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The Generalized Quasi-linearization Method for Reaction Diffusion Equations on an Unbounded Domain

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The method of generalized quasi-linearization has been well developed for ordinary differential equations. In this paper, we extend the method of generalized quasi-linearization to reaction diffusion equations on an unbounded domain. The iterates, which are solutions of linear equations starting from lower and upper solutions, converge uniformly and monotonically to the unique solution of the nonlinear reaction diffusion equation in an unbounded domain. Initially an existence theorem for the linear nonhomogeneous reaction diffusion equation in an unbounded domain has been proved under improved conditions. The quadratic convergence has been proved by using a comparison theorem of reaction diffusion equations with ordinary differential equations. This avoids the computational complexity of the quasi-linearization method, since the computation of Green's function at each stage of the iterates is avoided.

Key Words: Unbounded domain; generalized quasi-linearization; reaction diffusion equation; quadratic convergence.

1. INTRODUCTION

It is well known that the method of quasi-linearization [1] is a constructive method of proving the existence of solutions of initial and boundary value problems. The main advantage of the method in addition to quadratic convergence is that the iterates are solutions of linear and mildly nonlinear equations. See [11] for first-order ordinary differential equations and [7] for reaction diffusion equations. In [6, 7] we refer to the method of quasi-linearization as generalized quasi-linearization, since we no longer require the forcing function to be convex or concave, as the original

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quasi-linearization method. Furthermore, the method of upper and lower solutions is used as in the monotone method to obtain simultaneously increasing and decreasing sequences. Further the iterates are solutions of linear equations. In this paper, we extend the generalized quasi-linearization method to reaction diffusion equations in an unbounded domain. We develop the method of generalized quasi-linearization such that the forcing function $f(t,x,u)$ is split as $f_1(t,x,u)+f_2(t,x,u)+f_3(t,x,u)$, where $f_1(t,x,u)$ can be made convex, $f_2(t,x,u)$ can be made concave, and $f_3(t,x,u)$ is bounded and Lipschitzian. In this case, the iterates are solutions of simple nonlinear equations. However, if $f_3(t,x,u)\equiv 0$, the iterates are solutions of linear equations. These linear iterates converge quadratically to the unique solution of the nonlinear reaction diffusion equation. However, the rate of convergence is linear in [4, 7, 8, 9].

2. PRELIMINARIES

In this section we list the assumptions and recall some known existence and comparison results that are needed to develop our main result. See [3–7] for details.

We define the closed set

$$\Lambda = \left\{(t,x,u): v_0(t,x) \le u \le w_0(t,x),\ (t,x) \in \overline{Q_T} \equiv [0,T] \times R^N\right\}.$$

We consider the reaction diffusion equation with the Cauchy problem of the form

$$\begin{aligned} \frac{\partial u}{\partial t} - Lu &= f(t,x,u) \qquad \text{in } Q_T = (0,T] \times R^N, \\ u(0,x) &= u_0(x) \qquad \text{in } R^N, \end{aligned} \tag{2.1}$$

where L is a strictly uniform elliptic operator defined by

$$L = \sum_{i,j=1}^{N} a_{i,j}(t,x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{N} b_i(t,x)\frac{\partial}{\partial x_i} + c_0(t,x).$$

Here $f(t,x,u) = f_1(t,x,u) + f_2(t,x,u) + f_3(t,x,u)$.

We list the following assumptions for convenience.

(A_0) Let $f_1(t,x,u), f_2(t,x,u), f_3(t,x,u)$ be such that $F(t,x,u) = f_1(t,x,u) + \Phi(t,x,u)$ and $\Phi(t,x,u)$ are uniformly convex in u on Λ (i.e., $f_{1,uu} + \Phi_{uu} \ge 0$ and $\Phi_{uu} \ge 0$). Also let $G(t,x,u) = f_2(t,x,u) + \Psi(t,x,u)$ and $\Psi(t,x,u)$ be uniformly concave in u on Λ (i.e., $f_{2,uu} + \Psi_{uu} \le 0$ and

$\Psi_{uu} \leq 0$), and $f_{3,u}$ is a bounded continuous in u on Λ. In addition, let $F(t,x,u), G(t,x,u)$, and $f_3(t,x,u) \in C^{\alpha/2,\alpha}([0,T] \times R \times R^N, R)$. Furthermore, let

$$\left| F_u(t,x,\xi) + G_u(t,x,\eta) - \Phi_u(t,x,\eta) - \Psi_u(t,x,\xi) \right| \leq M\left(|x|^2 + 1\right), \tag{2.2}$$

where (t,x,ξ) and $(t,x,\eta) \in \Lambda$.

In fact, it is enough to assume that

$$F_u(t,x,w_0) + G_u(t,x,v_0) - \Phi_u(t,x,v_0) - \Psi_u(t,x,w_0) \leq M\left(|x|^2 + 1\right),$$
$$F_u(t,x,v_0) + G_u(t,x,w_0) - \Phi_u(t,x,w_0) - \Psi_u(t,x,v_0) \geq -M\left(|x|^2 + 1\right),$$

instead of (2.2).

(A_1) (i) a_{ij}, b_i, c_0 are bounded. For each $t \in (0,T)$, $a_{ij} \in C^{2+\alpha}(\overline{Q_T}, R)$, $b_i \in C^{1+\alpha}(\overline{Q}_T, R)$. Here a_{ij} satisfies

$$d_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(t,x)\xi_i\xi_j \leq d_1|\xi|^2,$$

where d_0 and d_1 are independent of (t,x).

(ii) $f(t,x,u)$ is holder continuous in t and x, and $|f(t,x,u)| \leq M_0 e^{\beta|x|^2}$, where M_0 and β are constants $M_0 > 0$, $\beta < (4Td_1)^{-1}$.

(iii) $u_0(x) \in C^{2+\alpha}$ and $|u_0(x)| \leq A_0 e^{\beta|x|^2}$.

DEFINITION 2.1. We say a function $v_0 \in C^{1,2}[\overline{Q_T}, R]$ is called a *lower solution* of (2.1) if

$$\frac{\partial v_0}{\partial t} - Lv_0 \leq f(t,x,v_0),$$
$$v_0(0,x) \leq u_0(x),$$

and an *upper solution* of (2.1) if the reversed inequality holds.

We state the following maximum principle from [3] and comparison theorem from [5], which we need in the main result to prove the monotonicity of iterates and quadratic convergence part, respectively.

THEOREM 2.1. *uLet (t, x) satisfy*

$$\frac{\partial u}{\partial t} - Lu + cu \geq 0 \quad \text{in } \overline{Q_T}$$

$$u(0, x) \geq 0,$$

where L is the elliptic operator defined by

$$L = \sum_{i,j=1}^{N} a_{i,j}(t, x) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{N} b_i(t, x) \frac{\partial}{\partial x_i} + c_0(t, x)$$

such that

$$|a_{i,j}| \leq M, \quad |b(t, x)| \leq M(|x| + 1),$$

and $|c(t, x)| \leq M(|x|^2 + 1)$, *and* $u(t, x) \geq -B \exp(\beta |x|)$, *where* M, B, β *are positive constants. Then we have*

$$u(t, x) \geq 0 \quad \text{in } \overline{Q_T}.$$

See [3] for details of the proof.

THEOREM 2.2. *Suppose that*

(i) $m \in C^{1,2}(Q_T, R_+)$ *such that* $m_t - Lm \leq f(t, x, m)$, *where* $f(t, x, u) \in C[Q_T \times R, R]$ *and the operator L is elliptic.*

(ii) $g \in C([0, T] \times R_+, R)$ *and let* $r(t, 0, y_0) \geq 0$ *is the maximal solution of the differential equations*

$$y' = g(t, y), \qquad y(0) = y_0 \geq 0$$

existing for $t \geq 0$ *and*

$$f(t, x, z) \leq g(t, z), \qquad z \geq 0.$$

(iii) $m(0, x) \leq r(0, 0, y_0)$ *for* $x \in R$.

Then $m(t, x) \leq r(t, 0, y_0)$ *on* Q_T.

See [5] for details.

In the following, we prove the existence of a unique solution of the following Cauchy problem, where the coefficient $c(t, x)$ is unbounded but satisfies some growth condition such as $|x|^2$.

First we recall some known results from [3].

Let $\Gamma(t, x; \tau, \xi)$ be the fundamental solution of the uniformly parabolic operator

$$L = \frac{\partial}{\partial t} - \left(\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + c_0 \right).$$

Let $f(t,x)$ be a function defined from Q to R, where $Q = (0,T) \times R^N$. We consider the potential function to be

$$V(t,x) = \int_0^t \int_{R^N} \Gamma(t,x;\tau,\xi) f(\tau,\xi)\, d\xi\, d\tau.$$

THEOREM 2.3. *Assume that f is a measurable function and $|f| \le Me^{\beta|x|^2}$; then $V(t,x)$ is a continuous function in Q.*

THEOREM 2.4. *Assume that $f(t,x)$ is a continuous function and $|f| \le Me^{\beta|x|^2}$ in Q. Then $V(t,x)$, where $(t,x) \in Q$, has the continuous first-order partial derivative with respect to x and*

$$\frac{\partial V(t,x)}{\partial x_i} = \int_0^t \int_{R^N} \frac{\partial}{\partial x_i} \Gamma(t,x;\tau,\xi) f(\tau,\xi)\, d\xi\, d\tau.$$

THEOREM 2.5. *If $f(t,x)$ is continuous in Q, f is locally Holder continuous in $x \in R^N$ uniformly with respect to t, and $|f| \le Me^{\beta|x|^2}$, then $V(t,x)$ has the continuous second-order partial derivative in x.*

THEOREM 2.6. *Let the assumptions of Theorem* 2.5 *hold. Then $\partial V(t,x)/\partial t$ exists and is continuous in Q.*

Now we prove the main result of this section. We prove the existence of the solution of the following Cauchy problem, using the monotone method.

Consider the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} - Lu + cu &= g(t,x,u) \quad \text{in } \overline{Q_T}, \\ u(0,x) &= u_0(x), \end{aligned} \tag{2.3}$$

where

$$L = \sum_{i,j=1}^{N} a_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + c_0,$$

where c_0 is the bounded function and

$$|a_{i,j}| \le M, \quad |b(t,x)| \le M, \quad |c(t,x)| \le M(|x|^2 + 1),$$

where M is a positive constant and $g(t,x,u)$ is a continuous function in Λ.

THEOREM 2.7. *Assume* (A_1)(i) *and* (A_1)(iii) *hold and there exists v_0 and $w_0 \in C^{1,2}[\overline{Q_T}, R]$, which are lower and upper solutions of* (2.3) *such that $v_0(t,x) \le w_0(t,x)$ on $\overline{Q_T}$. Furthermore, let $g(t,x)$ satisfy*

$$g(t,x,\alpha) - g(t,x,\beta) \ge -M_1(\alpha - \beta)$$

for $v_0 \le \beta \le \alpha \le w_0$, *and let* M_1 *be a positive constant. Then the Cauchy problem* (2.3) *has a unique solution belonging to* $C^{1,2}[\overline{Q_T}, R]$ *such that*

$$v_0(t,x) \le u(t,x) \le w_0(t,x) \quad \text{on } \overline{Q_T}.$$

Proof. We construct the sequences $\{v_n\}$ and $\{w_n\}$ as solutions of the linear equations

$$\frac{\partial v_n}{\partial t} - Lv_n + M_1 v_n = (-c + M_1) v_{n-1} + g(t, x, v_{n-1}),$$

$$v_n(0,x) = u_0(x),$$

and

$$\frac{\partial w_n}{\partial t} - Lw_n + M_1 w_n = (-c + M_1) w_{n-1} + g(t, x, w_{n-1}),$$

$$w_n(0,x) = u_0(x),$$

where $n = 1, 2, 3, \dots$.

We can show as in [7, 3] that $v_n, w_n \in C^{1,2}(Q)$ for $n = 1, 2, 3, \dots,$ are such that

$$v_0 \le v_1 \le \cdots \le v_n \le w_n \le \cdots \le w_1 \le w_0.$$

It is easy to show that $v_n(t,x), w_n(t,x)$ converge uniformly and monotonically such that $\lim_{n\to\infty} v_n(t,x) = v(t,x)$, $\lim_{n\to\infty} w_n(t,x) = w(t,x)$. Furthermore, we can show that v and w satisfy

$$v(t,x) = \int_0^t \int_{R^N} \Gamma(t,x;\tau,\xi)(-c(\xi,\tau)v(\xi,\tau) + g(\tau,\xi,v))\, d\xi\, d\tau + J^{(0)}(t,x),$$

$$w(t,x) = \int_0^t \int_{R^N} \Gamma(t,x;\tau,\xi)(-c(\xi,\tau)w(\xi,\tau) + g(\tau,\xi,v))\, d\xi\, d\tau + J^{(0)}(t,x),$$

where

$$J^{(0)}(t,x) = \int_{R^N} \Gamma(t,x;0,\xi) u_0(\xi)\, d\xi.$$

Clearly, $J^{(0)}(t,x) \in C^{1,2}(Q)$. Furthermore, v and w are Lebesgue measurable functions such that $v_0 \le v \le w \le w_0$. Using Theorems 2.3, 2.4, 2.5, and 2.6, it is easy to prove that v and w are the classical solutions of (2.3). Finally, we have to prove that $v \equiv w$. Let $\alpha = v - w$; then we have

$$\frac{\partial \alpha}{\partial t} - L\alpha + c\alpha = g(t,x,v) - g(t,x,w) \ge -M_1 \alpha,$$

$$\alpha(x,0) = 0.$$

Using the Maximum Principle, we have $\alpha \ge 0$, i.e., $v \ge w$. Hence, $v \equiv w$.

Remark 1. The above result is an improvement compared with the results of [6], since we have not assumed $c(t, x)$ to be bounded. We note that the above method is a constructive method of proving the existence of the unique solution of (2.3). However, the order of convergence of the sequences is linear.

3. GENERALIZED QUASI-LINEARIZATION

In this section we develop the method of generalized quasi-linearization. The method of generalized quasi-linearization yields monotone sequences that converge uniformly and monotonically to the unique solution of (2.1). Furthermore, we can show that the convergence is quadratic when the concerned functions are bounded.

THEOREM 3.1. *Let v_0 and w_0 be lower and upper solutions of* (2.1) *and, furthermore, let assumptions* (A_0) *and* (A_1) *hold. Then there exist monotone sequences* $\{v_n\}$ *and* $\{w_n\}$ *that converge uniformly and monotonically to the unique solution u of* (2.1).

Proof. From (A_0) it is easy to see that $f(t, x, u)$ satisfies

$$f(t,x,u) \geq f_1(t,x,v) + f_2(t,x,v) + f_3(t,x,u) + [F_u(t,x,v) + G_u(t,x,u) - \Phi_u(t,x,u) - \Psi_u(t,x,v)](u - v). \quad (3.1)$$

Let v_1, w_1 be the solutions of the IVP of the form

$$\frac{\partial v_1}{\partial t} - Lv_1 = f_1(t,x,v_0) + f_2(t,x,v_0) + f_3(t,x,v_1) + [F_u(t,x,v_0) + G_u(t,x,w_0) - \Phi_u(t,x,w_0) - \Psi_u(t,x,v_0)](v_1 - v_0) \quad (3.2)$$

$$v_1(0,x) = u_0(x)$$

and

$$\frac{\partial w_1}{\partial t} - Lw_1 = f_1(t,x,w_0) + f_2(t,x,w_0) + f_3(t,x,w_1) + [F_u(t,x,w_0) + G_u(t,x,v_0) - \Phi_u(t,x,v_0) - \Psi_u(t,x,w_0)](w_1 - w_0) \quad (3.3)$$

$$w_1(0,x) = u_0(x).$$

First, we will show that v_0 and w_0 are the lower and upper solutions of (3.2) and (3.3), respectively. Clearly, v_0 is the lower solution of (3.2). Similarly, using (3.1), we have

$$\begin{aligned}\frac{\partial w_0}{\partial t} - Lw_0 &\geq f(t,x,w_0)\\ &= f_1(t,x,w_0) + f_2(t,x,w_0) + f_3(t,x,w_0)\\ &\geq f_1(t,x,v_0) + f_2(t,x,v_0) + f_3(t,x,w_0)\\ &\quad + \big[F_u(t,x,v_0) + G_u(t,x,w_0) - \Phi_u(t,x,w_0)\\ &\qquad - \Psi_u(t,x,v_0)\big](w_0 - v_0),\end{aligned}$$

and

$$w_0(0,x) \geq u_0(x).$$

This proves w_0 is an upper solution of (3.2). Similarly, we can prove v_0 and w_0 are the lower and upper solutions of (3.3). Therefore, the solutions of (3.2) and (3.3) exist and are unique, by Theorem 2.7. This proves $v_0 \leq v_1$, $w_1 \leq w_0$.

Next we show that $v_0 \leq v_1 \leq w_1 \leq w_0$. Let $\gamma = v_1 - w_1$; then we have

$$\begin{aligned}\frac{\partial \gamma}{\partial t} - L\gamma = f_1(t,x,v_0) - f_1(t,x,w_0)&\\ + \big[F_u(t,x,v_0) - \Phi_u(t,x,w_0)\big]\big[(v_1 - w_1) + (w_0 - v_0)\big]&\\ + f_2(t,x,v_0) - f_2(t,x,w_0)&\\ + \big[G_u(t,x,w_0) - \Psi_u(t,x,v_0)\big]\big[(v_1 - w_1) + (w_0 - v_0)\big]&\\ + f_3(t,x,v_1) - f_3(t,x,w_1).&\end{aligned}$$

Using the fact that

$$f_1(t,x,v_0) - f_1(t,x,w_0) + \big[F_u(t,x,v_0) - \Phi_u(t,x,w_0)\big](w_0 - v_0) \leq 0$$
$$f_2(t,x,v_0) - f_2(t,x,w_0) + \big[G_u(t,x,w_0) - \Psi_u(t,x,v_0)\big](w_0 - v_0) \leq 0,$$

and using the Mean Value Theorem on $f_3(t,x,u)$, we have

$$\begin{aligned}\frac{\partial \gamma}{\partial t} - L\gamma \leq \big[F_u(t,x,v_0) + G_u(t,x,w_0) - \Phi_u(t,x,w_0) - \Psi_u(t,x,v_0)\big]\gamma&\\ + f_{3u}(t,x,\zeta)\gamma&\end{aligned}$$

for some $\zeta \in \Lambda$, so that $|f_{3u}(t, x, \zeta)| \le M$ by (A_0) and $\gamma(0, x) \le 0$. By Maximum Principle Theorem 2.1, we have $\gamma(t, x) \le 0$. Therefore, we have $v_0 \le v_1 \le w_1 \le w_0$.

Now, we assume that for some $k > 1$, we have

$$v_0 \le v_1 \le \cdots \le v_i \le \cdots \le v_k \le w_k \le \cdots \le w_i \le \cdots \le w_1 \le w_0,$$

where v_i and w_i are solutions of the system

$$\begin{aligned} \frac{\partial v_i}{\partial t} - Lv_i &= f_1(t, x, v_{i-1}) + f_2(t, x, v_{i-1}) + f_3(t, x, v_i) \\ &\quad + \big[F_u(t, x, v_{i-1}) + G_u(t, x, w_{i-1}) - \Phi_u(t, x, w_{i-1}) \\ &\quad - \Psi_u(t, x, v_{i-1})\big](v_i - v_{i-1}) \\ v_i(0, x) &= u_0(x) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \frac{\partial w_i}{\partial t} - Lw_i &= f_1(t, x, w_{i-1}) + f_2(t, x, w_{i-1}) + f_3(t, x, w_i) \\ &\quad + \big[F_u(t, x, w_{i-1}) + G_u(t, x, v_{i-1}) - \Phi_u(t, x, v_{i-1}) \\ &\quad - \Psi_u(t, x, w_{i-1})\big](w_i - w_{i-1}), \\ w_i(0, x) &= u_0(x), \end{aligned} \tag{3.5}$$

for $i = 1, 2, \ldots, k$.

Let v_{k+1} and w_{k+1} be solutions of the mildly nonlinear equations

$$\begin{aligned} \frac{\partial v_{k+1}}{\partial t} - Lv_{k+1} &= f_1(t, x, v_k) + f_2(t, x, v_k) + f_3(t, x, v_{k+1}) \\ &\quad + \big[F_u(t, x, v_k) + G_u(t, x, w_k) - \Phi_u(t, x, w_k) \\ &\quad - \Psi_u(t, x, v_k)\big](v_{k+1} - v_k) \\ v_{k+1}(0, x) &= u_0(x) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \frac{\partial w_{k+1}}{\partial t} - Lw_{k+1} &= f_1(t, x, w_k) + f_2(t, x, w_k) + f_3(t, x, w_{k+1}) \\ &\quad + \big[F_u(t, x, w_k) + G_u(t, x, v_k) - \Phi_u(t, x, v_k) \\ &\quad - \Psi_u(t, x, w_k)\big](w_{k+1} - w_k), \\ w_{k+1}(0, x) &= u_0(x), \end{aligned} \tag{3.7}$$

respectively.

We will prove that $v_k \le v_{k+1} \le w_{k+1} \le w_k$ on $\overline{Q_T}$.

First, we show that v_k and w_k are lower and upper solutions of (3.6), respectively. Using (3.1), we have

$$\begin{aligned}\frac{\partial v_k}{\partial t} - Lv_k &= f_1(t,x,v_{k-1}) + f_2(t,x,v_{k-1}) + f_3(t,x,v_k)\\ &\quad + \left[F_u(t,x,v_{k-1}) + G_u(t,x,w_{k-1}) - \Phi_u(t,x,w_{k-1})\right.\\ &\qquad \left. - \Psi_u(t,x,v_{k-1})\right](v_k - v_{k-1})\\ &\le f_1(t,x,v_k) + f_2(t,x,v_k) + f_3(t,x,v_k),\end{aligned}$$

which prove that v_k is the lower solution of (3.6). Similarly, using (3.1), we also have

$$\begin{aligned}\frac{\partial w_k}{\partial t} - Lw_k &= f_1(t,x,w_{k-1}) + f_2(t,x,w_{k-1}) + f_3(t,x,w_k)\\ &\quad + \left[F_u(t,x,w_{k-1}) + G_u(t,x,v_{k-1}) - \Phi_u(t,x,v_{k-1})\right.\\ &\qquad \left. - \Psi_u(t,x,w_{k-1})\right](w_k - w_{k-1})\\ &\ge f_1(t,x,w_k) + f_2(t,x,w_k) + f_3(t,x,w_k)\\ &\quad + F_u(t,x,v_k)(w_{k-1} - v_k) + \Phi_u(t,x,w_{k-1})(v_k - w_{k-1})\\ &\quad + \left[G_u(t,x,w_{k-1}) - \Psi_u(t,x,v_k)\right](w_{k-1} - v_k)\\ &\quad + \left[F_u(t,x,w_{k-1}) + G_u(t,x,v_{k-1}) - \Phi_u(t,x,v_{k-1})\right.\\ &\qquad \left. - \Psi_u(t,x,w_{k-1})\right](w_k - w_{k-1})\\ &\ge f_1(t,x,w_k) + f_2(t,x,w_k) + f_3(t,x,w_k)\\ &\quad + F_u(t,x,v_k)(w_{k-1} - v_k) + \Phi_u(t,x,w_k)(v_k - w_{k-1})\\ &\quad + \left[G_u(t,x,w_k) - \Psi_u(t,x,v_k)\right](w_{k-1} - v_k)\\ &\quad + \left[F_u(t,x,v_k) + G_u(t,x,w_k) - \Phi_u(t,x,w_k)\right.\\ &\qquad \left. - \Psi_u(t,x,v_k)\right](w_k - w_{k-1})\\ &\ge f_1(t,x,w_k) + f_2(t,x,w_k) + f_3(t,x,w_k)\\ &\quad + \left[F_u(t,x,v_k) + G_u(t,x,w_k) - \Phi_u(t,x,w_k)\right.\\ &\qquad \left. - \Psi_u(t,x,v_k)\right](w_k - v_k),\end{aligned}$$

and

$$w_k(0,x) = u_0(x).$$

This proves that w_k is the upper solution of (3.6). Therefore, using Theorem 2.7, there exists a unique solution v_{k+1} such that $v_k \le v_{k+1} \le w_k$.

Similarly, we can prove that there exists a unique solution w_{k+1} such that $v_k \le w_{k+1} \le w_k$. To prove that $v_{k+1} \le w_{k+1}$, let $\gamma = v_{k+1} - w_{k+1}$. Then we have

$$\frac{\partial \gamma}{\partial t} - L\gamma \le \left[F_u(t,x,v_k) + G_u(t,x,w_k) - \Phi_u(t,x,w_k) - \Psi_u(t,x,v_k) \right]\gamma + f_{3u}(t,x,\zeta)\gamma,$$

where $\zeta \in \Lambda$, so that $|f_{3u}(t,x,\zeta)| \le M$ by A_0 and $\gamma(0,x) \le 0$. By Maximum Principle Theorem 2.1, we have $\gamma(t,x) \le 0$. Therefore, $v_k \le v_{k+1} \le w_{k+1} \le w_k$.

Using mathematical induction, it is easy to see that v_n and w_n satisfy

$$v_0 \le v_1 \le \cdots \le v_n \le w_n \le \cdots \le w_1 \le w_0, \quad \text{for all } n.$$

Now using standard arguments as in the proof of Theorem 2.7, one can show that the sequences $\{v_n\}$ and $\{w_n\}$ converge uniformly and monotonically to the unique solution of (2.1) on $\overline{Q_T}$.

THEOREM 3.2. *Let the assumptions of Theorem* 3.1 *hold. Furthermore, suppose* v_0 *and* w_0 *are bounded functions and functions* $|f_u|$, $|f_{3,u}|$, $|F_u|$, $|G_u|$, $|F_{uu}|$, $|G_{uu}|$, $|\Phi_u|$, $|\Phi_{uu}|$, $|\Psi_u|$, $|\Psi_{uu}|$ *are bounded by* M *for* t, x, u *on* Λ, *where* M *is a positive constant. Then the convergence of the sequences of Theorem* 3.1 *is quadratic.*

Proof. Let $\alpha_{k+1} = u(t,x) - v_{k+1} \ge 0$, $\beta_{k+1} = w_{k+1} - u \ge 0$. Then we have

$$\alpha_{k+1}(0,x) = 0, \qquad \beta_{k+1}(0,x) = 0$$

Hence

$$\begin{aligned}
\frac{\partial \alpha_{k+1}}{\partial t} - L\alpha_{k+1}
&= f(t,x,u) - \big\{ f_1(t,x,v_k) + f_2(t,x,v_k) + f_3(t,x,v_{k+1}) \\
&\qquad + \left[F_u(t,x,v_k) + G_u(t,x,w_k) - \Phi_u(t,x,w_k) \right. \\
&\qquad \left. - \Psi_u(t,x,v_k) \right] (v_{k+1} - v_k) \big\} \\
&= f(t,x,u) - f_1(t,x,v_k) - f_2(t,x,v_k) - f_3(t,x,v_{k+1}) \\
&\quad - \left[F_u(t,x,v_k) + G_u(t,x,w_k) - \Phi_u(t,x,w_k) \right. \\
&\qquad \left. - \Psi_u(t,x,v_k) \right] (v_{k+1} - u + u - v_k)
\end{aligned}$$

$$\begin{aligned}
&\le [F_u(t,x,u) - F_u(t,x,v_k)]\alpha_k + [G_u(t,x,v_k) - G_u(t,x,w_k)]\alpha_k \\
&\quad + [F_u(t,x,v_k) + G_u(t,x,w_k) - \Phi_u(t,x,w_k) - \Psi_u(t,x,v_k)]\alpha_{k+1} \\
&\quad + [\Phi_u(t,x,w_k) - \Phi_u(t,x,v_k)]\alpha_k \\
&\quad + [\Psi_u(t,x,v_k) - \Psi_u(t,x,u)]\alpha_k + f_{3,u}(t,x,\xi)\alpha_{k+1} \\
&\le M\alpha_k^2 + M[\beta_k + \alpha_k]\alpha_k + 5M\alpha_{k+1} + M[\beta_k + \alpha_k]\alpha_k + M\alpha_k^2 \\
&= 4M\alpha_k^2 + 5M\alpha_{k+1} + 2M\alpha_k\beta_k \\
&\le 5M\alpha_k^2 + 5M\alpha_{k+1} + M\beta_k^2.
\end{aligned}$$

Similarly,

$$\frac{\partial \beta_{k+1}}{\partial t} - L\beta_{k+1} \le M\beta_k^2 + 5M\beta_{k+1} + M\alpha_k^2.$$

Using Comparison Theorem 2.2, we get

$$0 \le \alpha_{k+1}(t,x) \le \int_0^t e^{5M(t-s)} \max_{Q_T}\{5M\alpha_k^2 + M\beta_k^2\}\, ds.$$

Therefore,

$$\max|u - v_{k+1}| \le \tfrac{1}{5} e^{5MT} \max\left\{5M|u - v_k|^2 + M|u - w_k|^2\right\}.$$

Similarly,

$$\max|u - w_{k+1}| \le \tfrac{1}{5} e^{5MT} \max\left\{M|u - w_k|^2 + M|u - v_k|^2\right\}.$$

This completes the proof.

The population genetics problem of [7, p. 38] clearly illustrates the application of Theorems 3.1 and 3.2.

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