosets of X, then V is a cozeroset.*

Proof. Suppose $V = \bigcup_{n<\omega} Z_n$, where each Z_n is a zeroset. By assumption, for each $n < \omega$, there is an $f_n \in C(X)$ such that $f_n[Z_n] = \{1\}$ and $f_n[X\backslash V] = \{0\}$. Clearly $X\backslash V = \bigcap_{n<\omega} Z[f_n] \in \mathcal{Z}(X)$, so V is a cozeroset. □

The following is 15.14 of [24].

3.20. Lemma. *An open perfect surjection is zeroset preserving.*

The next result is a step towards answering 3.1(b).

3.21. Theorem. *If X is compact, Y is δ-normally separated, and $X \times Y$ is cozero complemented, then Y is cozero complemented.*

Proof. If $W \in \mathcal{C}oz(Y)$, then $X \times W \in \mathcal{C}oz(X \times Y)$ and hence by assumption has a cozero complement $V \in \mathcal{C}oz(X \times Y)$ which is a countable union of zerosets Z_n of $X \times Y$. So the open set $\pi_Y[V]$ is the countable union of the $\pi_Y[Z_n]$, which are zerosets by 3.20. (As X is compact, π_Y is a perfect map.) It follows then from 3.19 that $\pi_Y[V] \in \mathcal{C}oz[Y]$ and it is easy to verify that $\pi_Y[V]$ is a cozero complement of W in Y. □

Another partial answer to 3.1(b) follows:

3.22. Theorem. *If $X \times Y$ is weakly Lindelöf and cozero complemented, then X and Y are cozero complemented.*

Proof. As $X \times Y$ is weakly Lindelöf and dense in $\beta X \times \beta Y$, it follows by 2.4 that $X \times Y$ is $\mathcal{Z}^{\#}$-embedded in $\beta X \times \beta Y$. Because $X \times Y$ is also cozero complemented, $\beta X \times \beta Y$ is cozero complemented by 2.9. It then follows from 3.21 that each of βX and βY is cozero complemented. So X and Y are cozero complemented by 1.4(a). □

As was shown in 3.10, a product of two compact cozero complemented spaces need not be cozero complemented. Another question related to the questions in 3.1 is:

3.23. Question. Must the product of a P-space and a cozero complemented space be cozero complemented?

The last example of this section is peculiar in that its conclusion does not follow from any of our general theorems on when a product of cozero complemented spaces is cozero complemented.

3.24. Products of spaces of ordinals. Let Ω denote the first uncountable ordinal, and Ω^* $(= \beta\Omega)$ its one-point compactification. We will show that $\Omega^* \times \Omega^*$ is cozero complemented. Hence since $\beta(\Omega \times \Omega) = \beta\Omega \times \beta\Omega$, we will know also that $\Omega \times \Omega$, $\Omega^* \times \Omega$, and $\Omega \times \Omega^*$ are cozero complemented. If $\alpha < \omega_1$, let

$$A(\alpha) = [\alpha + 1, \omega_1] \times [\alpha + 1, \omega_1],$$

$$B_1(\alpha) = [0, \alpha] \times [\alpha + 1, \omega_1],$$

$$B_2(\alpha) = [0, \alpha] \times [0, \alpha], \quad \text{and}$$

$$B_3(\alpha) = [\alpha + 1, \omega_1] \times [0, \alpha].$$

Note that each of these four sets is clopen in $\Omega^* \times \Omega^*$.

By 8LM of [11], if $V \in \mathcal{C}oz(\Omega^* \times \Omega^*)$, then either:

Case 1. $p = (\omega_1, \omega_1) \in V$ and there is an $\alpha < \omega_1$ such that $A(\alpha) \subset V$, or

Case 2. $p \notin V$ and there is an $\alpha < \omega_1$ such that $V \cap A(\alpha) = \emptyset$.

In case 1, note that $B_2(\alpha)$ is cozero complemented because it is a compact metrizable space. Each of $B_1(\alpha)$ and $B_3(\alpha)$ contains as a dense subspace the product of Ω^* and a countable discrete space. Recall from 1.6(f) that Ω^* is cozero complemented, so these latter dense subspaces are cozero complemented by 1.5, as well as being Lindelöf. So by 1.4(d) or 2.6, each of the spaces $B_2(\alpha)$ and $B_3(\alpha)$ is cozero complemented. It is easy to verify that the union for $i = 1, 2, 3$ of a cozero complement of V in $V \cap B_i(\alpha)$ is a cozero complemented of V in $\Omega^* \times \Omega^*$.

In case 2, one may proceed as in case 1 to combine cozero complements in each of the rectangles $B_i(\alpha)$ to complete the proof that $\Omega^* \times \Omega^*$ is cozero complemented.

4. Extensions and cozero complementation

First, we consider when being a cozeroset is preserved under extension. A minor alteration of the proof of 3.15 in [11] yields:

4.1. Lemma. *If X is dense in T and V is a locally compact open subset of X, then V is open in T.*

4.2. Lemma. *If V is subspace of a space X, the following are equivalent*:

(a) *V is open, locally compact, and σ-compact.*
(b) *If X is dense in a space T, then $V \in \mathcal{C}oz(T)$.*

Proof. (a) implies (b). By 4.1, V is open in T as well as locally compact and σ-compact. Let $V = \bigcup_{n<\omega} K(n)$ where each $K(n)$ is compact. Since T is a Tychonoff space, there is for each $n < \omega$ an $f_n \in C(T)$ such that $f_n[K(n)] = 1$ and $f_n[T \setminus V] = 0$. Clearly $V = \bigcup_{n<\omega} \operatorname{coz}(f_n) \in \mathcal{C}oz(T)$.

(b) implies (a). Note that X is dense in the compact space $T = \beta X$, so by (b), $V \in \mathcal{C}oz(\beta X)$. So V is locally compact, σ-compact, open in βX, and hence in X. □

4.3. Corollary. *If V is a locally compact σ-compact cozeroset of X with no cozero complement in X, then no extension of X is cozero complemented.*

Proof. If T were a cozero complemented extension of X, then by 4.2, $V \in \mathcal{C}oz(T)$. So there exists $W \in \mathcal{C}oz(T)$ that is a cozero complement of V in T. Because X is dense in T, $W \cap X$ is a cozero complement of V in X, which is a contradiction. □

4.4. Examples. (a) Using 4.3, we see that the Michael line of 1.7(d) has no cozero complemented extension.

(b) Any locally compact σ-compact noncompact space V can be embedded as a cozeroset in a space with no cozero complemented extension. To see this, suppose V is a locally compact σ-compact space that is not compact, let $b \in \beta V \setminus V$, let λ denote a cardinal with uncountable cofinality, and let $L(\lambda) = D \cup \{p\}$ topologized as follows: each point of D is isolated, and if $p \in A \subset L(\lambda)$, then A is open in $L(\lambda)$ if and only if $|D \setminus A| < \lambda$. Finally, let $X(\lambda)$ be the quotient space obtained from the free union $(V \cup \{b\}) \oplus L(\lambda)$ by identifying b and p to a point we denote by r. Clearly V is open in $X(\lambda)$ and is locally compact and σ-compact. So, by 4.2, if T is an extension of $X(\lambda)$, then $V \in \mathcal{C}oz(T)$.

If V has a cozero complement W in T, then $V \cap W = \emptyset$ and $V \cup W$ is dense in T. Then $V \cap (W \cap X(\lambda)) = \emptyset$ and $V \cup (W \cap X(\lambda))$ is dense in $X(\lambda)$. Because $r \in \operatorname{cl}_{X(\lambda)} V$, it follows that $W \cap X(\lambda) = D$, and because $W \cap X(\lambda) \in \mathcal{C}oz(X(\lambda))$ is an F_σ-set in $X(\lambda)$, there is a sequence of closed subsets $\{A_n: n < \omega\}$ of $X(\lambda)$ whose union is $W \cap X(\lambda)$. Since λ has uncountable cofinality, $|A_k| = \lambda$ for some $k < \omega$. But then $(D \cup \{r\}) \setminus A_k$ is an open subset of $L(\lambda)$ that contains r and has cardinality λ, contrary to the way in which the topology of $L(\lambda)$ was obtained. We conclude that no extension of $X(\lambda)$ is cozero complemented. (A special case of this construction appears in 5.7 of [13].) Note that if $\lambda \geqslant \aleph_2$, then $L(\lambda)$ is not weakly Lindelöf.

The next lemma will be used to characterize a class of spaces such that each of its extensions is cozero complemented.

4.5. Lemma. *If* (i) *X is locally compact.* (ii) *X has a σ-compact subspace S whose closure is not compact, and* (iii) *X is not weakly Lindelöf, then X has a compactification that is not cozero complemented.*

Proof. By (iii) X has an open cover $\mathcal{V}$ no countable subfamily of which has a dense union. Let $V \in \mathcal{V}$. By (i), for each $x \in V$, there is a compact set K_x and a cozeroset C_x such that $x \in C_x \subset K_x \subset V$. Then $C_x \in \mathcal{C}oz(K_x)$ is σ-compact. Thus each member of $\mathcal{V}$ is a union of σ-compact cozerosets, so we may assume that $\mathcal{V}$ consists of σ-compact cozerosets.

By (ii), there is a countable subfamily $\mathcal{C}$ of $\mathcal{V}$ whose union W contains S. Then W is a locally compact and σ-compact cozeroset of X. Let $T = X \cup \{p\}$ denote the one-point compactification of X. Then by 4.2, $W \in \mathcal{C}oz\,T$. Now $\mathrm{cl}_T\, S$ is compact, and by (ii) $\mathrm{cl}_X\, S$ is not compact. As $\mathrm{cl}_X\, S \subset \mathrm{cl}_T\, S \subset \mathrm{cl}_X\, S \cup \{p\}$, it follows that $p \in \mathrm{cl}_T\, S$. Then $p \in \mathrm{cl}_T\, W$ because $S \subset W$.

Suppose W has a cozero complement U in T. Because $U \cap W = \emptyset$ and $p \in \mathrm{cl}_T\, W$, we know that $p \notin U$, so $U \subset X$. But the cozeroset $U \in \mathcal{C}oz\,T$ is σ-compact, so there is a countable subfamily $\mathcal{A}$ of $\mathcal{V}$ whose union contains U. If $W \cup U$ were dense in T, then $\mathcal{C} \cup \mathcal{A}$ would be a countable subfamily of $\mathcal{V}$ whose union is dense in X, contrary to the choice of $\mathcal{V}$. Thus W has no cozero complement, so T is not cozero complemented. □

4.6. Theorem. *Suppose X is cozero complemented and not pseudocompact. Then the following are equivalent*:

(a) *Every compactification of X is cozero complemented.*
(b) *Every extension of X is cozero complemented.*
(c) *X is weakly Lindelöf.*

Proof. That (c) implies (b) is immediate from Corollary 2.9. That (b) implies (a) is trivial. We now prove that if (c) fails, then (a) fails. Because X is not pseudocompact, it follows from 1.21 and 8A1 of [11] that X contains a closed countable discrete C-embedded subspace D. Then $K = \mathrm{cl}_{\beta X}\, D \backslash X$ is a compact subspace of $\beta X \backslash X$. Because X is not weakly Lindelöf, there is a collection $\mathcal{C}$ of open subsets of βX whose union covers X such that no countable subcollection intersects X in a dense subspace of X. For each $x \in X$, find $C_x \in \mathcal{C}$ containing x. Because $x \notin K$,there is a $V_x \in \mathcal{C}oz(\beta X)$ containing x and disjoint from K Let $W_x = C_x \cap V_x$ and $\mathcal{W} = \{W_x\colon\, x \in X\}$. Because $\mathcal{W}$ is a refinement of $\mathcal{C}$, no countable subfamily of $\mathcal{W}$ has a union whose intersection with X is dense in X. Consequently the union Y of the members of $\mathcal{W}$ is an open and hence locally compact subset of βX that is not weakly Lindelöf. Furthermore $K \cap Y = \emptyset$, so Y has a σ-compact subset D whose closure in Y is not compact. Finally, X is dense in Y.

So, by 4.5 (applied to Y), Y has a compactification that is not cozero complemented and is a compactification of X. □

This leads us to ask:

4.7. Question. Which pseudocompact but not compact spaces have the property that each of their extensions is cozero complemented?

5. Local versus global

The main purpose of this section is to address the question of when a space that is locally cozero complemented must be cozero complemented. That this is not always the case will be shown below. The main results of this section are 5.2 and 5.5.

Recall that every cozeroset is z- and hence $\mathcal{Z}^{\#}$-embedded in any space containing it. Thus by 2.5, if X is cozero complemented, then every neighborhood of each point of X contains a cozero complemented neighborhood.

5.1. Proposition. *If T is a countable union of cozero complemented cozerosets, then T is cozero complemented.*

Proof. Suppose $T = \bigcup_{n<\omega} V_n$, where each $V_n \in \mathcal{C}oz(T)$ and is cozero complemented, and let $C \in \mathcal{C}oz(T)$. Because a finite union of cozerosets is a cozeroset, we may assume that $V_n \subset V_{n+1}$ for all $n < \omega$. By assumption, for each n, there is a $W_n \in \mathcal{C}oz(V_n)$ such that $(C \cap V_n) \cap W_n = \emptyset$ and $(C \cap V_n) \cup W_n$ is dense in V_n. By 1.5(a), $W = \bigcup_{n<\omega} W_n \in \mathcal{C}oz(T)$. It will be shown that W is a cozero complement of C in T; that is (i) $C \cup W$ is dense in T and (ii) $C \cap W = \emptyset$. For, if S is a nonempty open subset of T, then $S \cap V_m \neq \emptyset$ for some $m < \omega$, and hence $S \cap [(C \cap V_m) \cup W_m] \neq \emptyset$, so $S \cap (C \cup W) \neq \emptyset$ and (i) holds. To see that (ii) holds, note that

$$C \cap W = \left(\bigcup_{n<\omega} C \cap V_n\right) \cap \left(\bigcup_{j<\omega} (W \cap V_j)\right) = \bigcup_{n<\omega} \bigcup_{j<\omega} (C \cap V_n \cap W \cap V_j)$$
$$= \bigcup_{n<\omega} \bigcup_{j<\omega} (C \cap V_{\min(n,j)}) \cap W_{\min(n,j)}) = \emptyset. \qquad \Box$$

5.2. Theorem. *If a space X is weakly Lindelöf, then it is cozero complemented if and only if it is locally cozero complemented.*

Proof. As noted above, we need only prove the sufficiency. If $U(x)$ is a cozero complemented neighborhood of x in X, and $V(x)$ is a cozeroset such that $x \in V(x) \subset U(x)$, then $V(x)$ is cozero complemented by the remarks preceding 5.1. By assumption the open cover $\{V(x): x \in X\}$ has a countable subfamily whose union V is dense in X. Moreover, V is cozero complemented by 5.1. So, by 2.4 and 2.8, X is cozero complemented. $\Box$

The example that follows shows that the assumption in 5.2 that X is weakly Lindelöf may not be dropped altogether.

5.3. Example. A locally cozero complemented space that is not cozero complemented.

Recall from Example 3.11 that if D is a discrete space of cardinality ω_1 and $U = \bigcup\{\mathrm{cl}_{\beta D} A: A \subset D \text{ and } |A| = \omega\}$, then $\beta D \times U$ fails to be cozero complemented. It is easy to verify, however, that each point of this space has a clopen neighborhood homeomorphic with $\beta D \times \beta\omega$, which is cozero complemented by 3.3(b). So $\beta D \times U$ is locally cozero complemented.

Next, we exhibit another way that a locally cozero complemented is forced to become cozero complemented.

5.4. Lemma. *The union X of a locally finite family $\{Y_\alpha: \alpha \in \Gamma\}$ of cozero complemented cozerosets is cozero complemented.*

Proof. If $V \in \mathcal{C}oz(X)$, then $V \cap Y_\alpha \in \mathcal{C}oz(Y_\alpha)$ for all $\alpha \in \Gamma$ and hence there is a $C_\alpha \in \mathcal{C}oz(Y_\alpha)$ such that $C_\alpha \cap (V \cap Y_\alpha) = \emptyset$ and $C_\alpha \cup (V \cap Y_\alpha)$ is dense in Y_α. Clearly,

$C_\alpha = \mathrm{coz}(f_\alpha)$ for some $f_\alpha \geqslant 0$ in $C(Y_\alpha)$. Observe that the local finiteness of the collection $\{Y_\alpha\colon \alpha \in \Gamma\}$ enables us to define $f : X \to \mathbb{R}$ by letting $f(x) = \sum\{f_\alpha(x)\colon x \in C_\alpha\}$ for each $x \in X$. We claim that $f \in C(X)$. For if $p \in X$, there is an open neighborhood U_p of p such that $\{\alpha \in \Gamma\colon U_p \cap C_\alpha \neq \emptyset\}$ is a finite set. Thus, the restriction of f to U_p is continuous. It follows from 1A(2) of [11] that $f \in C(X)$. Thus,

$$V \cap \mathrm{coz}(f) = V \cap \left(\bigcup\{C_\alpha\colon \alpha \in \Gamma\}\right) = \bigcup\{V \cap C_\alpha\colon \alpha \in \Gamma\} = \emptyset.$$

If W is open in X, then some Y_α meets it, so $W \cap (V \cup C_\alpha) \neq \emptyset$, and hence $W \cap (V \cup \mathrm{coz}(f)) \neq \emptyset$. Thus $V \cup \mathrm{coz}(f)$ is dense in X, so $\mathrm{coz}(f)$ is a cozero complement of V. □

5.5. Theorem. *A locally cozero complemented paracompact space X is cozero complemented.*

Proof. Because X is locally cozero complemented, it has a cover $\mathcal{C}$ consisting of cozero complemented cozerosets. Because X is paracompact, $\mathcal{C}$ has a locally finite partition of unity Φ subordinated to it. Thus for all $f \in \Phi, \mathrm{coz}(f) \subset C$ for some $C \in \mathcal{C}$. Since cozerosets of cozero complemented spaces are cozero complemented, so is $\mathrm{coz}(f)$. Finally, because Φ is locally finite, $\{\mathrm{coz}(f)\colon f \in \Phi\}$ is a locally finite cover of X. So by 5.4, X is cozero complemented. □

Recall from 1.6(d) that every ccc-space is cozero complemented. So 5.5 yields immediately:

5.6. Corollary. *Every paracompact space in which each point has a neighborhood that satisfies the countable chain condition* (*in particular, any paracompact space that is locally separable*) *is cozero complemented.*

A space X is called *hereditarily Lindelöf* if each of its subspaces is Lindelöf. It is well known and easy to see that a space is hereditarily Lindelöf if and only if each of its open subspaces is Lindelöf. As is noted in 3P of [22], this latter condition implies that X is a ccc-space and hence is cozero complemented.

5.7. Corollary. *Every hereditarily Lindelöf space is cozero complemented.*

5.8. Question. Can the hypothesis "paracompact" be replaced by "realcompact" in 5.5?

6. Open questions: products of ccc-spaces

A number of open questions have been stated above. We summarize and add to the more important questions in what follows.

6.1. Suppose $X \times Y$ is cozero complemented. Must X or Y be cozero complemented? In this generality, we cannot answer this question even if $X = Y$, but we showed in Section 3 that both factors are cozero complemented if the product is weakly Lindelöf and that if X has an isolated point, then Y is cozero complemented.

Recall from [18] that a space for which one and hence all of its compactification is separable is said to be *dense separable.* Such a space need not be separable but is cozero complemented. One may deduce easily from this paper that if $X \times Y$ is dense separable, then both X and Y are cozero complemented.

6.2. Because the space of minimal prime ideals of $C(X)$ is determined completely by this algebra, we may assume without loss of generality that the spaces X we consider are realcompact. While we find this hypothesis difficult to use, we may ask how many of the results given above might be improved or changed by adding this hypothesis. For example, must a dense (or dense or open) realcompact subspace of a (compact) cozero complemented space be cozero complemented? Recall that the subspace given in Example 3.11 above is not realcompact.

6.3. Is there a class of mapping between spaces which preserve cozero complementation directly or inversely that makes it possible to create new cozero complemented spaces from old ones?

6.4. In 1.6(e), it is shown that every almost P-point of a cozero complemented space is a P-point. Note that both the Alexandroff double of $[0, 1]$ and the Michael line of Example 1.7, neither of which is cozero complemented, have nowhere dense closed almost P-sets that fail to be P-sets. This leads us to ask if every nowhere dense closed almost P-set of a cozero complemented space must be a P-set—at least if the space is realcompact?

6.5. Characterize those spaces all of whose subspaces are cozero complemented.

It was shown in 1.6(d) that ccc-spaces are cozero complemented; indeed they are fraction dense. It is known that if Martin's Axiom and the negation of the continuum hypothesis (CH), hold then arbitrary products of ccc-spaces are ccc (and hence cozero complemented). So it consistent with Zermelo–Fraenkel set theory together with the axiom of choice (ZFC) to believe that an arbitrary product of ccc-spaces is cozero complemented. There are, however, models of set theory in which a product of a ccc-space with itself is not ccc. More precisely, a totally ordered ccc-space that fails to be separable is called a *Souslin line*, and there are models of set theory in which CH holds and Souslin lines exist. Moreover, ccc fails to hold in any product of a Souslin line with itself, although there are models for ZFC in which ccc can hold for products of some pairs of Souslin lines. (For background and definitions of unfamiliar terminology in the above, see [2], Chapter II of [17], and 3T of [22].)

In a recent communication, Gary Gruenhage announced that if S is a connected Souslin line, then $S \times S$ is cozero complemented. However, the more general question remains.

6.6. Must every finite or infinite product of ccc-spaces be cozero complemented in all models of ZFC?

We are indebted to the referee for making a large number of constructive suggestions for improving the quality of our exposition.

Added March 9, 2004. Gary Gruenhage has announced some results about when a product of two spaces is cozero complemented. In particular, Question 3.1(b), 3.14, 3.23 and 6.6 above have negative answers, 3.8 has an affirmative answer, and it is shown that the product of a cozero complemented space and a separable metric space is cozero complemented.

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