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Higher-order semi-implicit Taylor schemes for Itô stochastic differential equations

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ABSTRACT

The paper considers the derivation of families of semi-implicit schemes of weak order $N = 3.0$ (general case) and $N = 4.0$ (additive noise case) for the numerical solution of Itô stochastic differential equations. The degree of implicitness of the schemes depends on the selection of N parameters which vary between 0 and 1 and the families contain as particular cases the 3.0 and 4.0 weak order explicit Taylor schemes. Since the implementation of the multiple integrals that appear in these theoretical schemes is difficult, for the applications they are replaced by simpler random variables, obtaining simplified schemes. In this way, for the multidimensional case with one-dimensional noise, we present an infinite family of semi-implicit simplified schemes of weak order 3.0 and for the multidimensional case with additive one-dimensional noise, we give an infinite family of semi-implicit simplified schemes of weak order 4.0. The mean-square stability of the 3.0 family is analyzed, concluding that, as in the deterministic case, the stability behavior improves when the degree of implicitness grows. Numerical experiments confirming the theoretical results are shown.

1. Introduction

Analytical solutions of stochastic differential equations (SDEs) are, in general, not available and investigators are forced to use numerical methods that give approximated solutions. There are mainly two ways of measuring the accuracy of a numerical solution of an SDE: mean-square (strong) convergence is suitable when the sample paths of the solutions need to be approximated; if one is only interested in the moments of the solution, weak approximations are used. Many numerical methods have been proposed for both types of convergence, see for example [1] and the references therein. As in the deterministic case, many problems lead to stiff SDEs, characterized, see for example [1], by having a linearized system with Lyapunov exponents $\lambda_d \le \cdots \le \lambda_2 \le \lambda_1 < 0$ verifying $\lambda_d \ll \lambda_1$. The integration of stiff SDEs requires the use of schemes with good stability properties, such as implicit or predictor–corrector methods. A number of implicit schemes have been proposed in the literature, see for example [1–7]. To increase the efficiency without losing the stability properties, following the pattern laid down by Platen, see [1] or [6], in this work, a general approach to construct semi-implicit Taylor methods is developed. The paper is organized as follows: In Section 2, weak convergence is defined and some basic results are recalled. Section 3 is devoted to explicit weak Taylor schemes. Truncated stochastic Taylor expansions are used in Section 4 to derive a family of third weak order semi-implicit schemes for general Itô SDEs and a fourth order family suitable for equations with additive noise. In addition, in the scalar noise case, simplified versions of the obtained families are given. In Section 5, we show that the stability behavior of each semi-implicit simplified scheme depends on its degree of implicitness. Numerical results that confirm the convergence and stability properties are given in the last section.

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2. Weak approximations

Consider a filtered probability space $(\Omega, \mathcal{A}_t, P)$, an m-dimensional Wiener process $\{W_t\}$ with components $W_t^1, \ldots, W_t^m$, and an Itô stochastic differential equation

$$dX_t = a(t, X_t)\,dt + \sum_{j=1}^{m} b^j(t, X_t)\,dW_t^j, \tag{1}$$

with d-dimensional drift vector a with components $a^1, \ldots, a^d$ and diffusion coefficients $b^j, j = 1, \ldots, m$, with components $b^{1j}, \ldots, b^{dj}$. The functions $a = a(t, x)$ and $b^j = b^j(t, x)$ are assumed to be defined and measurable in $[t_0, T] \times \mathbb{R}^d$ and to satisfy both Lipschitz and linear growth bound conditions in x. These assumptions ensure the existence of a unique solution of the sde (1) with the initial condition $X_{t_0} = X_0$ if X_0 is $\mathcal{F}_{t_0}$-measurable, see [8]. We shall suppose that all of the initial moments $E[|X_0|^r] < \infty, r = 1, 2, \ldots$ exist; so, the moments of every X_t will exist (see [8]). Let $X_{t,x}$ denote the solution of (1) starting at time $t \in [t_0, T]$ at $x \in \mathbb{R}^d$. Let $\mathcal{C}_P$ denote the space of functions $f(t, x)$ defined in $[t_0, T] \times \mathbb{R}^d$ that have polynomial growth (with respect to x) and $\mathcal{C}_P^N, N = 1, 2, \ldots,$ the subspace of functions $f \in \mathcal{C}_P$ with all partial derivatives up to order N in $\mathcal{C}_P$.

Together with Eq. (1) we consider the *one-step approximation*

$$\overline{X}_{t,x}(t + h) = x + A(t, x, h, \xi), \tag{2}$$

where A is some $\mathbb{R}^d$-valued function and ξ a random vector. We shall say that the one-step approximation $\overline{X} = \overline{X}_{t,x}$ *converges weakly to* $X = X_{t,x}$ *with local order* $N + 1$ if there exists a function $K(x) \in \mathcal{C}_P$ such that

$$\left| E\left[\prod_{j=1}^{l}(\overline{X}^{i_j} - x^{i_j}) - \prod_{j=1}^{l}(X^{i_j} - x^{i_j})\right]\right| \le K(x)h^{N+1}, \quad i_j = 1, \ldots, d,\ l = 1, \ldots, 2N + 2, \tag{3}$$

where z^i denotes the ith component of the vector z. From (3) it is obvious that the differences between the moments, from the first up to $(2N + 2)$th inclusively of the vector X and the corresponding moments of its approximation $\overline{X}$ have $N + 1$ order of smallness in h.

Given an equidistant discretization $\{t_0, t_1, \ldots, t_M\}$ of the time interval $[t_0, T]$ with step size $\Delta = (T - t_0)/M$, the one-step approximation (2) gives the *scheme*:

$$Y_0 = X_0$$
$$Y_{n+1} = Y_n + A(t_n, Y_n, \Delta, \xi_n), \quad n = 0, \ldots, M - 1.$$

The scheme $Y = \{Y_0, Y_1, \ldots, Y_M\}$ is said to *converge weakly to* X *with order* N if for each $g \in \mathcal{C}_P^{2N+2}$ there exist constants $K_g \ge 0$ and $\Delta_0 > 0$ such that

$$|E[g(Y_M) - g(X_T)]| \le K_g\, \Delta^N$$

for all $\Delta \in (0, \Delta_0)$.

The number N in the above definition is the *order* of the scheme on an interval. Based on a theorem due to Milstein (see [9]) one can obtain schemes of order N by means of one-step approximations of local order $N + 1$.

We shall say that two one-step approximations $\overline{X}_{t,x}$ and $\overline{\overline{X}}_{t,x}$ of the solution $X_{t,x}$ are N-*equivalent* if there exists a function $K(x) \in \mathcal{C}_P$ such that

$$\left| E\left[\prod_{j=1}^{l}(\overline{X}^{i_j} - x^{i_j}) - \prod_{j=1}^{l}(\overline{\overline{X}}^{i_j} - x^{i_j})\right]\right| \le K(x)h^{N+1}, \quad i_j = 1, \ldots, d,\ l = 1, \ldots, 2N + 2.$$

To denote that the one-step approximations $\overline{X}_{t,x}(t+h)$ and $\overline{\overline{X}}_{t,x}(t+h)$ are N-equivalent we write $\overline{X}_{t,x} \overset{(N)}{\simeq} \overline{\overline{X}}_{t,x}$ or $\overline{Y}_{n+1} \overset{(N)}{\simeq} \overline{\overline{Y}}_{n+1}$ if $\overline{Y}$ and $\overline{\overline{Y}}$ are the corresponding schemes.

It is obvious that if the one-step approximations $\overline{X}_{t,x}$ and $\overline{\overline{X}}_{t,x}$ are N-equivalent then either both or none of them have local order $N + 1$. Notice also that, with this notation, $\overline{X}_{t,x} \overset{(N)}{\simeq} X_{t,x}$ for any local order $N + 1$ one-step approximation $\overline{X}_{t,x}$ of the solution $X_{t,x}$.

3. Weak Taylor schemes

Stochastic Taylor expansions, see [10], provide a systematic way to construct weak Taylor schemes of any order. Following Kloeden and Platen [1], together with Eq. (1) define the differential operators

$$L^{(0)} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} a^i \frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{m} b^{ik}b^{jk}\frac{\partial^2}{\partial x^i \partial x^j};$$

$$L^{(k)} = \sum_{i=1}^{d} b^{ik}\frac{\partial}{\partial x^i} \quad k = 1, \ldots, m. \tag{4}$$

For each multi-index $\alpha = (j_1, \ldots, j_l)$ with length $l(\alpha) = l > 1$ and components $j_1, \ldots, j_l \in \{0, 1, \ldots, m\}$, define $L^\alpha = L^{(j_1)}L^{(j_2)} \cdots L^{(j_l)}$. Theorem 5.11.1 in [1] proves that for an Itô stochastic differential equation with drift and diffusion components a^k and b^{kj} in $\mathcal{C}_P^{2(N+1)}$ and satisfying Lipschitz and linear growth conditions, the weak order $N \in \mathbb{N}$ truncated Itô–Taylor expansion of $f(t,x) = x$ (obtained retaining in the stochastic Taylor expansion the terms corresponding to multi-indices of length up to N) provides a one-step approximation of X_t of local order $N+1$:

$$f(t, X_t) \overset{(N)}{\simeq} f(t_0, X_{t_0}) + \sum_{1 \le l(\alpha) \le N} (L^\alpha f)(t_0, X_{t_0})\, I_{\alpha,t_0,t}, \tag{5}$$

where $I_{\alpha,t_0,t} = \int_{t_0}^{t} (\int_{t_0}^{s_l} \cdots \int_{t_0}^{s_3} (\int_{t_0}^{s_2} \mathrm{d}W_{s_1}^{j_1})\mathrm{d}W_{s_2}^{j_2} \ldots)\mathrm{d}W_{s_l}^{j_l}$ with $\mathrm{d}W_t^0 = \mathrm{d}t$. As a consequence, the corresponding scheme

$$Y_{n+1} = Y_n + \sum_{l(\alpha)=1}^{N} L^\alpha f \cdot I_\alpha \tag{6}$$

where I_α stands for $I_{\alpha,t_n,t_{n+1}}$ and $L^\alpha f$ stands for $(L^\alpha f)(t_n, Y_n)$, has weak order N and is called the *weak order N Taylor scheme*. From now on, for simplicity, when a function g in a scheme is evaluated at (t_n, Y_n) this point is omitted and we write g_n or simply g.

As explained in Section 2 the scheme (6) still achieves weak order N if the Itô multiple integrals I_α are replaced by simpler random variables $\hat{I}_\alpha$ with the same first $2N+1$ moments, i.e. satisfying the conditions

$$\left| E\left[\prod_{k=1}^{l} I_{\alpha_k} - \prod_{k=1}^{l} \hat{I}_{\alpha_k} \right] \right| \le K\, \Delta^{N+1} \tag{7}$$

for all choices of multi-indices $\alpha_1, \ldots, \alpha_l$, $l = 1, \ldots, 2N+1$, with $1 \le l(\alpha_k) \le N$. The obtained schemes in this way

$$Y_{n+1} = Y_n + \sum_{l(\alpha)=1}^{N} L^\alpha f \hat{I}_\alpha$$

are called *simplified weak order N Taylor schemes*.

For example, taking $N = 1$ in (6) we obtain the Euler scheme

$$Y_{n+1} = Y_n + a\,\Delta + \sum_{j=1}^{m} b^j\, \Delta W_n^j, \tag{8}$$

where $\Delta W_n^j = I_{(j)}, j = 1, \ldots, m$, are the Gaussian increment components of the Wiener process on $[t_n, t_{n+1}]$. It achieves weak order 1.0 and the variables $I_{(j)} = \Delta W^j$ in (8), $j = 1, \ldots, m$, can be replaced by random variables $\hat{I}_{(j)} = \Delta\hat{W}^j$ satisfying conditions (7) with $N = 1$, which can be summarized in the moment condition

$$|E(\Delta\hat{W}^j)| + |E((\Delta\hat{W}^j)^3)| + |E(\Delta\hat{W}^j)^2 - \Delta| \le K\, \Delta^2. \tag{9}$$

The Euler scheme is the simplest Taylor scheme. The second weak order Taylor scheme was originally proposed in [11] and can be obtained by the above general procedure taking $N = 2$, see [1].

To obtain the weak order 3.0 Taylor scheme one takes $N = 3$ in (6). Platen proposed, see [1], the following simplified weak order 3.0 Taylor scheme in the scalar case with scalar noise ($d = m = 1$):

$$\begin{aligned} Y_{n+1} = {} & Y_n + a\,\Delta + b\,\Delta\hat{W} + \frac{1}{2}L^1 b((\Delta\hat{W})^2 - \Delta) + L^1 a\,\Delta\hat{Z} + L^0 a\frac{\Delta^2}{2} \\ & + L^0 b(\Delta\hat{W}\Delta - \Delta\hat{Z}) + L^{(0,0)}a\frac{\Delta^3}{6} + \frac{1}{6}(L^{(0,0)}b + L^{(0,1)}a + L^{(1,0)}a)\Delta\hat{W}\Delta^2 \\ & + \frac{1}{6}(L^{(1,1)}a + L^{(0,1)}b + L^{(1,0)}b)((\Delta\hat{W})^2 - \Delta)\Delta + \frac{1}{6}L^{(1,1)}b((\Delta\hat{W})^2 - 3\Delta)\Delta\hat{W}, \end{aligned} \tag{10}$$

where $\Delta\hat{W}$ and $\Delta\hat{Z}$ are correlated Gaussian random variables with

$$\Delta\hat{W} \sim N(0, \Delta), \qquad \Delta\hat{Z} \sim N\left(0, \frac{1}{3}\Delta^3\right) \quad \text{and} \quad E(\Delta\hat{W}\,\Delta\hat{Z}) = \frac{1}{2}\Delta^2. \tag{11}$$

For a more efficient implementation, the scheme (10) is proposed in [12] using simpler variables.

To derive the weak order 4.0 Taylor scheme one includes all the fourth order multiple Itô integrals from the Itô Taylor expansion. For the one-dimensional additive noise case ($m = 1$, $b(t,x) = b(t)$ for all (t,x)), the explicit 4.0 order weak

Taylor scheme is

$$\begin{aligned} Y_{n+1} = {} & Y_n + a\,\Delta + b\,I_{(1)} + L^{(0)}a\frac{\Delta^2}{2} + L^{(1)}a\,I_{(1,0)} + L^{(0)}b\,I_{(0,1)} + L^{(0,0)}a\frac{\Delta^3}{3!} \\ & + L^{(1,0)}a\,I_{(1,0,0)} + L^{(0,1)}a\,I_{(0,1,0)} + L^{(0,0)}b\,I_{(0,0,1)} + L^{(1,1)}a\,I_{(1,1,0)} + L^{(0,0,0)}a\frac{\Delta^4}{4!} \\ & + L^{(1,0,0)}a\,I_{(1,0,0,0)} + L^{(0,1,0)}a\,I_{(0,1,0,0)} + L^{(0,0,1)}a\,I_{(0,0,1,0)} + L^{(0,0,0)}b\,I_{(0,0,0,1)} \\ & + L^{(1,1,0)}a\,I_{(1,1,0,0)} + L^{(1,0,1)}a\,I_{(1,0,1,0)} + L^{(0,1,1)}a\,I_{(0,1,1,0)} + L^{(1,1,1)}a\,I_{(1,1,1,0)}. \end{aligned} \tag{12}$$

In [13] a simplified explicit 4.0 weak order Taylor scheme was obtained replacing the fifteen I_α's that appear in (12) by the following variables $\hat{I}_\alpha$'s satisfying (7) with $N = 4$:

$$\begin{aligned} & \hat{I}_{(1)} = \Delta\hat{W}; \qquad I_{(1,0)} = \Delta\hat{Z}; \qquad \hat{I}_{(0,1)} = \Delta\hat{W}\,\Delta - \Delta\hat{Z}; \qquad \hat{I}_{(0,1,0)} = \frac{1}{6}\,\Delta\hat{W}\,\Delta^2; \\ & \hat{I}_{(1,1,0)} = \frac{1-\sqrt{2}}{2}\,\Delta\hat{W}\,\Delta\hat{Z} + \frac{3\sqrt{2}-2}{24}\,\Delta\hat{W}^2\,\Delta - \frac{4+\sqrt{2}}{24}\,\Delta^2 + \frac{\sqrt{2}}{2}\,\Delta\hat{Z}^2/\Delta; \\ & \hat{I}_{(1,0,0)} = \frac{1}{2}\,\Delta\hat{Z}\,\Delta - \frac{1}{12}\,\Delta\hat{W}\,\Delta^2; \qquad \hat{I}_{(0,0,1)} = -\frac{1}{2}\,\Delta\hat{Z}\,\Delta + \frac{5}{12}\,\Delta\hat{W}\,\Delta^2; \\ & \hat{I}_{(1,1,1,0)} = \frac{1}{24}(\Delta\hat{W}^2 - 3\,\Delta)\Delta\hat{W}\,\Delta; \qquad \hat{I}_{(1,1,0,0)} = \hat{I}_{(0,1,1,0)} = \hat{I}_{(1,0,1,0)} = \frac{1}{24}(\Delta\hat{W}^2 - \Delta)\Delta^2; \\ & \hat{I}_{(1,0,0,0)} = \hat{I}_{(0,1,0,0)} = \hat{I}_{(0,0,1,0)} = \hat{I}_{(0,0,0,1)} = \frac{1}{24}\,\Delta\hat{W}\,\Delta^3, \end{aligned} \tag{13}$$

where $\Delta\hat{W}$ and $\Delta\hat{Z}$ are as in (11).

4. Semi-implicit Taylor schemes

As it was stated, the introduction of implicitness in the terms of a stochastic scheme is a useful tool to overcome stability problems. The simplest method of this kind is the implicit Euler scheme

$$Y_{n+1} = Y_n + a(t_{n+1}, Y_{n+1})\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j \tag{14}$$

where $\Delta\hat{W}^j$ are independent random variables verifying the moment condition (9). From the explicit scheme (8) and the implicit scheme (14), a family of semi-implicit Euler schemes can be defined by

$$Y_{n+1} = Y_n + ((1-\theta)a + \theta\,a(t_{n+1}, Y_{n+1}))\Delta + \sum_{j=1}^{m} b^j\,\Delta\hat{W}^j \tag{15}$$

where $\theta \in [0, 1]$ can be interpreted as the degree of implicitness. These schemes, also called θ-methods, have order 1.0 in the weak sense and their stability properties have been studied in [14–17]. Milstein presented in [4], see also [9], a two-parameter family of semi-implicit weak order 2.0 methods where the parameters can be chosen in [0, 1] to determine the degree of implicitness. The stability properties of this family of schemes have been studied in [18]. Platen [6], see also [1], proposed a way to obtain these 1.0 and 2.0 families of semi-implicit schemes using truncated Itô–Taylor expansions. Systematizing this procedure we propose in this section families of semi-implicit schemes of third and fourth order containing the explicit schemes (10) and (12) respectively as the members with null degree of implicitness.

4.1. Semi-implicit weak order 3.0 Taylor schemes

Using (5) with $N = 3$ one obtains for any smooth function $f(t, x)$ that

$$f_{n+1} \overset{(3)}{\simeq} f_n + \sum_{j=0}^{m} f_{(j),n}\,I_{(j)} + \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2),n}\,I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3),n}\,I_{(j_1,j_2,j_3)}, \tag{16}$$

where $f_{\alpha,n}$ means $(L^\alpha f)(t_n, X_n)$ for each multi-index α. In particular, if $f(t, x) = x$ we have

$$X_{n+1} - X_n \overset{(3)}{\simeq} \sum_{j=0}^{m} f_{(j),n}\,I_{(j)} + \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2),n}\,I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3),n}\,I_{(j_1,j_2,j_3),n} \tag{17}$$

where $f_{(0)} = a, f_{(j)} = b^j, j = 1, \dots, m, f_{(0,0)} = L^{(0)}a$, etc.

Applying the expansion (16) to the coefficient function $f_{(0)}$ we obtain

$$f_{(0),n+1} \overset{(3)}{\simeq} f_{(0),n} + \sum_{j=0}^{m} f_{(j,0),n}\, I_{(j)} + \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2,0),n}\, I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3,0),n}\, I_{(j_1,j_2,j_3)}. \tag{18}$$

Since any product $I_{(j_1,j_2,j_3),n}\ I_{(0)}$ is a linear combination of multiple integrals of length 4 we have $I_{(j_1,j_2,j_3),n}\ I_{(0)} \overset{(3)}{\simeq} 0$ for $j_1, j_2, j_3 = 0, \dots, m$. Then from (18)

$$f_{(0),n}\,\Delta \overset{(3)}{\simeq} f_{(0),n+1}\,\Delta - \sum_{j=0}^{m} f_{(j,0),n}\, I_{(j)}\,\Delta - \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2,0),n}\, I_{(j_1,j_2)}\,\Delta. \tag{19}$$

If $\theta \in [0, 1]$ from (17) and (19) we have

$$\begin{aligned}
X_{n+1} - X_n \overset{(3)}{\simeq}\ & \theta f_{(0),n}\,\Delta + (1-\theta) f_{(0),n}\,\Delta + \sum_{j=1}^{m} f_{(j),n}\, I_{(j)} + \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2),n}\, I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3),n}\, I_{(j_1,j_2,j_3)} \\
\overset{(3)}{\simeq}\ & \theta \left(f_{(0),n+1}\Delta - \sum_{j=0}^{m} f_{(j,0),n}\, I_{(j)}\,\Delta - \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2,0),n}\, I_{(j_1,j_2)}\,\Delta \right) + (1-\theta) f_{(0),n}\,\Delta + \sum_{j=1}^{m} f_{(j),n}\, I_{(j)} \\
& + \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2),n}\, I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3),n}\, I_{(j_1,j_2,j_3)} \\
=\ & (\theta f_{(0),n+1} + (1-\theta) f_{(0),n})\Delta + f_{(0,0),n}(1-2\theta)\frac{\Delta^2}{2} + \sum_{j=1}^{m} f_{(j),n}\, I_{(j)} - \theta \sum_{j=1}^{m} f_{(j,0),n} I_{(j)}\Delta \\
& - \theta \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2,0),n}\, I_{(j_1,j_2)}\,\Delta + \sum_{\substack{j_1,j_2=0 \\ j_1+j_2\neq 0}}^{m} f_{(j_1,j_2),n}\, I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3),n}\, I_{(j_1,j_2,j_3)}.
\end{aligned} \tag{20}$$

We can now use the expansion (16) with the function $f_{(0,0)}$ and multiply by Δ^2; since $I_{(j_1,j_2)}\ \Delta^2 \overset{(3)}{\simeq} 0$ and $I_{(j_1,j_2,j_3)}\ \Delta^2 \overset{(3)}{\simeq} 0$ for $j_1, j_2, j_3 = 0, \dots, m$ we have

$$f_{(0,0),n+1}\Delta^2 \overset{(3)}{\simeq} f_{(0,0),n}\Delta^2 + \sum_{j=0}^{m} f_{(j,0,0),n}\, I_{(j)}\ \Delta^2.$$

Then, if $\beta \in [0, 1]$ we can write

$$\begin{aligned}
f_{(0,0),n}\Delta^2 =\ & \beta f_{(0,0),n}\Delta^2 + (1-\beta) f_{(0,0),n}\Delta^2 \\
\overset{(3)}{\simeq}\ & \beta f_{(0,0),n+1}\Delta^2 - \beta \sum_{j=0}^{m} f_{(j,0,0),n}\, I_{(j)}\Delta^2 + (1-\beta) f_{(0,0),n}\Delta^2
\end{aligned}$$

and inserting this expression in (20) we obtain

$$\begin{aligned}
X_{n+1} - X_n \overset{(3)}{\simeq}\ & (\theta f_{(0),n+1} + (1-\theta) f_{(0),n})\Delta + (1-2\theta)(\beta f_{(0,0),n+1} + (1-\beta) f_{(0,0),n})\frac{\Delta^2}{2} \\
& + (1 - 3\theta - 3\beta + 6\theta\beta) f_{(0,0,0),n}\frac{\Delta^3}{6} + \sum_{j=1}^{m} f_{(j),n}\, I_{(j)} - \theta \sum_{j=1}^{m} f_{(j,0),n}\, I_{(j)}\Delta \\
& - \frac{1}{2}(1-2\theta)\beta \sum_{j=1}^{m} f_{(j,0,0),n}\, I_{(j)}\Delta^2 + \sum_{\substack{j_1,j_2=0 \\ j_1+j_2\neq 0}}^{m} f_{(j_1,j_2),n}\, I_{(j_1,j_2)} \\
& - \theta \sum_{\substack{j_1,j_2=0 \\ j_1+j_2\neq 0}}^{m} f_{(j_1,j_2,0),n}\, I_{(j_1,j_2)}\,\Delta + \sum_{\substack{j_1,j_2,j_3=0 \\ j_1+j_2+j_3\neq 0}}^{m} f_{(j_1,j_2,j_3),n}\, I_{(j_1,j_2,j_3)}.
\end{aligned} \tag{21}$$

Finally we can use the expansion (16) with the function $f_{(0,0,0)}$ and multiply by Δ^3, obtaining

$$f_{(0,0,0),n+1}\Delta^3 \overset{(3)}{\simeq} f_{(0,0,0),n}\Delta^3.$$

Then, if $\gamma \in [0, 1]$ we have

$$f_{(0,0,0),n}\Delta^3 = \gamma f_{(0,0,0),n}\Delta^3 + (1-\gamma)f_{(0,0,0),n}\Delta^3 \overset{(3)}{\simeq} \gamma f_{(0,0,0),n+1}\Delta^3 + (1-\gamma)f_{(0,0,0),n}\Delta^3$$

and inserting this expression in (21) we get

$$\begin{aligned} X_{n+1} - X_n \overset{(3)}{\simeq} & (\theta f_{(0),n+1} + (1-\theta)f_{(0),n})\Delta + (1-2\theta)(\beta f_{(0,0),n+1} + (1-\beta)f_{(0,0),n})\frac{\Delta^2}{2!} \\ & + (1-3\theta-3\beta(1-2\theta))(\gamma f_{(0,0,0),n+1} + (1-\gamma)f_{(0,0,0),n})\frac{\Delta^3}{3!} + \sum_{j=1}^{m} f_{(j),n}\, I_{(j)} \\ & - \theta \sum_{j=1}^{m} f_{(j,0),n}\, I_{(j)}\Delta - \frac{1}{2}(1-2\theta)\beta \sum_{j=1}^{m} f_{(j,0,0),n}\, I_{(j)}\Delta^2 + \sum_{\substack{j_1,j_2=0 \\ j_1+j_2\neq 0}}^{m} f_{(j_1,j_2),n}\, I_{(j_1,j_2)} \\ & - \theta \sum_{\substack{j_1,j_2=0 \\ j_1+j_2\neq 0}}^{m} f_{(j_1,j_2,0),n}\, I_{(j_1,j_2)}\,\Delta + \sum_{\substack{j_1,j_2,j_3=0 \\ j_1+j_2+j_3\neq 0}}^{m} f_{(j_1,j_2,j_3),n}\, I_{(j_1,j_2,j_3)} \end{aligned} \tag{22}$$

where $f(t, x) = x$. From this equivalence we have:

Theorem 1. *Suppose that the drift and diffusion components $a^k, b^{kj}, k = 1, \dots, d, j = 1 \dots, m$ of Eq. (1) belong to $\mathcal{C}_P^8$ and satisfy Lipschitz and linear growth bound conditions. Then the scheme defined by (22) has order 3.0 in the weak sense.*

In the multidimensional case with scalar noise, $d = 1, 2, \dots, m = 1$, (22) gives the family of semi-implicit 3.0 weak order Taylor schemes

$$\begin{aligned} Y_{n+1} = & Y_n + (\theta\, a_{n+1} + (1-\theta)a)\Delta + (1-2\theta)(\beta(L^{(0)}a)_{n+1} + (1-\beta)\, L^{(0)}a)\frac{\Delta^2}{2!} \\ & + (1-3\theta-3\beta(1-2\theta))(\gamma\,(L^{(0,0)}a)_{n+1} + (1-\gamma)\, L^{(0,0)}a)\frac{\Delta^3}{3!} \\ & + \left(b - \theta\, L^{(1)}a\,\Delta - \frac{1}{2}(1-2\theta)\beta\, L^{(1,0)}a\,\Delta^2\right) I_{(1)} + (L^{(1)}b - \theta\, L^{(1,1)}a\,\Delta)I_{(1,1)} \\ & + (L^{(1)}a - \theta\, L^{(1,0)}a\,\Delta)I_{(1,0)} + (L^{(0)}b - \theta\, L^{(0,1)}a\,\Delta)I_{(0,1)} \\ & + L^{(1,0)}a\, I_{(1,0,0)} + L^{(0,1)}a\, I_{(0,1,0)} + L^{(0,0)}b\, I_{(0,0,1)} \\ & + L^{(1,1)}a\, I_{(1,1,0)} + L^{(1,0)}b\, I_{(1,0,1)} + L^{(0,1)}b\, I_{(0,1,1)} + L^{(1,1)}b\, I_{(1,1,1)}, \end{aligned} \tag{23}$$

where the parameters $\theta, \beta, \gamma \in [0, 1]$ determine the degree of implicitness.

It can be seen that taking $\Delta\hat{W}$ and $\Delta\hat{Z}$ as in (11) the variables

$$\begin{aligned} & \hat{I}_{(1)} = \Delta\hat{W}, \qquad \hat{I}_{(1,0)} = \Delta\hat{Z}, \qquad \hat{I}_{(0,1)} = \Delta\,\Delta\hat{W} - \Delta\hat{Z}, \\ & \hat{I}_{(1,1)} = \frac{1}{2}(\Delta\hat{W}^2 - \Delta), \qquad \hat{I}_{(0,0,1)} = \hat{I}_{(0,1,0)} = \hat{I}_{(1,0,0)} = \frac{1}{6}\Delta^2\Delta\hat{W}, \\ & \hat{I}_{(1,1,0)} = \hat{I}_{(1,0,1)} = \hat{I}_{(0,1,1)} = \frac{1}{6}\Delta(\Delta\hat{W}^2 - \Delta), \qquad \hat{I}_{(1,1,1)} = \frac{1}{6}\Delta\hat{W}(\Delta\hat{W}^2 - 3\Delta) \end{aligned}$$

satisfy (7) with $N = 3$. Then the multiple integrals I_α in (23) can be replaced by these variables $\hat{I}_\alpha$, obtaining the three-parameter family of simplified semi-implicit 3.0 weak order Taylor schemes for the $m = 1$ case:

$$\begin{aligned} Y_{n+1} = & Y_n + (\theta\, a_{n+1} + (1-\theta)a)\Delta + (1-2\theta)(\beta(L^{(0)}a)_{n+1} + (1-\beta)\, L^{(0)}a)\frac{\Delta^2}{2!} \\ & + (1-3\theta-3\beta(1-2\theta))(\gamma\,(L^{(0,0)}a)_{n+1} + (1-\gamma)\, L^{(0,0)}a)\frac{\Delta^3}{3!} \\ & + \left(b - \theta\, L^{(1)}a\,\Delta - \frac{1}{2}(1-2\theta)\beta\, L^{(1,0)}a\,\Delta^2\right)\Delta\hat{W} \\ & + \frac{1}{2}(L^{(1)}b - \theta\, L^{(1,1)}a\,\Delta)(\Delta\hat{W}^2 - \Delta) + (L^{(1)}a - \theta\, L^{(1,0)}a\,\Delta)\Delta\hat{Z} \\ & + (L^{(0)}b - \theta\, L^{(0,1)}a\,\Delta)(\Delta\,\Delta\hat{W} - \Delta\hat{Z}) + \frac{1}{6}(L^{(1,0)}a + L^{(0,1)}a + L^{(0,0)}b)\Delta^2\Delta\hat{W} \\ & + \frac{1}{6}(L^{(1,1)}a + L^{(1,0)}b + L^{(0,1)}b)\Delta(\Delta\hat{W}^2 - \Delta) + \frac{1}{6}L^{(1,1)}b\Delta\hat{W}(\Delta\hat{W}^2 - 3\Delta), \end{aligned} \tag{24}$$

with $\Delta\hat{W}$, $\Delta\hat{Z}$ as in (11) and parameters $\theta, \beta, \gamma \in [0, 1]$. Notice that the values $\theta = \beta = \gamma = 0$ in (24) results in the simplified explicit 3.0 weak order Taylor scheme (10).

4.2. Semi-implicit weak order 4.0 Taylor schemes

The above procedure can be repeated with $N = 4$ to achieve a family of semi-implicit Taylor schemes of weak order 4.0. In this case we omit all terms that are not relevant for a fourth order weak approximation: for a sufficiently smooth function $f(t, x)$ we have from (5)

$$f_{n+1} \overset{(4)}{\simeq} f_n + \sum_{j=0}^{m} f_{(j),n}\, I_{(j)} + \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2),n}\, I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3),n}\, I_{(j_1,j_2,j_3)} + \sum_{j_1,j_2,j_3,j_4=0}^{m} f_{(j_1,j_2,j_3,j_4),n}\, I_{(j_1,j_2,j_3,j_4)}. \tag{25}$$

In particular, if $f(t, x) = x$ we have

$$X_{n+1} - X_n \overset{(4)}{\simeq} \sum_{j=0}^{m} f_{(j),n}\, I_{(j)} + \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2),n}\, I_{(j_1,j_2)} + \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3)}\, I_{(j_1,j_2,j_3),n} + \sum_{j_1,j_2,j_3,j_4=0}^{m} f_{(j_1,j_2,j_3,j_4),n}\, I_{(j_1,j_2,j_3,j_4)} \tag{26}$$

where $f_{(0)} = a, f_{(j)} = b^j, j = 1, \ldots, m$, etc. Applying the expansion (25) to the coefficient functions $f_{(0)}, f_{(0,0)}, f_{(0,0,0)}, f_{(0,0,0,0)}$ and multiplying by Δ, Δ^2, Δ^3 and Δ^4 respectively, one obtains

$$\begin{aligned}
f_{(0),n}\, \Delta &\overset{(4)}{\simeq} f_{(0),n+1}\Delta - \sum_{j=0}^{m} f_{(j,0),n}\, I_{(j)}\, \Delta - \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2,0),n}\, I_{(j_1,j_2)}\Delta - \sum_{j_1,j_2,j_3=0}^{m} f_{(j_1,j_2,j_3,0)}\, I_{(j_1,j_2,j_3),n}\Delta \\
f_{(0,0),n}\, \Delta^2 &\overset{(4)}{\simeq} f_{(0,0),n+1}\Delta^2 - \sum_{j=0}^{m} f_{(j,0,0),n}\, I_{(j)}\, \Delta^2 - \sum_{j_1,j_2=0}^{m} f_{(j_1,j_2,0,0),n}\, I_{(j_1,j_2)}\Delta^2 \\
f_{(0,0,0),n}\, \Delta^3 &\overset{(4)}{\simeq} f_{(0,0,0),n+1}\Delta^3 - \sum_{j=0}^{m} f_{(j,0,0,0),n}\, I_{(j)}\Delta^3 \\
f_{(0,0,0,0),n}\, \Delta^4 &\overset{(4)}{\simeq} f_{(0,0,0,0),n+1}\Delta^4.
\end{aligned} \tag{27}$$

As in the third order case we can control the degree of implicitness of the scheme inserting in (26) the equivalences of (27) with parameters $\theta, \beta, \gamma, \alpha \in [0, 1]$. In this way we obtain:

Theorem 2. *Suppose that the drift and diffusion components a^k, b^{kj}, $k = 1, \ldots, d, j = 1, \ldots, m$ of Eq. (1) belong to $\mathcal{C}_P^{10}$ and satisfy Lipschitz and linear growth bound conditions. For each selection of the parameters $\theta, \beta, \gamma, \alpha \in [0, 1]$ the scheme*

$$\begin{aligned}
Y_{n+1} = {} & Y_n + (\theta f_{(0),n+1} + (1-\theta) f_{(0),n})\Delta + (1-2\theta)(\beta f_{(0,0),n+1} + (1-\beta) f_{(0,0),n})\frac{\Delta^2}{2!} \\
& + (1-3\theta-3\beta(1-2\theta))(\gamma f_{(0,0,0),n+1} + (1-\gamma) f_{(0,0,0),n})\frac{\Delta^3}{3!} \\
& + (1-4\theta-6\beta(1-2\theta)-4\gamma(1-3\theta-3\beta(1-2\theta)))(\alpha f_{(0,0,0,0),n+1} + (1-\alpha) f_{(0,0,0,0),n})\frac{\Delta^4}{4!} \\
& + \sum_{j=1}^{m}\left(f_{(j),n} - \theta\, f_{(j,0),n}\, \Delta - \frac{\beta}{2}(1-2\theta)\, f_{(j,0,0),n}\, \Delta^2 - \frac{\gamma}{6}(1-3\theta-3\beta(1-2\theta)) f_{(j,0,0,0),n}\, \Delta^3\right) I_{(j)} \\
& + \sum_{\substack{j_1,j_2=0\\ j_1+j_2\neq 0}}^{m}\left(f_{(j_1,j_2),n} - \theta\, f_{(j_1,j_2,0),n}\, \Delta - \frac{1}{2}(1-2\theta)\beta\, f_{(j_1,j_2,0,0),n}\, \Delta^2\right) I_{(j_1,j_2)} \\
& + \sum_{\substack{j_1,j_2,j_3=0\\ j_1+j_2+j_3\neq 0}}^{m} (f_{(j_1,j_2,j_3),n} - \theta\, f_{(j_1,j_2,j_3,0),n}\, \Delta)\, I_{(j_1,j_2,j_3)} + \sum_{\substack{j_1,j_2,j_3,j_4=0\\ j_1+j_2+j_3+j_4\neq 0}}^{m} f_{(j_1,j_2,j_3,j_4),n}\, I_{(j_1,j_2,j_3,j_4)}
\end{aligned} \tag{28}$$

has order 4.0 in the weak sense.

For SDEs with one-dimensional additive noise, if $f(t,x)=x$ we have $f_{(1,1)}=L^{(1,1)}f=L^{(1)}b=0$; then the terms f_α with α ended by $(1,1)$ do not appear in (28); in a similar way the terms with indices ended by $(1,0,1)$ and $(1,0,0,1)$ do not appear either. If in addition we take $\theta=0$ in (28) we obtain the three-parameter family of semi-implicit schemes

$$\begin{aligned} Y_{n+1} = {} & Y_n + a\,\Delta + (\beta(L^{(0)}a)_{n+1} + (1-\beta)L^{(0)}a)\frac{\Delta^2}{2} + (1-3\beta)(\gamma(L^{(0,0)}a)_{n+1} + (1-\gamma)L^{(0,0)}a)\frac{\Delta^3}{3!} \\ & + (1-6\beta-4\gamma(1-3\beta))(\alpha(L^{(0,0,0)}a)_{n+1} + (1-\alpha)L^{(0,0,0)}a)\frac{\Delta^4}{4!} \\ & + \left(b - \frac{1}{2}\beta L^{(1,0)}a\Delta^2 - \frac{1}{6}\gamma(1-3\beta)L^{(1,0,0)}a\Delta^3\right)I_{(1)} - \frac{1}{2}\beta L^{(1,1,0)}a\Delta^2 I_{(1,1)} \\ & + \left(L^{(1)}a - \frac{1}{2}\beta L^{(1,0,0)}a\Delta^2\right)I_{(1,0)} + \left(L^{(0)}b - \frac{1}{2}\beta L^{(0,1,0)}a\Delta^2\right)I_{(0,1)} \\ & + L^{(1,0)}a\,I_{(1,0,0)} + L^{(0,1)}a\,I_{(0,1,0)} + L^{(0,0)}b\,I_{(0,0,1)} + L^{(1,1)}a\,I_{(1,1,0)} \\ & + L^{(1,0,0)}a\,I_{(1,0,0,0)} + L^{(0,1,0)}a\,I_{(0,1,0,0)} + L^{(0,0,1)}a\,I_{(0,0,1,0)} + L^{(0,0,0)}b\,I_{(0,0,0,1)} \\ & + L^{(1,1,0)}a\,I_{(1,1,0,0)} + L^{(1,0,1)}a\,I_{(1,0,1,0)} + L^{(0,1,1)}a\,I_{(0,1,1,0)} + L^{(1,1,1)}a\,I_{(1,1,1,0)}. \end{aligned} \tag{29}$$

Notice that taking $\beta=\gamma=\alpha=0$ in (29) one obtains the explicit 4.0 Taylor scheme (12).

The semi-implicit schemes (29) contain sixteen random integrals I_α: the fifteen that appear in the explicit scheme (12) plus $I_{(1,1)}$. In order to obtain from (29) semi-implicit simplified schemes we shall replace these fifteen I_α's by the corresponding $\hat{I}_\alpha$'s in (13) and $I_{(1,1)}$ by $\hat{I}_{(1,1)}=\frac{1}{2}(\Delta\hat{W}^2-\Delta)$. It is straightforward to show that this sixteen-element set of multiple integrals fulfills (7) with $N=4$: Since the conditions have been proven in [13] for the fifteen-element subset in (12), it reduces to check that every product $\hat{I}_{\alpha_1}\cdots\hat{I}_{\alpha_l}$ with mean-square order up to 4.5 in which $I_{(1,1)}$ appears has the same expectation that the corresponding product $I_{\alpha_1}\cdots I_{\alpha_l}$, i.e.:

Order 1.0:

$$\mathrm{E}[\hat{I}_{(1,1)}]=0.$$

Order 2.0:

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}^2_{(1)}]=\Delta^2;\qquad \mathrm{E}[\hat{I}^2_{(1,1)}]=\frac{1}{2}\,\Delta^2.$$

Order 3.0:

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}^4_{(1)}]=6\,\Delta^3;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1)}\,\hat{I}_{(1,0)}]=\frac{1}{2}\,\Delta^3;\qquad \mathrm{E}[\hat{I}^2_{(1,1)}\,\hat{I}^2_{(1)}]=\frac{5}{2}\,\Delta^3;$$

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1)}\,\hat{I}_{(0,1)}]=\frac{1}{2}\,\Delta^3;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1,1,0)}]=\frac{1}{6}\,\Delta^3;\qquad \mathrm{E}[\hat{I}^3_{(1,1,1)}]=\Delta^3.$$

Order 4.0:

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1,1,0,0)}]=\frac{1}{24}\,\Delta^4;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1,1,0)}\,\hat{I}^2_{(1)}]=\frac{5}{6}\,\Delta^4;\qquad \mathrm{E}[\hat{I}^2_{(1,1)}\,\hat{I}^4_{(1)}]=\frac{39}{2}\,\Delta^4;$$

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(0,1,1,0)}]=\frac{1}{24}\,\Delta^4;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}^2_{(1,0)}]=\frac{1}{4}\,\Delta^4;\qquad \mathrm{E}[\hat{I}^2_{(1,1)}\,\hat{I}_{(1,0)}\,\hat{I}_{(1)}]=\frac{5}{4}\,\Delta^4;$$

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1,0,1,0)}]=\frac{1}{24}\,\Delta^4;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1,0)}\,\hat{I}_{(0,1)}]=\frac{1}{4}\,\Delta^4;\qquad \mathrm{E}[\hat{I}^2_{(1,1)}\,\hat{I}_{(0,1)}\,\hat{I}_{(1)}]=\frac{5}{4}\,\Delta^4;$$

$$\mathrm{E}[\hat{I}_{(1,1)}\hat{I}_{(1,1,1,0)}\,\hat{I}_{(1)}]=\frac{1}{8}\,\Delta^4;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}^2_{(0,1)}]=\frac{1}{4}\,\Delta^4;\qquad \mathrm{E}[\hat{I}^2_{(1,1)}\,\hat{I}_{(1,1,0)}]=\frac{1}{3}\,\Delta^4;$$

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1,0,0)}\,\hat{I}_{(1)}]=\frac{1}{6}\,\Delta^4;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(1,0)}\,\hat{I}^3_{(1)}]=3\,\Delta^4;\qquad \mathrm{E}[\hat{I}^3_{(1,1)}\,\hat{I}^2_{(1)}]=\frac{17}{2}\,\Delta^4;$$

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(0,1,0)}\,\hat{I}_{(1)}]=\frac{1}{6}\,\Delta^4;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(0,1)}\,\hat{I}^3_{(1)}]=3\,\Delta^4;\qquad \mathrm{E}[\hat{I}^4_{(1,1)}]=\frac{15}{4}\,\Delta^4;$$

$$\mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}_{(0,0,1)}\,\hat{I}_{(1)}]=\frac{1}{6}\,\Delta^4;\qquad \mathrm{E}[\hat{I}_{(1,1)}\,\hat{I}^6_{(1)}]=45\,\Delta^4.$$

together with $\mathrm{E}[\hat{I}_{\alpha_1}\cdots\hat{I}_{\alpha_l}]=0$ for every product of mean-square orders 1.5, 2.5, 3.5 and 4.5. Then the proposed family of 4.0 weak order semi-implicit schemes for SDEs with additive noise is

$$Y_{n+1} = Y_n + a\,\Delta + (\beta(L^{(0)}a)_{n+1} + (1-\beta)L^{(0)}a)\frac{\Delta^2}{2} + (1-3\beta)(\gamma(L^{(0,0)}a)_{n+1} + (1-\gamma)L^{(0,0)}a)\frac{\Delta^3}{3!}$$

$$+ (1 - 6\beta - 4\gamma(1 - 3\beta))(\alpha(L^{(0,0,0)}a)_{n+1} + (1 - \alpha)L^{(0,0,0)}a)\frac{\Delta^4}{4!}$$
$$+ \left(b - \frac{1}{2}\beta L^{(1,0)}a\Delta^2 - \frac{1}{6}\gamma(1 - 3\beta)L^{(1,0,0)}a\Delta^3\right)\Delta\hat{W} - \frac{1}{4}\beta L^{(1,1,0)}a\Delta^2(\Delta\hat{W}^2 - \Delta)$$
$$+ \left(L^{(1)}a - \frac{1}{2}\beta L^{(1,0,0)}a\Delta^2\right)\Delta\hat{Z} + \left(L^{(0)}b - \frac{1}{2}\beta L^{(0,1,0)}a\Delta^2\right)(\Delta\hat{W}\,\Delta - \Delta\hat{Z})$$
$$+ \frac{1}{2}(L^{(1,0)}a - L^{(0,0)}b)\Delta\hat{Z}\,\Delta + \frac{1}{6}\left(L^{(0,1)}a - \frac{1}{2}L^{(1,0)}a + \frac{5}{2}L^{(0,0)}b\right)\Delta\hat{W}\,\Delta^2$$
$$+ L^{(1,1)}a\left(\frac{1-\sqrt{2}}{2}\Delta\hat{W}\,\Delta\hat{Z} + \frac{3\sqrt{2}-2}{24}\Delta\hat{W}^2\,\Delta - \frac{4+\sqrt{2}}{24}\Delta^2 + \frac{\sqrt{2}}{2}\Delta\hat{Z}^2/\Delta\right)$$
$$+ \frac{1}{24}(L^{(1,0,0)}a + L^{(0,1,0)}a + L^{(0,0,1)}a + L^{(0,0,0)}b)\Delta\hat{W}\,\Delta^3$$
$$+ \frac{1}{24}(L^{(1,1,0)}a + L^{(1,0,1)}a + L^{(0,1,1)}a)(\Delta\hat{W}^2 - \Delta)\Delta^2 + \frac{1}{24}L^{(1,1,1)}a(\Delta\hat{W}^2 - 3\,\Delta)\Delta\hat{W}\,\Delta \quad (30)$$

where $\beta, \gamma, \alpha \in [0, 1]$ and $\Delta\hat{W}$ and $\Delta\hat{Z}$ are as in (11).

5. MS-stability analysis of semi-implicit 3.0 schemes

In this section the stability properties of the proposed weak order 3.0 semi-implicit schemes is investigated. We shall see that, as in the deterministic case, the introduction of implicitness in Taylor schemes improves the numerical stability behavior. In the deterministic case the stability properties of a method are clarified applying it to the linear test equation

$$dX_t = \lambda X_t\, dt \quad (31)$$

with $\lambda \in \mathbb{C}$ and $\Re(\lambda) < 0$ and studying the asymptotic stability of the resulting difference equation. Then, to analyze the linear stability of a numerical scheme for the stochastic case, the test equation (31) needs to be generalized and the concept of asymptotic stability as well as its characterization in terms of the equation parameters must be established. For SDEs with multiplicative noise the usual linear scalar test equation, see for example [3,14,16,19–21], is

$$dX_t = \lambda X_t\, dt + \mu X_t\, dW_t \quad (32)$$

with $\lambda, \mu \in \mathbb{C}$ and $X_0 = x_0 \in \mathbb{R}$, $x_0 \neq 0$. The exact solution of (32) is given by $X_t = x_0 \exp\{(\lambda - \frac{1}{2}\mu^2)t + \mu\, W_t\}$. For this solution

$$E[|X_t|^2] = |x_0|^2 \exp\{(2\Re(\lambda) + |\mu|^2)t\},$$

and then the following property holds:

$$\lim_{t\to\infty} E[|X_t|^2] = 0 \quad \text{if and only if} \quad \Re(\lambda) + \frac{1}{2}|\mu|^2 < 0. \quad (33)$$

The condition on the left is known as mean-square (MS) stability, see e.g. [14,16,17]. The set

$$\mathcal{D} = \left\{(\lambda, \mu) \in \mathbb{C} \times \mathbb{C} : \Re(\lambda) + \frac{1}{2}|\mu|^2 < 0\right\} \quad (34)$$

is called the MS-stability domain of the stochastic equation (32). Notice that if $\mu = 0$ the condition on the right of (33) reduces to the deterministic A-stability condition $\Re(\lambda) < 0$ of the linear equation (31), see [22].

By analogy with the deterministic case, to analyze the stability of a stochastic scheme S we apply it to test equation (32), obtaining a recurrence of the form

$$Y_{n+1} = r_S(\lambda, \mu, \Delta, \Delta W)\, Y_n, \quad (35)$$

and study the behavior of the numerical solution $\{Y_n\}$. In the case of MS-stability we ask for the conditions leading to

$$\lim_{n\to\infty} E[|Y_n|^2] = 0. \quad (36)$$

Taking mean-square norm in (35) we obtain a difference equation

$$E[|Y_{n+1}|^2] = R_S(\lambda, \mu, \Delta)\, E[|Y_n|^2]$$

where $R_S(\lambda, \mu, \Delta) = E|r_S(\lambda, \mu, \Delta, \Delta W)|^2$. Then the analysis of condition (36) reduces to find for what values of the parameters $\lambda, \mu \in \mathbb{C}$ and of the step size $\Delta > 0$ the inequality $|R_S(\lambda, \mu, \Delta)| < 1$ is fulfilled. So, $R_S(\lambda, \mu, \Delta)$ is called the *stability function* of the scheme and the set

$$\mathcal{D}_S(\Delta) = \{(\lambda, \mu) \in \mathbb{C} \times \mathbb{C} : |R_S(\lambda, \mu, \Delta)| < 1\} \quad (37)$$

is called the *stability domain* of the stochastic method S (applied with step Δ). In [16] there appear the stability functions of a number of schemes.

In our case, applying the scheme (24) to test equation (32), the obtained recurrence is $Y_{n+1} = (A/B)\,Y_n$ with

$$\begin{aligned} A \;=\; & 1+(1-\theta)\lambda\Delta+\frac{1}{2}(1-\beta)(1-2\theta)\lambda^2\Delta^2+\frac{1}{6}(1-\gamma)(1-3\theta-3\beta(1-2\theta))\lambda^3\Delta^3 \\ & +\mu\left(1+(1-\theta)\lambda\Delta+\frac{1}{2}(1-\beta)(1-2\theta)\lambda^2\Delta^2\right)\Delta\hat{W} \\ & +\frac{1}{2}\,\mu^2\,(1+(1-\theta)\lambda\Delta)\,(\Delta\hat{W}^2-\Delta)+\frac{1}{6}\,\mu^3(\Delta\hat{W}^3-3\Delta\,\Delta\hat{W}); \\ B = & \; 1-\theta\lambda\Delta-\frac{1}{2}\,\beta(1-2\theta)\lambda^2\Delta^2-\frac{1}{6}\,\gamma(1-3\theta-3\beta(1-2\theta))\lambda^3\Delta^3. \end{aligned}$$

Then

$$\mathrm{E}[|Y_{n+1}|^2] = R_{(\theta,\beta,\gamma)}(\lambda,\mu,\Delta)\;\mathrm{E}[|Y_n|^2] \tag{38}$$

with $R_{(\theta,\beta,\gamma)}(\lambda,\mu,\Delta) = U/V$,

$$\begin{aligned} U \;=\; & \left|1+(1-\theta)\lambda\Delta+\frac{1}{2}(1-\beta)(1-2\theta)\lambda^2\Delta^2+\frac{1}{6}(1-\gamma)(1-3\theta-3\beta(1-2\theta))\lambda^3\Delta^3\right|^2 \\ & +\left|1+(1-\theta)\lambda\Delta+\frac{1}{2}(1-\beta)(1-2\theta)\lambda^2\Delta^2\right|^2|\mu|^2\Delta+\frac{1}{2}|1+(1-\theta)\lambda\Delta|^2|\mu|^4\Delta^2+\frac{1}{6}|\mu|^6\Delta^3; \\ V = & \left|1-\theta\lambda\Delta-\frac{1}{2}\beta(1-2\theta)\lambda^2\Delta^2-\frac{1}{6}\gamma(1-3\theta-3\beta(1-2\theta))\lambda^3\Delta^3\right|^2. \end{aligned} \tag{39}$$

In particular the simplified explicit 3.0 Taylor scheme ($\theta=\beta=\gamma=0$) in (10) has stability function

$$R_{(0,0,0)}(\lambda,\mu,\Delta) = \left|1+\lambda\Delta+\frac{1}{2}\lambda^2\Delta^2+\frac{1}{6}\lambda^3\Delta^3\right|^2+\left|1+\lambda\Delta+\frac{1}{2}\lambda^2\Delta^2\right|^2|\mu|^2\Delta+\frac{1}{2}|1+\lambda\Delta|^2|\mu|^4\Delta^2+\frac{1}{6}|\mu|^6\Delta^3$$

and the simplified implicit 3.0 Taylor scheme ($\theta=\beta=\gamma=1$) has stability function

$$R_{(1,1,1)}(\lambda,\mu,\Delta) = \frac{1+|\mu|^2\Delta+\frac{1}{2}|\mu|^4\Delta^2+\frac{1}{6}|\mu|^6\Delta^3}{\left|1-\lambda\Delta+\frac{1}{2}\lambda^2\Delta^2-\frac{1}{6}\lambda^3\Delta^3\right|^2}.$$

As in the deterministic case, Higham [14,15] emphasizes that the interest of the stability analysis of a numerical method lies not only in finding the parameters values for which the scheme is stable, but in the comparison of its stability domains $\mathcal{D}_S(\Delta)$ with the MS-stability domain of the stochastic test equation $\mathcal{D}$. In particular, the inclusion $\mathcal{D}\subseteq\mathcal{D}_S(\Delta)$ for a given scheme S and any $\Delta>0$ means that whenever the stochastic differential equation is stable then so is the scheme. In this case we can say that the scheme S is MS-stable and MS-stability can be seen as a generalization of deterministic A-stability, see [22], p. 224. On the other hand, the inclusion $\mathcal{D}_S(\Delta)\subseteq\mathcal{D}$ denotes that if the SDE is unstable then so is the numerical method applied with step size Δ.

Since the parameters λ,μ belong to $\mathbb{C}$, the stability domains are difficult to visualize. If λ,μ are restricted to real numbers the sets $\mathcal{D}_S(\Delta)$ and $\mathcal{D}$ are called regions instead domains; they are denoted by $\mathcal{R}_S(\Delta)$ and $\mathcal{R}$ respectively and can be represented in the real plane. Here, following [14], when $\lambda,\mu\in\mathbb{R}$ the stability regions will be drawn in the x–y plane with $x=\lambda\Delta$, $y=\mu^2\Delta$. For example, the stability region $\mathcal{R}$ in the x–y plane corresponds to the region $0<y<-2x$, represented by the (unbounded) squared area in the pictures of Figs. 1–3. And the stability regions $\mathcal{R}_{(0,0,0)}$ and $\mathcal{R}_{(1,1,1)}$ are the shaded areas shown in Fig. 1, left and right, respectively. A different representation of stability regions can be seen in [16], where $\hat{h}=\Delta\lambda$ and $k=-\mu^2/\lambda$ are taken as plane coordinates.

The following results show some instances of semi-implicit 3.0 schemes that are MS-stable:

Proposition 1. *The simplified semi-implicit* 3.0 *weak order Taylor schemes* (24) *with* $\theta=1$, $\beta=1$ *and* $\frac{1}{2}\le\gamma\le\frac{3}{4}$ *are MS-stable, i.e. these schemes give numerical MS-stable solutions for any step size* $\Delta>0$ *whenever they are applied to a linear test equation with mean-square stable solution.*

Proof. We have to prove that $\mathcal{D}\subset\mathcal{D}_{(1,1,\gamma)}(\Delta)$ for any $\Delta>0$ if $\frac{1}{2}\le\gamma\le\frac{3}{4}$. From (38)–(39) the stability function of schemes (24) with $\theta=1$ and $\beta=1$ is

$$R_{(1,1,\gamma)}(\lambda,\mu,\Delta) = \frac{\left|1+\frac{1-\gamma}{6}\lambda^3\Delta^3\right|^2+|\mu|^2\Delta+\frac{1}{2}|\mu|^4\Delta^2+\frac{1}{6}|\mu|^6\Delta^3}{\left|1-\lambda\Delta+\frac{1}{2}\lambda^2\Delta^2-\frac{\gamma}{6}\lambda^3\Delta^3\right|^2}.$$

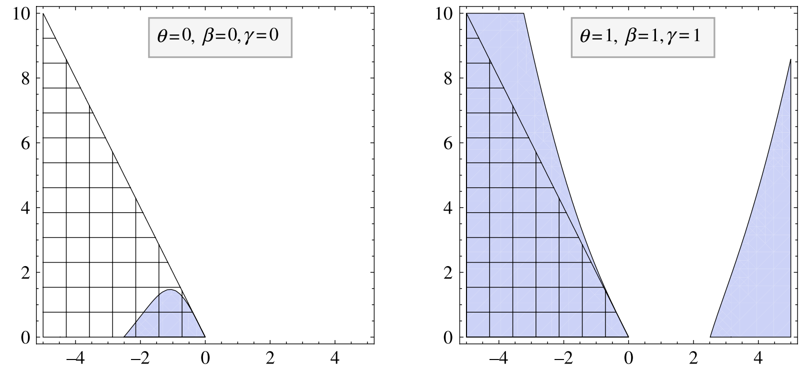

Fig. 1. Comparison between MS-stability region $\mathcal{R}$ (squared area) and MS-stability regions of semi-implicit 3.0 Taylor schemes (shaded area) with $\theta = \beta = \gamma = 0$ (left) and $\theta = \beta = \gamma = 1$ (right).

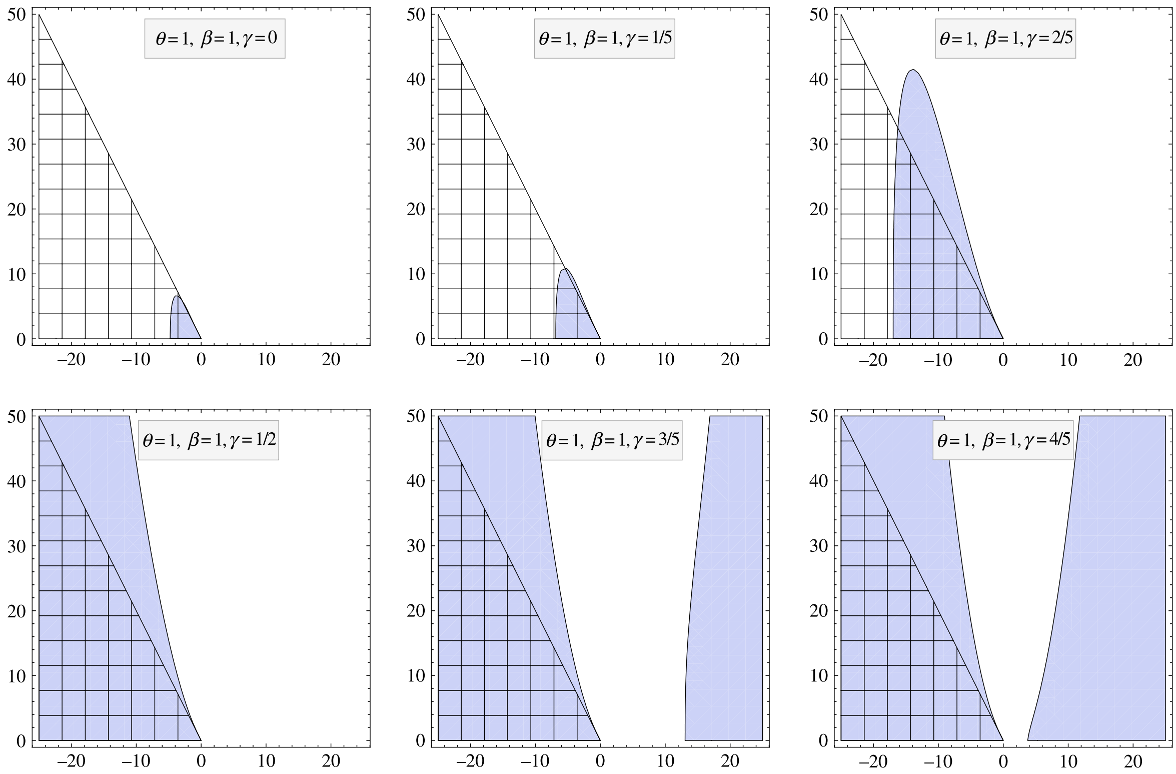

Fig. 2. Stability regions of semi-implicit 3.0 Taylor schemes (shaded) for $\theta = 1, \beta = 1$ and $\gamma = 0, \frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{4}{5}$ compared with the stability region of test equation (squared).

After some algebraic manipulation, the condition $R_{(1,1,\gamma)}(\lambda, \mu, \Delta) < 1$ can be written

$$2\left(\Re(\lambda) + \frac{1}{2}|\mu|^2\right)\left(1 + \frac{1}{2}|\mu|^2\Delta - \Re(\lambda)\Delta + \frac{2}{3}\Re(\lambda)^2\Delta^2 - \frac{1}{3}\Re(\lambda)|\mu|^2\Delta^2 + \frac{1}{6}|\mu|^4\Delta^2\right)$$
$$- \frac{1}{12}|\lambda|^2\Delta((3-4\gamma)|\lambda|^2\Delta^2 + 8\gamma\Re(\lambda)^2\Delta^2) + \frac{\gamma}{6}\Re(\lambda)|\lambda|^4\Delta^4 + \frac{1-2\gamma}{36}|\lambda|^6\Delta^5 < 0. \quad (40)$$

If $\Re(\lambda) + \frac{1}{2}|\mu|^2 < 0$, the product in the first line of (40) is negative. If in addition $\frac{1}{2} \le \gamma \le \frac{3}{4}$ all the remaining addends in (40) are negative. Then we have proven that for any $\Delta > 0$

$$\Re(\lambda) + \frac{1}{2}|\mu|^2 < 0 \Longrightarrow R_{(1,1,\gamma)}(\lambda, \mu, \Delta) < 1 \quad \text{if } \frac{1}{2} \le \gamma \le \frac{3}{4},$$

i.e. $\mathcal{D} \subset \mathcal{D}_{(1,1,\gamma)}(\Delta)$ if $\frac{1}{2} \le \gamma \le \frac{3}{4}$. □

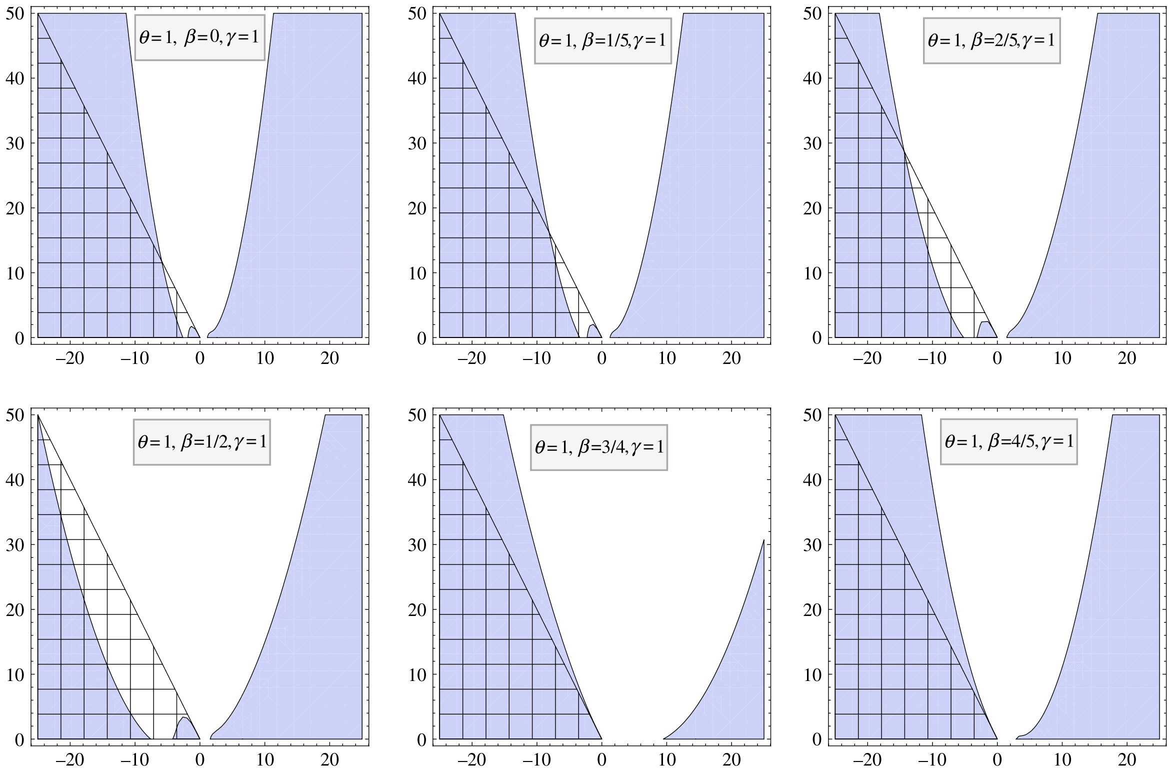

Fig. 3. Stability regions of semi-implicit 3.0 Taylor schemes (shaded) for $\theta = 1, \gamma = 1$ and $\beta = 0, \frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{3}{4}, \frac{4}{5}$ compared with the stability region of test equation (squared).

Notice that if $\gamma > 3/4$ there are values of $\lambda, \mu \in \mathbb{C}$ such that $\Re(\lambda) + \frac{1}{2}|\mu|^2 < 0$ and (40) is positive. For example, if $\gamma = 1$ the values $\lambda = -\frac{3}{5} + 10\mathrm{i}, \mu = 1$ and $\Delta = 10^{-1}$ give $\Re(\lambda) + \frac{1}{2}|\mu|^2 = -\frac{1}{10}$ and

$$R_{(1,1,1)}\left(-\frac{3}{5} + 10\mathrm{i}, 1, \frac{1}{10}\right) = \frac{621656250000}{609813683629} \approx 1.01942 > 1. \tag{41}$$

This proves that there exist values $\lambda, \mu \in \mathbb{C}$ such that the test stochastic equation (32) is stable but the method (24) with $\theta = 1, \beta = 1, \gamma = 1$ is not stable for some $\Delta > 0$.

On the other hand if $\lambda \in \mathbb{R}$ we have that the term

$$-\frac{1}{12}|\lambda|^2\Delta((3-4\gamma)|\lambda|^2\Delta^2 + 8\gamma\Re(\lambda)^2\Delta^2) = -\frac{4\gamma+3}{12}\lambda^4\Delta^3 < 0$$

in (40). Then if $\lambda \in \mathbb{R}, \gamma \geq 1/2$ and $\lambda + \frac{1}{2}|\mu|^2 < 0$, condition (40) holds. We have proven:

Corollary 1. *For any values $\lambda, \mu \in \mathcal{D}$ with $\lambda \in \mathbb{R}$ and any step size $\Delta > 0$ the simplified semi-implicit* 3.0 *weak order Taylor schemes* (24) *with $\theta = 1, \beta = 1$ and $\gamma \geq \frac{1}{2}$ give numerical MS-stable solutions.*

In particular, we have that $\mathcal{R} \subseteq \mathcal{R}_{(1,1,\gamma)}(\Delta)$ for any $\Delta > 0$ if $\gamma \geq \frac{1}{2}$: the MS-stability region of any semi-implicit 3.0 Taylor scheme with $\theta = 1, \beta = 1$ and $\gamma \geq \frac{1}{2}$ contains the region of MS-stability of the linear test equation. This result is confirmed in Fig. 2 comparing the region $\mathcal{R}$ (squared) with the mean-square stability regions $\mathcal{R}_{(1,1,\gamma)}$ (shaded) plotted for $\gamma = \frac{1}{2}, \frac{3}{5}, \frac{4}{5}$. See also the region $\mathcal{R}_{(1,1,1)}$ on the right plot of Fig. 1. Fig. 2 shows also that $\mathcal{R} \not\subseteq \mathcal{R}_{(1,1,\gamma)}(\Delta)$ for $\gamma = 0, \frac{1}{5}, \frac{2}{5}$.

Proposition 2. *The simplified semi-implicit* 3.0 *weak order Taylor schemes* (24) *with $\theta = 1, \gamma = 1$ and $\frac{3}{4} \leq \beta \leq \frac{5}{6}$ are MS-stable, i.e. these schemes give numerical MS-stable solutions for any step size $\Delta > 0$ whenever they are applied to a linear test equation with mean-square stable solution.*

Proof. We have to prove that $\mathcal{D} \subset \mathcal{D}_{(1,\beta,1)}(\Delta)$ for any $\Delta > 0$ if $\frac{3}{4} \leq \beta \leq \frac{5}{6}$. From (38)–(39) the stability function of schemes (24) with $\theta = 1$ and $\gamma = 1$ is

$$R_{(1,\beta,1)}(\lambda, \mu, \Delta) = \frac{\left|1 - \frac{1-\beta}{2}\lambda^2\Delta^2\right|^2 + \left|1 - \frac{1-\beta}{2}\lambda^2\Delta^2\right|^2 |\mu|^2\Delta + \frac{1}{2}|\mu|^4\Delta^2 + \frac{1}{6}|\mu|^6\Delta^3}{\left|1 - \lambda\Delta + \frac{\beta}{2}\lambda^2\Delta^2 - \frac{3\beta-2}{6}\lambda^3\Delta^3\right|^2}.$$

After some algebraic manipulation, the condition $R_{(1,\beta,1)}(\lambda,\mu,\Delta) < 1$ can be written

$$2\left(\Re(\lambda)+\frac{1}{2}|\mu|^2\right)\left(1-\Re(\lambda)\Delta+\frac{1}{2}|\mu|^2\Delta+(1-\beta)\Im(\lambda)^2\Delta^2+\frac{3\beta-1}{3}\Re(\lambda)^2\Delta^2-\frac{1}{3}\Re(\lambda)|\mu|^2\Delta^2+\frac{1}{6}|\mu|^4\Delta^2\right)$$
$$+\frac{6\beta-5}{12}|\lambda|^4\Delta^3+2\left(\frac{2}{3}-\beta\right)|\lambda|^2\Delta^3\Re(\lambda)^2+\frac{1}{2}|\lambda|^4\Delta^4\left(\frac{\beta(3\beta-2)}{3}\Re(\lambda)+\frac{(1-\beta)^2}{2}|\mu|^2\right)$$
$$-\frac{(3\beta-2)^2}{36}|\lambda|^6\Delta^5<0. \qquad (42)$$

If $\Re(\lambda)+\frac{1}{2}|\mu|^2<0$ and $\beta\geq 1/3$ it is clear that the product in the first line of (42) is negative. If in addition $6\beta-5\leq 0$, $2/3-\beta\leq 0$ and $\beta(3\beta-2)\geq 3(1-\beta)^2$ then all the remaining addends in (42) are negative. This proves that if $\frac{3}{4}\leq\beta\leq\frac{5}{6}$ then $\mathcal{D}\subset\mathcal{D}_{(1,\beta,1)}(\Delta)$ for any $\Delta>0$. □

Notice that if $\lambda\in\mathbb{R}$ the inequality (42) becomes

$$2\left(\lambda+\frac{1}{2}|\mu|^2\right)\left(1-\lambda\Delta+\frac{1}{2}|\mu|^2\Delta+\frac{3\beta-1}{3}\lambda^2\Delta^2-\frac{1}{3}\lambda|\mu|^2\Delta^2+\frac{1}{6}|\mu|^4\Delta^2\right)+\frac{11-18\beta}{12}\lambda^4\Delta^3$$
$$+\frac{1}{2}\lambda^4\Delta^4\left(\frac{\beta(3\beta-2)}{3}\lambda+\frac{(1-\beta)^2}{2}|\mu|^2\right)-\frac{(3\beta-2)^2}{36}\lambda^6\Delta^5<0,$$

which is obviously true if $\lambda+\frac{1}{2}|\mu|^2<0$ and $\beta\geq 3/4$. Then we have proven

Corollary 2. *For any values $\lambda,\mu\in\mathcal{D}$ with $\lambda\in\mathbb{R}$ and any step size $\Delta>0$ the simplified semi-implicit 3.0 weak order Taylor schemes (24) with $\theta=1,\gamma=1$ and $\beta\geq\frac{3}{4}$ give numerical MS-stable solutions.*

In particular, the MS-stability region of any semi-implicit 3.0 Taylor scheme with $\theta=1,\gamma=1$ and $\beta\geq\frac{3}{4}$ contains the MS-stability region of the linear test equation:

$$\mathcal{R}\subseteq\mathcal{R}_{(1,\beta,1)}\quad\text{if }\beta\geq\frac{3}{4}.$$

This result is confirmed in Fig. 3 comparing the region $\mathcal{R}$ (squared) with the mean-square stability regions $\mathcal{R}_{(1,\beta,1)}$ (shaded) plotted for $\beta=\frac{3}{4},\frac{4}{5}$. See also the region $\mathcal{R}_{(1,1,1)}$ on the right of Fig. 1. Notice also in Fig. 3 that if $\beta=0,\frac{1}{5},\frac{2}{5},\frac{1}{2}$. then $\mathcal{R}\not\subseteq\mathcal{R}_{(1,\beta,1)}(\Delta)$.

6. Numerical experiments

To verify the previous order and stability analysis, a number of numerical experiments have been carried out and some of them are presented in this section. In the first subsection we compare some of the proposed schemes with the Euler and the simplified weak order 2.0 Taylor methods, see [1], to verify their weak order of convergence. In the second subsection different examples that confirm the stability behavior of the order 3.0 schemes (24), denoted here by $M_{(\theta,\beta,\gamma)}$, are shown.

6.1. Numerical experiments confirming the weak order

To verify the weak order of convergence of the new schemes we have used the problems

(a) $dX_t=\left(\frac{1}{3}X_t^{1/3}+6X_t^{2/3}\right)dt+X_t^{2/3}dW_t;\ X_0=1;$

(b) $\begin{pmatrix}dX_t^1\\ dX_t^2\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}X_t^1\\X_t^2\end{pmatrix}dt+\begin{pmatrix}0\\1\end{pmatrix}dW_t;\ X_0^1=X_0^2=0.$

For equation (a) we have estimated the exact value $E[X_1]=28$ with Euler, Taylor 2.0, $M_{(0,0,0)}$ and $M_{(1,1,1)}$ schemes using 50,000 trials with steps $\Delta=2^{-1},2^{-2},2^{-3},2^{-4},2^{-5}$. The left-hand picture in Fig. 4 shows $\log_2|\mu|$ against $\log_2(\Delta)$ where $\mu=\mathrm{E}[Y_1]-\mathrm{E}[X_1]$ is the mean error. For equation (b) the noise is additive; we have estimated $E[(X_1^1)^2+(X_1^2)^2]=1$ using Euler, Taylor 2.0, $M_{(0,0,0)}$, $M_{(1,1,1)}$ and $M_{(0,0,0,0)}$ schemes. In this case we have used $\Delta=2^{-2},2^{-3},2^{-4},2^{-5}$. The right-hand picture in Fig. 4 shows $\log_2|\mu|$ against $\log_2(\Delta)$ where $\mu=\mathrm{E}[(Y_1^1)^2+(Y_2^1)^2]-\mathrm{E}[(X_1^1)^2+(X_2^1)^2]$.

6.2. Stability numerical experiments with semi-implicit 3.0 schemes

For each problem we have used 5000 simulations and a constant step size Δ to calculate $E[|X_t|^2]$ or $E[X_t]$. The computations were made with 16 significant digits. We have applied $M_{(0,0,0)}$, $M_{(1,\frac{3}{4},1)}$, $\mathrm{M}_{(1,1,\frac{1}{2})}$ and $M_{(1,1,1)}$ to calculate $\|X_t\|^2=E[|X_t|^2]$, where X_t is the solution of test equation (32) with initial condition $X_0=1$ for the following cases:

(i) $\lambda=-10+5i,\mu=2,\Delta=0.1$
(ii) $\lambda=-10+5i,\mu=4,\Delta=0.1.$
(iii) $\lambda=-\frac{3}{5}+10i,\mu=1,\Delta=0.1.$

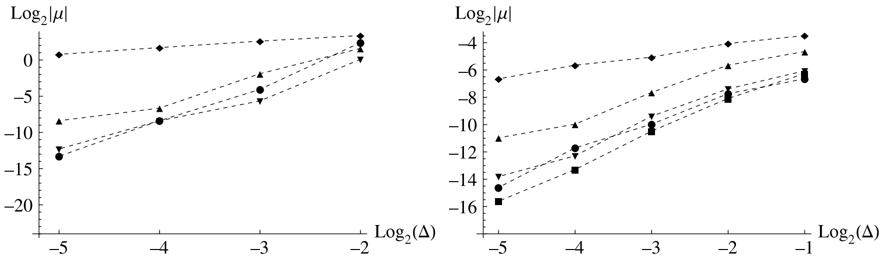

Fig. 4. Comparison of the weak order of convergence of Euler (♦), Taylor 2.0 (▲), $M_{(0,0,0)}$ (▼), $M_{(1,1,1)}$(●) and $M_{(0,0,0,0)}$ (■) schemes using problem (a), left-hand plot; problem (b), right-hand plot.

Table 1
Values of $E|X_t|^2$ using weak order 3.0 schemes $M_{(0,0,0)}$, $M_{(1,\frac{3}{4},1)}$, $M_{(1,1,\frac{1}{2})}$ and $M_{(1,1,1)}$ for test examples (i)–(ii).

t	$\|X\|^2 = E\|X_t\|^2$							
	Problem (i)				Problem (ii)			
	$M_{(0,0,0)}$	$M_{(1,\frac{3}{4},1)}$	$M_{(1,1,\frac{1}{2})}$	$M_{(1,1,1)}$	$M_{(0,0,0,0)}$	$M_{(1,\frac{3}{4},1)}$	$M_{(1,1,\frac{1}{2})}$	$M_{(1,1,1)}$
0.2	0.061472	0.038971	0.040436	0.037943	2.466417	0.280343	0.26601	0.243947
0.4	0.003817	0.001542	0.001663	0.001465	1.381881	0.048333	0.045591	0.038516
0.6	0.000917	0.000344	0.000373	0.000317	1.423551	0.031015	0.030338	0.024744
0.8	1.1006E−5	1.623E−6	1.930E−6	1.510E−6	0.235228	3.047E−5	2.218E−5	1.545E−5
1.0	7.3844E−7	8.105E−8	1.01239E−7	7.430E−8	0.344000	7.683E−7	9.808E−7	6.025E−7

Table 2
Values of $E|X_t|^2$ using weak order 3.0 schemes $M_{(0,0,0)}$, $M_{(1,\frac{3}{4},1)}$, $M_{(1,1,\frac{1}{2})}$ and $M_{(1,1,1)}$ for test example (iii).

t	$\|X\|^2 = E\|X_t\|^2$			
	Problem (iii)			
	$M_{(0,0,0)}$	$M_{(1,\frac{3}{4},1)}$	$M_{(1,1,\frac{1}{2})}$	$M_{(1,1,1)}$
0.2	0.868594	0.892386	0.817473	1.02465
0.4	0.773260	0.814360	0.683251	1.073865
0.6	0.724794	0.772012	0.620358	1.091651
0.8	0.576438	0.648247	0.456552	1.126469
1.0	0.516061	0.583764	0.376955	1.16737
2.0	0.176409	0.271111	0.114564	1.075440
3.0	0.055511	0.129726	0.035351	1.008479

The results are summarized in Tables 1 and 2. We have the following analysis:

(i) Since $|R_{(0,0,0)}(-10+5i, 2, 0.1)| = 0.250545$, $|R_{(1,\frac{3}{4},1)}(-10+5i, 2, 0.1)| = 0.206235$, $|R_{(1,1,\frac{1}{2})}(-10+5i, 2, 0.1)| = 0.209467$, and $|R_{(1,1,1)}(-10+5i, 2, 0.1)| = 0.202781$, the four schemes are MS-stable.

(ii) In this case $|R_{(0,0,0)}(-10+5i, 4, 0.1)| = 1.3913$, $|R_{(1,\frac{3}{4},1)}(-10+5i, 4, 0.1)| = 0.676149$, $|R_{(1,1,\frac{1}{2})}(-10+5i, 4, 0.1)| = 0.649436$, and $|R_{(1,1,1)}(-10+5i, 4, 0.1)| = 0.620676$, which means that the explicit scheme $M_{(0,0,0)}$ is unstable and the three semi-implicit schemes are stable.

(iii) This example illustrates the discussion in (41). From (41) the scheme $M_{(1,1,1)}$ gives an unstable solution. The other three are stable because $|R_{(0,0,0)}(-\frac{3}{5}+10i, 1, 0.1)| = 0.940856$, $|R_{(1,\frac{3}{4},1)}(-\frac{3}{5}+10i, 1, 0.1)| = 0.951007$, $|R_{(1,1,\frac{1}{2})}(-\frac{3}{5}+10i, 1, 0.1)| = 0.910336$. Notice that the convergence of these schemes is slow, due to their stability function values are near to 1.

7. Conclusions

The integration of stiff SDEs requires the use of schemes with good stability properties, such as implicit methods. In this paper, using stochastic Taylor expansions (as Platen in [11]) and weak equivalences we have obtained general schemes of weak order 3.0 and 4.0 for the multidimensional case with multidimensional noise. These schemes contain multiple integrals that are difficult to implement. Then, for applications, these integrals need to be replaced by simpler random variables, obtaining the so-called simplified schemes. In this way, for the multidimensional case with one-dimensional noise we present an infinite family of semi-implicit simplified schemes of weak order 3.0. Notice that the explicit Taylor scheme of weak order 3.0 proposed in [1] is a member of this family. For the multidimensional case with additive one-dimensional noise we present an infinite family of semi-implicit simplified schemes of weak order 4.0 containing the explicit scheme presented in [18]. Using a linear test equation with multiplicative noise, the mean-square stability of the 3.0 family has been analyzed, concluding that, as in the 1.0 Taylor schemes, see [15], for higher order schemes the stability behavior improves when the degree of implicitness grows. These theoretical results have been verified with numerical experiments.

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