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Integrated Semigroups

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The Banach space valued inhomogeneous Cauchy problem

$$u'(t) = Au(t) + f(t)$$
$$u(0) = x$$

for a (non-densely defined) linear operator A is treated with the help of the theory of integrated semigroups. New well-posedness results are obtained for differential operators and in particular the Schrödinger operator $i\Delta$.

INTRODUCTION

In this paper we shall be concerned with linear Cauchy problems

$$u'(t) = Au(t)$$
$$u(0) = x$$

for functions $t \to u(t)$ with values in a Banach space E. Thereby we present a concept introduced recently by W. Arendt [5]. His idea may be described as follows: Let $(T_t)_{t \geq 0}$ be a C_0-semigroup on E. Then $S_t := \int_0^t T_s\, ds$ defines a family $(S_t)_{t \geq 0}$ of bounded operators having the following three properties.

(a) $S_0 = 0$.

(b) $t \to S_t$ is strongly continuous.

(c) $S_s S_t = \int_0^s (S_{r+t} - S_r)\, dr \quad$ for $\quad s, t \geq 0$.

He called "integrated semigroup" an operator family satisfying (a)–(c). The

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first application of this concept was a result about resolvent positive operators (see [3] and [5]).

The aim of this paper is twofold: We make some contributions to the structure theory of integrated semigroups and characterize (non-densely defined) operators satisfying the Hille–Yosida condition as generators of locally Lipschitz continuous integrated semigroups. As an application we give an easy proof of a theorem due to Da Prato and Sinestrari [10] on the inhomogeneous Cauchy problem associated to such operators (Section 2). In Section 3 we consider bounded perturbations of generators of integrated semigroups.

The remaining section is devoted to applications. We mention a typical result: Whereas the Schrödinger operator A given by $Af = if''$ generates a C_0-semigroup on $L^p(\mathbb{R})$ only if $p=2$, it generates an integrated semigroup on $L^p(\mathbb{R})$ for all $1 \leqslant p \leqslant \infty$.

1. Integrated Semigroups and Their Generators

Definition 1.1. Let E be a Banach space. An *integrated semigroup* is a family $(S_t)_{t \geqslant 0}$ of bounded linear operators S_t on E with the following properties:

(a) $S_0 = 0$.

(b) $t \to S_t$ is strongly continuous.

(c) $S_s S_t = \int_0^s (S_{r+t} - S_r)\, dr$ for $s, t \geqslant 0$.

Examples, Remarks

(1) Let $(T_t)_{t \geqslant 0}$ be a C_0-semigroup on a Banach space E. Then $(S_t)_{t \geqslant 0} := (\int_0^t T_s\, ds)_{t \geqslant 0}$ defines an integrated semigroup on E.

(2) Let again $(T_t)_{t \geqslant 0}$ be a C_0-semigroup on a Banach space E. Then $(S_t)_{t \geqslant 0} := (\int_0^t T_s^*\, ds)_{t \geqslant 0}$ (where T_s^* is the adjoint of T_s) is an integrated semigroup on E'. In general $(S_t)_{t \geqslant 0}$ is not an integrated C_0-semigroup.

(3) Let $(\mathrm{Cos}(t))_{t \in \mathbb{R}}$ be a cosine operator function on a Banach space E (see, e.g., [11]). If we define $\mathrm{Sin}(t) := \int_0^t \mathrm{Cos}(s)\, ds$ then by

$$S_t = \begin{pmatrix} \mathrm{Sin}(t) & \int_0^t \mathrm{Sin}(s)\, ds \\ \mathrm{Cos}(t) - I & \mathrm{Sin}(t) \end{pmatrix}$$

an integrated semigroup $(S_t)_{t \geqslant 0}$ is given on $E \times E$.

(4) A short calculation shows that the functional equation 1.1(c) can also be written as

$$S_s S_t = \int_0^{s+t} S_r\,dr - \int_0^s S_r\,dr - \int_0^t S_r\,dr \qquad \text{for} \quad s, t \geqslant 0.$$

A consequence of this is $S_s S_t = S_t S_s$ for $s, t \geqslant 0$.

(5) Let P be a bounded projection on a Banach space E. Then $(S_t)_{t \geqslant 0}$ for $S_t := tP$ is an integrated semigroup. If $P \neq I$ then $(S_t)_{t \geqslant 0}$ is not an integrated C_0-semigroup. Moreover $(S_t)_{t \geqslant 0}$ has a property that plays a disturbing role: $(S_t)_{t \geqslant 0}$ vanishes on a nontrivial subspace. Later we will call that property "degeneracy" and exclude these integrated semigroups from further considerations.

From now on let E be a Banach space and $(S_t)_{t \geqslant 0}$ an integrated semigroup on E.

Motivated from the Laplace transform theory we want to define the generator of an integrated semigroup as

$$A := \lambda - R_\lambda^{-1} \qquad \text{where} \quad R_\lambda = \lambda \int_0^\infty e^{-\lambda t} S_t\,dt.$$

The following example shows that this integral does not exist in general.

EXAMPLE 1.2. Let $E = l^2$, $S_t\colon (x_n) \to (\int_0^t \exp(a_n s)\,ds\,x_n)$, where $a_n := n + 2^{n^2}\pi i$. We show first that $(S_t)_{t \geqslant 0}$ is an integrated semigroup.

(1) $S_t \in L(E)$: $|\int_0^t \exp(a_n s)\,ds| = |\exp(a_n t) - 1|/|a_n| \leqslant (e^{nt} + 1)/e^{n^2 \log 2} \leqslant 2e^{nt}/e^{n^2 \log 2} = 2e^{nt - n^2 \log 2} \leqslant 2e^{t^2/4 \log 2}$.

(2) $t \to S_t$ is strongly continuous. Obviously $t \to S_t e_n$, where $e_n = (\delta_{in})$, is continuous. Since $\{e_n : n \in \mathbb{N}\}$ is total in l^2 and $(S_t)_{t \geqslant 0}$ is uniformly bounded on compact t-intervals (see (1)) the strong continuity of $(S_t)_{t \geqslant 0}$ follows.

(3) The functional equation is clearly fulfilled and $S_0 = 0$.

Therefore $(S_t)_{t \geqslant 0}$ is an integrated semigroup. But for all $\lambda \in \mathbb{C}$ the Laplace integral does not even converge as a weak integral.

For given $\lambda \in \mathbb{C}$ $\langle \int_0^\alpha e^{-\lambda t} S_t\,dt\,x, y \rangle$ diverges (as $\alpha \to \infty$) for $x = (1/\overline{a_n})$ and $v = (1/(a_n - \lambda))$:

$$\left\langle \int_0^\alpha e^{-\lambda} S_t\,dt\,x, y \right\rangle$$
$$= \sum_{n=1}^\infty \int_0^\alpha e^{-\lambda t} \int_0^t \exp(a_n s)\,ds\,dt / \overline{a_n (a_n - \lambda)})$$

$$= \sum_{n=1}^{\infty} \exp((a_n-\lambda)\,\alpha)/(|a_n|\,|a_n-\lambda|)^2 - \sum_{n=1}^{\infty} 1/(|a_n|\,|a_n-\lambda|)^2$$

$$- \sum_{n=1}^{\infty} \int_0^{\alpha} e^{-\lambda t}\,dt/(|a_n|^2\,(\overline{a_n-\lambda})).$$

The last two terms converge for $\alpha \to \infty$, therefore we only consider the first term. For $\alpha \in \mathbb{N}$ we obtain by the definition of a_n

$$\sum_{n=1}^{\infty} \exp((a_n-\lambda)\,\alpha)/(|a_n|\,|a_n-\lambda|)^2$$

$$= e^{-i(\operatorname{Im}\lambda)\alpha} \sum_{n=1}^{\infty} e^{(n-\operatorname{Re}\lambda)\alpha}/(|a_n|\,|a_n-\lambda|)^2.$$

For $\alpha \to \infty$ every summand with index $n > \operatorname{Re}\lambda$ diverges to infinity and therefore the series diverges.

A similar estimate shows that this integrated semigroup is even continuous with respect to the operator norm.

We introduce now the appropriate concept in order to obtain a neat theory.

An integrated semigroup $(S_t)_{t\geqslant 0}$ is called *exponentially bounded*, if there exist constants $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that $\|S_t\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$. Moreover $(S_t)_{t\geqslant 0}$ is called *non-degenerate*, if $S_t x = 0$ for all $t \geqslant 0$ implies $x = 0$. (Compare Remark 5 after Definition 1.1).

Then the Laplace integral $R_\lambda := \lambda \int_0^\infty e^{-\lambda t} S_t\,dt$ exists for all λ with $\operatorname{Re}\lambda > \omega$, but R_λ is injective if and only if $(S_t)_{t\geqslant 0}$ is non-degenerate. In this case there exists a unique operator A satisfying $(\omega, \infty) \subset \rho(A)$ (the resolvent set of A), such that $R_\lambda = (\lambda - A)^{-1}$ for all $\lambda > \omega$. This operator is called the *generator* of $(S_t)_{t\geqslant 0}$.

Usually the given object is the operator. An operator A is called generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $(S_t)_{t\geqslant 0}$ of bounded operators such that $S(0) = 0$ and $(\lambda - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S_t\,dt$.

The following proposition shows that in that case $(S_t)_{t\geqslant 0}$ is automatically an integrated semigroup.

PROPOSITION 1.3 (Arendt [4, Thm. 3.1]). *Let $(S_t)_{t\geqslant 0}$ be a strongly continuous family of bounded operators such that $S(0) = 0$ and $\|S_t\| \leqslant Me^{\omega t}$ for all $t \geqslant 0$.*

Then $(S_t)_{t\geqslant 0}$ is an integrated semigroup if and only if $R_\lambda = \lambda \int_0^\infty e^{-\lambda t} S_t\,dt$ is a pseudo-resolvent.

Moreover, we will need the following relation between integrated semigroups and their generators.

PROPOSITION 1.4 (Arendt [4, prop. 3.3]). *Let A be the generator of an integrated semigroup $(S_t)_{t \geqslant 0}$.*

Then for all $x \in E$ and $t \geqslant 0$

$$\int_0^t S_s x \, ds \in D(A) \qquad \textit{and} \qquad S_t x = A \int_0^t s_s x \, ds + tx.$$

The following theorem characterizes densely defined generators of integrated semigroups.

THEOREM 1.5 (Arendt [4]; Neubrander [17]). *Let $A: D(A) \to E$ be densely defined. The following assertions are equivalent:*

(a) *A is the generator of an integrated semigroup.*

(b) *There are real constants M, ω such that $(\omega, \infty) \subset \rho(A)$ and*

$$\left\| \left(\frac{d}{d\lambda} \right)^n (\lambda - A)^{-1}/\lambda) \right\| \leqslant n! M/(\lambda - \omega)^{n+1} \qquad \textit{for} \quad n \in \mathbb{N} \cup \{0\} \quad \textit{and} \quad \lambda > \omega.$$

Remarks 1.6. (a) Condition (b) is necessary for A to be the generator of an integrated semigroup even if A is not densely defined (Arendt [4]).

(b) We state explicitly two properties of R_λ which will be useful in proving perturbation theorems: If A is the generator of an integrated semigroup, then there exist constants C, $C' \geqslant 0$ such that $\| R_\lambda \| \leqslant C$ and $\| R_\lambda^2 \| \leqslant C'/(\lambda - \omega)$.

Generators of integrated semigroups are of interest since the Cauchy problem associated with them has a unique solution for a large class of initial values. More precisely the following holds.

Let A be the generator of an integrated semigroup $(S_t)_{t \geqslant 0}$ on E and $b > 0$. Further let $f: [0, b] \to E$ be a continuous function. By a solution of

$$\begin{aligned} u'(t) &= Au(t) + f(t), \qquad t \in [0, b] \\ u(0) &= x \end{aligned} \tag{CP}$$

we understand a function $u \in C^1([0, b], E)$, such that $u(t) \in D(A)$ and (CP) holds for all $t \in [0, b]$.

Consider the function $v \in C([0, b], E)$ given by

$$v(t) = S_t x + \int_0^t S_{t-s} f(s) \, ds.$$

Using this notation the following holds.

THEOREM 1.7 (Arendt [4, Thm. 5.2]). *If* $v \in C^2([0, b], E)$, *then* $u := v'$ *is the unique solution of* (CP).

In particular, if $f = 0$, the theorem implies that there is a unique solution of (CP) whenever $x \in D(A^2)$.

2. LOCALLY LIPSCHITZ CONTINUOUS INTEGRATED SEMIGROUPS

In this section we present a direct proof of a well-posedness result due to Da Prato and Sinestrari [10] for Cauchy problems with non-densely defined operators.

DEFINITION 2.1. An integrated semigroup $(S_t)_{t \geq 0}$ is called locally Lipschitz continuous if for all $b > 0$ there exists a constant L, such that $\|S_t - S_s\| \leq L|t - s|$ for all $s, t \in [0, b]$.

The following proposition shows that locally Lipschitz continuous integrated semigroups have an exponentially bounded Lipschitz constant.

PROPOSITION 2.2. *Suppose that* $(S_t)_{t \geq 0}$ *is a non-degenerate locally Lipschitz continuous integrated semigroup. Then*

$$\overline{\lim_{h \to 0}} \|S_{t+h} - S_t\|/h \leq Me^{\omega t} \qquad \textit{for} \quad t \geq 0$$

for suitable constants M *and* ω.

Proof. Let $E_1 := \{x \in E : t \to S_t x$ is continuously differentiable on $[0, \infty)\}$. Then E_1 is a closed subspace of E. In fact, let $b > 0$. Observe that the space $C_0^1 := \{f \in C^1[0, b], f(0) = 0\}$ is closed in the Banach space $\mathrm{Lip}_0 := \{f \in \mathrm{Lip}[0, b], f(0) = 0\}$ equipped with the Lipschitz norm. Now, if $x = \lim x_n$ in E, where $x_n \in E_1$, then the functions $f_n \in C_0^1$ defined by $f_n(t) := S_t x_n$ converge to $f \in \mathrm{Lip}_0$ given by $f(t) = S_t x$.

Let $x \in E_1$. Differentiating

$$S_s S_t x = \int_0^{s+t} S_r x \, dr - \int_0^s S_r x \, dr - \int_0^t S_r x \, dr$$

one sees that $(d/dt)\, S_s S_t x = S_{t+s} x - S_t x - S_t x$ $(s, t \geq 0)$. In particular $S_t' x := (d/dt)\, S_t x$ defines a bounded operator on E_1 such that

$$S_s S_t' x = S_{t+s} x - S_t x \qquad (x \in E_1).$$

Differentiating again, one obtains

$$S'_s S'_t = S'_{t+s} \qquad (t, s \geqslant 0)$$

and $S_s S'_0 x = S_s x$ $(s \geqslant 0)$, which implies $S'_0 = I$ on E_1.

Thus $(S'_s)_{s \geqslant 0}$ is a strongly continuous semigroup on E_1. So there exist $M \geqslant 0$, $\omega \in \mathbb{R}$, such that

$$\|(S'_s - S'_t)\, x\| \leqslant M e^{\omega t}\, \|x\|.$$

Let $0 \leqslant h \leqslant b$ and let L be the Lipschitz constant for the interval $[0, b]$. Then this implies that

$$\|S_{t+h} - S_t\| = \|S'_t S_h\| \leqslant M e^{\omega t} L h. \quad ∎$$

COROLLARY 2.3. *Every locally Lipschitz continuous integrated semigroup is exponentially bounded.*

Together with [4, Theorem 4.1] this shows that the Hille–Yosida condition (without the assumption of a dense domain) characterizes generators of locally Lipschitz continuous integrated semigroups. More precisely, the following holds:

THEOREM 2.4. *The following assertions are equivalent:*

(a) *A is the generator of a locally Lipschitz continuous integrated semigroup.*

(b) *There exist real constants M, ω such that $(\omega, \infty) \subset \rho(A)$ and*

$$\|(\lambda - A)^{-n}\| \leqslant M/(\lambda - \omega)^n \qquad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad \lambda > \omega.$$

With the help of Theorem 2.4, we obtain an easy proof of a theorem due to Da Prato and Sinestrari [10] on the inhomogeneous Cauchy Problem associated with a (non-densely defined) operator satisfying the Hill–Yosida condition.

THEOREM 2.5 (Da Prato and Sinestrari [10]). *Let $A: D(A) \to E$ be a linear operator, f: $[0, 1] \to E$, and $x \in D(A)$ such that*

(a) *there exist real constants M, ω such that $(\omega, \infty) \subset \rho(A)$ and*

$$\|(\lambda - A)^{-n}\| \leqslant M/(\lambda - \omega)^n \qquad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad \lambda > \omega.$$

(b) *$f(t) = f(0) + \int_0^t g(s)\, ds$ for some Bochner-integrable function g.*

(c) *$Ax + f(0) \in \overline{D(A)}$.*

Then there exists a unique function $u \in C^1([0, 1], E)$, *such that* $u(t) \in D(A)$ *and*

$$u'(t) = Au(t) + f(t), \qquad t \in [0, 1]$$
$$u(0) = x.$$

Proof. Assumption (a) means by Theorem 2.4 that A generates a locally Lipschitz continuous integrated semigroup $(S_t)_{t \geqslant 0}$.

By Theorem 1.7, it suffices to show that the function v given by

$$v(t) = S_t x + \int_0^t S_s f(t-s)\, ds$$

is twice continuously differentiable. Using (b) one obtains

$$v(t) = S_t x + \int_0^t S_s f(0)\, ds + \int_0^t S_s \int_0^{t-s} g(r)\, dr\, ds.$$

Since $x \in D(A)$, by Proposition 1.4

$$v'(t) = x + S_t Ax + S_t f(0) + \int_0^t S_t\, g(t-s)\, ds.$$

The function $t \to S_t y$ is differentiable for all $y \in D(A)$ (by Proposition 1.4) and hence for all $y \in \overline{D(A)}$ because of the local Lipschitz continuity (cf. the proof of Proposition 2.2). Since $Ax + f(0) \in \overline{D(A)}$, it remains to show that $w(t) = \int_0^t S_t\, g(t-s)\, ds = \int_0^t S(t-s)\, g(s)\, ds$ is C^1. The function $s \to S_{t-s}\, g(s)$ is Bochner integrable.

Let L be the Lipschitz constant of $(S_t)_{0 \leqslant t \leqslant 1}$. Then

$$\begin{aligned}
\| w \|_{\mathrm{Lip}} &= \sup_{t, r \in [0, 1],\, r \neq t} \| w(t) - w(r) \|\, | t - r |^{-1} \\
&= \sup_{t, r \in [0, 1],\, r \neq t} \left\| \int_r^t S_{t-s}\, g(s)\, ds + \int_0^r (S_{t-s} - S_{r-s})\, g(s)\, ds \right\| | t - r |^{-1} \\
&\leqslant \sup_{t, r \in [0, 1],\, r \neq t} (L | t - r |\, \| g \|_{L^1} + L | t - r |\, \| g \|_{L^1}) | t - r |^{-1} \\
&= 2L\, \| g \|_{L^1}.
\end{aligned}$$

Further let (g_n) be a sequence of C^1-functions converging to g with respect to the L^1-norm. For $n \in \mathbb{N}$ and $t \in [0, 1]$ define

$$w_n(t) := \int_0^t S_{t-s}\, g_n(s)\, ds.$$

Then w_n is continuously differentiable and

$$\| w_n - w_m \|_{\text{Lip}} \leqslant 2L \, \| g_n - g_m \|_{L^1} \qquad (n, m \in \mathbb{N}).$$

Since C^1 is a closed subspace of Lip, it follows that $w \in C^1$. ∎

3. Perturbations

In this section we shall be concerned with additive perturbations of generators.

The new and surprising result, compared to C_0-semigroups and cosine functions, is that even bounded perturbations may fail to preserve the property "generator" (see Example 3.7). Nevertheless we will establish some positive perturbation results.

PROPOSITION 3.1 (Commuting Perturbations). *Let $A: D(A) \to E$ be the generator of an integrated semigroup $(S_t)_{t \geqslant 0}$. Further let $B \in L(\overline{D(A)})$ be such that $B(\lambda - A)^{-1} = (\lambda - A)^{-1} B$ for all large λ.*

Then $A + B$ is the generator of the integrated semigroup $(S_t^B)_{t \geqslant 0}$, given by

$$S_t^B = e^{tB} S_t - B \int_0^t e^{sB} S_s \, ds.$$

Proof. For certain constants M, ω we have $\| S_t \| \leqslant M e^{\omega t}$ and also $\| S_t^B \| \leqslant M e^{\omega t}$ for $t \geqslant 0$. Let, for $\lambda > \omega$, $R_\lambda^B := \lambda \int_0^\infty e^{-\lambda t} S_t^B \, dt$ and $R_\lambda := (\lambda - A)^{-1}$.

We show that $R_\lambda^B = R_\lambda (I - B R_\lambda)^{-1}$ for large λ. But $I - B R_\lambda$ is invertible for large λ since

$$\| (B R_\lambda)^2 \| = \| B^2 R_\lambda^2 \| \leqslant \| B^2 \| \, c/(\lambda - \omega) \qquad \text{(Remark 1.6(b))}.$$

Therefore we obtain

$$\begin{aligned} R_\lambda (I - B R_\lambda)^{-1} &= \sum_{i=0}^\infty B^i R_\lambda^{i+1} \\ &= \sum_{i=0}^\infty B^i (-1)^i \frac{1}{i!} \left(\frac{d}{d\lambda} \right)^i R_\lambda \\ &= \sum_{i=0}^\infty B^i (-1)^i \frac{1}{i!} \left(\frac{d}{d\lambda} \right)^i \left(\lambda \int_0^\infty e^{-\lambda t} S_t \, dt \right) \\ &= \sum_{i=0}^\infty B^i (-1)^i \frac{1}{i!} \lambda \left(\frac{d}{d\lambda} \right)^i \int_0^\infty e^{-\lambda t} S_t \, dt \\ &\quad + \sum_{i=0}^\infty B^i (-1)^i \frac{1}{i!} i \left(\frac{d}{d\lambda} \right)^{i-1} \int_0^\infty e^{-\lambda t} S_t \, dt \end{aligned}$$

$$= \lambda \sum_{i=0}^{\infty} B^i \frac{1}{i!} \int_0^\infty t^i e^{-\lambda t} S_t \, dt$$

$$- \sum_{i=1}^{\infty} B^i \frac{1}{(i-1)!} \int_0^\infty t^{i-1} e^{-\lambda t} S_t \, dt$$

$$= \lambda \int_0^\infty e^{-\lambda t} e^{tB} S_t \, dt - \int_0^\infty e^{-\lambda t} B e^{tB} S_t \, dt$$

$$= \lambda \int_0^\infty e^{-\lambda t} e^{tB} S_t \, dt - \lambda \int_0^\infty e^{-\lambda t} B \int_0^t e^{sB} S_s \, ds \, dt$$

$$= R_\lambda^B.$$

A short calculation now shows that R_λ^B is the resolvent of $A + B$. Together with Proposition 1.3 this implies that $(S_t^B)_{t \geqslant 0}$ is an integrated semigroup and $A + B$ is its generator.

Remark 3.2. A consequence of this theorem is as follows: If A is the generator of the integrated semigroup $(S_t)_{t \geqslant 0}$ then for any $\mu \in \mathbb{C}$ $A + \mu$ is the generator of the integrated semigroup $(S_t^\mu)_{t \geqslant 0}$ given by

$$S_t^\mu = e^{\mu t} S_t - \mu \int_0^t e^{\mu s} S_s \, ds.$$

PROPOSITION 3.3. *Let A be the generator of a locally Lipschitz continuous integrated semigroup and $B \in L(\overline{DA}, E)$.*

Then $A + B$ is the generator of a locally Lipschitz continuous integrated semigroup.

Proof. By Theorem 2.4 there exist constants $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that

$$\|(\lambda - \omega)^n (\lambda - A)^{-n}\| \leqslant M$$

for all $\lambda > \omega$ and $n \in \mathbb{N}$. By rescaling, we may suppose without loss of generality that $\omega = 0$. Further, according to the standard renorming lemma [18, Ch. I, Lemma 5.1], there exists an equivalent norm $\|\cdot\|_R$ on E such that

$$\|\lambda R(\lambda, A)\|_R \leqslant 1.$$

Note that $(\lambda - (A + B)) = (I - BR(\lambda, A))(\lambda - A)$. Since $\|BR(\lambda, A)\|_R \leqslant \|B\|_R/\lambda$, the operator $I - BR(\lambda, A)$ is invertible for large λ and

$$\|(I - BR(\lambda, A))^{-1}\|_R \, \|(\lambda - A)^{-1}\|_R \leqslant (1 - \|B\|_R \lambda^{-1})^{-1} \lambda^{-1} = (\lambda - \|B\|_R)^{-1}.$$

Therefore $\|\lambda R(\lambda, A+B-\|B\|_R)\|_R \leqslant 1$. Now Theorem 2.4 together with Remark 3.2 implies the assertion. ∎

Proposition 3.3 is closely related to the following fact, which can be proved by similar methods (see [14]).

PROPOSITION 3.4. *Let A be the generator of an integrated semigroup and $B \in L(E, D(A))$ (where $D(A)$ is equipped with the graph norm).*
Then $A+B$ generates an integrated semigroup.

As an application we consider generators of cosine functions (see [11], [12] for details). An operator A is the *generator of a cosine function*, if $(a, \infty) \subset \rho(A)$ *for some* $a \geqslant 0$ and there exists a strongly continuous exponentially bounded function C: $[0, \infty) \to L(E)$ such that

$$\lambda R(\lambda^2, A) = \int_0^\infty e^{-\lambda t} C(t)\, dt$$

for all sufficiently large λ. In that case, C is the cosine function generated by A.

THEOREM 3.5. *Let $A: D(A) \to E$ be a linear operator. The following assertions are equivalent*:

(i) *A is the generator of a cosine function.*

(ii) *$\mathscr{A} := \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ generates an integrated semigroup on the Banach space $E \times E$.*

Proof. (i)$\Rightarrow$(ii) Let $(\mathrm{Cos}(t))_{t \in \mathbb{R}}$ be the cosine function corresponding to the operator A and further let $\mathrm{Sin}(t) := \int_0^t \mathrm{Cos}(s)\, ds$. Then the family $(S_t)_{t \geqslant 0}$, given by

$$S_t := \begin{pmatrix} \mathrm{Sin}(t) & \int_0^t \mathrm{Sin}(s)\, ds \\ \mathrm{Cos}(t) - I & \mathrm{Sin}(t) \end{pmatrix}$$

satisfies $\|S_t\| \leqslant M' e^{\omega' t}$ for $t \geqslant 0$ (M' and ω' suitable). After a short calculation one obtains

$$\lambda \int_0^\infty e^{-\lambda t} S_t\, dt = \begin{pmatrix} \lambda & 1 \\ A & \lambda \end{pmatrix} (\lambda^2 - A)^{-1} \qquad \text{for} \quad \lambda > \omega'.$$

The term on the right-hand side is just $(\lambda - \mathscr{A})^{-1}$. Therefore $(S_t)_{t \geqslant 0}$ is an integrated semigroup and $\mathscr{A}$ its generator.

(ii)$\Rightarrow$(i) Suppose that $\mathscr{A}$ generates an integrated semigroup. Then $\lambda \to \lambda^{-1}(\lambda - \mathscr{A})^{-1}$ is a Laplace transform. It is not hard to see that the entry in the lower left-hand corner is $\lambda^{-1}A(\lambda^2 - A)^{-1}$. Hence $\lambda(\lambda^2 - A)^{-1}$ is a Laplace transform as well, ▮

COROLLARY 3.6. *If A is the generator of a cosine function, then $A + B$ generates a cosine function for every $B \in L(E)$.*

Proof. The operator-matrix associated with $A + B$ on $E \times E$ is given by

$$\mathscr{A} + \mathscr{B} := \begin{pmatrix} 0 & I \\ A+B & 0 \end{pmatrix} \quad \text{where} \quad \mathscr{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \quad \text{and} \quad \mathscr{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$$

with domains $D(\mathscr{A}) = D(A) \times E$ and $D(\mathscr{B}) = E \times E$. Since $\mathscr{B} \in L(E \times E, D(\mathscr{A}))$, the assertion follows from Proposition 3.4. ▮

EXAMPLE 3.7. If an operator A generates a cosine function, then $\left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ generates an integrated semigroup on $E \times E$ (Theorem 3.5). The bounded perturbation $\left(\begin{smallmatrix} 0 & -I \\ 0 & 0 \end{smallmatrix}\right)$ leads to $\left(\begin{smallmatrix} 0 & 0 \\ A & 0 \end{smallmatrix}\right) =: \mathscr{A}$. But this operator never generates an integrated semigroup if A is unbounded, since $\rho(\mathscr{A}) = \emptyset$.

4. HIGHER ORDER DIFFERENTIAL OPERATORS

The third derivative $(d/dx)^3$ is a generator on $L^2(\mathbb{R})$ but not on the other L^P-spaces. The same holds for the Schrödinger operator $i(d/dx)^2$ (see [13, Theorem 1.14]). But together with a large class of higher order differential operators $(d/dx)^3$ and $i(d/dx)^2$ generate integrated semigroups on these and other function spaces. This is what we shall show in this section.

In the following we use extensively the Fourier transformation and denote it by ^ i.e.,

$$\hat{f}(y) := (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx.$$

Its inverse will be denoted by ~.

We shall consider differential operators of the form $\sum_{j=0}^{k} a_j(d/dx)^j$ on the function spaces

$$C_0(\mathbb{R}),\ C_b(\mathbb{R}),\ UC_b(\mathbb{R}),\ L^p(\mathbb{R}) \text{ for } 1 \leqslant p \leqslant \infty. \tag{L}$$

($UC_b(\mathbb{R})$ is the space of uniformly continuous and bounded functions.) More precisely we make the following assumptions: Let $a_0, ..., a_k \in \mathbb{C}$ and let E be one of the spaces in (L). Then we consider the operator $A: D(A) \to E$, where

$$D(A) := \left\{ f \in E : \sum_{j=0}^{k} a_j D^j f \in E \text{ distributionally} \right\}$$

$$Af := \sum_{j=0}^{k} a_j D^j f.$$

A key role will be played by the polynomial $\sum_{j=0}^{k} a_j (ix)^j$ (i = imaginary unit). It will be abbreviated by $p(x)$.

With this notation the following theorem holds.

THEOREM 4.1. *Let $a_0, ..., a_k \in \mathbb{C}$ such that $\sup_{x \in \mathbb{R}} \operatorname{Re}(p(x)) < \infty$. Then $A : D(A) \to E$ generates a norm continuous integrated semigroup on all spaces E listed under* (L).

Remark 4.2. It should be noted that A is not densely defined on the spaces $C_b(\mathbb{R})$, and $L^\infty(\mathbb{R})$.

We first illustrate this result by an example.

EXAMPLE. Consider the following partial differential equation

$$\begin{aligned} u_t + u_{xxx} &= 0, \qquad t \geq 0, x \in \mathbb{R} \\ u(0, x) &= \phi(x). \end{aligned} \tag{$*$}$$

It is not difficult to see that the distributional domain of the third derivative in $L^p(\mathbb{R})$ $(1 \leq p \leq \infty)$ coincides with the third Sobolev space $W^{3,p}$. Hence, for each $\phi \in W^{6,p}$ there exists a unique solution

$$u \in C([0, \infty), W^{3,p}) \cap C^1([0, \infty), L^p)$$

of $(*)$.

We now prove Theorem 4.1 in a series of four lemmas. Subsequently we exclude the trivial case "$k = 0$" and assume $k \geq 1$ and $a_k \neq 0$. Further we use the abbreviation

$$w := \max(0, \sup_{x \in \mathbb{R}} \operatorname{Re}(p(x))).$$

Two operator families will be considered:

(1) $(S_t)_{t \geq 0}$, where $S_t f := (1/\sqrt{2\pi})\, \tilde{\phi}_t * f$ with $\phi_t(x) := \int_0^t e^{p(x)s}\, ds$. (This will be the integrated semigroup.)

(2) $(R_\lambda)_{\lambda > w}$, where $R_\lambda f := (1/\sqrt{2\pi})\, \tilde{r}_\lambda * f$ with $r_\lambda(x) := 1/(\lambda - p(x))$ (its resolvent).

First we have a closer look at the convolution kernels $\tilde{\phi}_t$ and $\tilde{r}_\lambda$.

LEMMA 4.3. *For $t \geqslant 0$ and $\lambda > w$ the following holds.*

(a) $\phi_t,\ r_\lambda \in H^1(\mathbb{R})$,

(b) $t \to \phi_t$ *is continuous with respect to* $\|\ \|_{H_1}$

(c) $\|\phi_t\|_{H_1} \leqslant (c + c't)\, e^{wt}$.

Proof. The assumption $k \geqslant 1$ implies that there is some $L_0 \geqslant 0$ such that $|p(x)| \geqslant |a_k x^k|/2$ for $|x| \geqslant L_0$.

(a) One verifies that ϕ_t, $(d/dx)\,\phi_t$, r_λ, $(d/dx)\, r_\lambda$ are bounded as functions in x and further that

$$|\phi_t(x)|, ..., \left|\left(\frac{d}{dx}\right) r_\lambda(x)\right| \leqslant c/|x| \qquad \text{(for suitable } c\text{)}.$$

This implies the assertion.

(b) Let $0 \leqslant s \leqslant t$,

$$\begin{aligned} \|\phi_t - \phi_s\|_2^2 &= \int_{-\infty}^{\infty} \left| \int_s^t e^{p(x)r}\, dr \right|^2 dx \\ &= \int_{-\infty}^{-L} |e^{p(x)t} - e^{p(x)s}|^2 / |p(x)|^2\, dx \\ &\quad + \int_{-L}^{L} \left| \int_s^t e^{p(x)r}\, dr \right|^2 dx \\ &\quad + \int_L^{\infty} |e^{p(x)t} - e^{p(x)s}|^2 / |p(x)|^2\, dx \qquad \text{for} \quad L > L_0 \\ &\leqslant \int_{-\infty}^{-L} 16 e^{2wt} / |a_k x^k|^2\, dx + 2L e^{2wt} |t - s|^2 \\ &\quad + \int_L^{\infty} 16 e^{2wt} / |a_k x^k|^2\, dx. \end{aligned}$$

If L is large the first and third term get small. By choosing $|t-s|$ small one can make the second term small, too. In a similar calculation one obtains

$$\begin{aligned} \left\| \left(\frac{d}{dx}\right) \phi_t - \left(\frac{d}{dx}\right) \phi_s \right\|_2^2 &\leqslant c t^2 e^{2wt} \int_{-\infty}^{-L} |x|^{-2}\, dx \\ &\quad + 2L t^2 e^{2wt} \Big(\sup_{|x| \leqslant L} |p'(x)| \Big)^2 |t - s|^2 \\ &\quad + c t^2 e^{2wt} \int_L^{\infty} |x|^{-2}\, dx. \end{aligned}$$

Choose again first L large and then $|t-s|$ small.

(c) can be obtained by choosing $L = L_0$ and $s = 0$. ∎

LEMMA 4.4. *Let $f \in H^1$. Then $\tilde{f} \in L^1(\mathbb{R})$ and there is a constant $c > 0$ such that*

$$\|\tilde{f}\|_1 \leqslant c \, \|f\|_{H_1} \qquad \text{for} \quad f \in H^1.$$

Proof. Let $f \in H^{1,2}$. Then $(f + (d/dx) f)^{\sim} \in L^2$. This implies that the function $y \to (1 + iy) \tilde{f}(y)$ is in L^2. Using the Cauchy–Schwarz inequality we obtain

$$\left(\int_{-\infty}^{\infty} |\tilde{f}(y)| \, dy\right)^2 = \left(\int_{-\infty}^{\infty} \frac{1}{|1+iy|} |(1+iy)\tilde{f}| \, dy\right)^2$$
$$\leqslant \int_{-\infty}^{\infty} |1+iy|^{-2} \, dy \int_{-\infty}^{\infty} \left|\left(f + \left(\frac{d}{dx}\right) f\right)^{\sim}\right|^2 dy.$$

This shows that $\tilde{f} \in L^1(\mathbb{R})$ and that the estimate holds with $c = \sqrt{2\pi}$. ∎

LEMMA 4.5. *Let E be one of the spaces listed under* (L). *Then the family $(S_t)_{t \geqslant 0}$, $S_t f = (1/\sqrt{2\pi}) \tilde{\phi}_t * f$ is an integrated semigroup on E continuous with respect to the operator norm. Further:*

(a) $\|S_t\|_{\mathscr{L}(E)} \leqslant (a + bt) e^{wt}$ *(with suitable constants a, $b \geqslant 0$).*

(b) *For $\lambda > w$ the pseudo-resolvent of $(S_t)_{t \geqslant 0}$ is given by $R_\lambda f = (1/\sqrt{2\pi}) \tilde{r}_\lambda * f$.*

Proof. Lemmas 4.3 and 4.4 imply

(1) $\tilde{\phi}_t \in L^1$ for $t \geqslant 0$.

(2) $t \to \tilde{\phi}_t$ is continuous with respect to $\| \ \|_1$.

(3) $\|\tilde{\phi}_t\|_1 \leqslant (a + bt) e^{wt}$ for suitable a, $b \geqslant 0$.

By (1) the convolution operators S_t are bounded operators on all listed spaces.

By the Young inequality

$$\|g * f\|_p \leqslant \|g\|_1 \, \|f\|_p \qquad \text{for} \quad g \in L^1, f \in L^p, 1 \leqslant p \leqslant \infty.$$

Statement (2) implies the norm continuity and (3) implies (i). To show that $(S_t)_{t \geqslant 0}$ fulfills the functional equation observe that $\phi_s \phi_t = \int_0^s (\phi_{r+t} - \phi_r) \, dr$. Therefore

$$(1/\sqrt{2\pi}) \, \tilde{\phi}_s * \tilde{\phi}_t = \int_0^s (\tilde{\phi}_{r+t} - \tilde{\phi}_r) \, dr.$$

Since the convolution is associative the functional equation follows.

(b) By (a) $\|\tilde{\phi}_t\| \leqslant Me^{\omega t}$ (and therefore $\|S_t\| \leqslant Me^{\omega t}$) for any $\omega > w$ (with suitably chosen M). Thus the Laplace integral exists for $\lambda > w$. We show first (by means of distributions)

$$\lambda \int_0^\infty e^{-\lambda t} \tilde{\phi}_t \, dt = \tilde{r}_\lambda \qquad \text{for} \quad \lambda > w.$$

Let Ψ be a function of rapid decrease. Then

$$\begin{aligned}
&\int_{-\infty}^\infty \lambda \int_0^\infty e^{-\lambda t} \tilde{\phi}_t(x) \, dt \, \Psi(x) \, dx \\
&\quad = \int_0^\infty \int_{-\infty}^\infty \lambda e^{-\lambda t} \tilde{\phi}_t(x) \, \Psi(x) \, dx \, dt \\
&\quad = \int_0^\infty \int_{-\infty}^\infty \lambda e^{-\lambda t} \phi_t(x) \, \tilde{\Psi}(x) \, dx \, dt \\
&\quad = \int_{-\infty}^\infty \lambda \int_0^\infty e^{-\lambda t} \phi_t(x) \, dt \, \tilde{\Psi}(x) \, dx \\
&\quad = \int_{-\infty}^\infty r_\lambda(x) \, \tilde{\Psi}(x) \, dx \\
&\quad = \int_{-\infty}^\infty \tilde{r}_\lambda(x) \, \Psi(x) \, dx.
\end{aligned}$$

This shows the assertion.

For the time being let $f \in L^\infty(\mathbb{R})$. Then for $z \in \mathbb{R}$ we have

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda t} S_t f \, dt(z) &= \lambda \int_0^\infty e^{-\lambda t} (1/\sqrt{2\pi})(\tilde{\phi}_t * f)(z) \, dt \\
&= (1/\sqrt{2\pi}) \, \lambda \int_{-\infty}^\infty e^{-\lambda t} \tilde{\phi}_t(z-y) f(y) \, dy \, dt.
\end{aligned}$$

The integrand is in $L^1(\mathbb{R}_+ \times \mathbb{R})$. Therefore one obtains by Fubini's theorem

$$= (1/\sqrt{2\pi}) \int_{-\infty}^\infty \lambda \int_0^\infty e^{-\lambda t} \tilde{\phi}_t(z-y) \, dt \, f(y) \, dy.$$

And from what we just showed we deduce

$$= (1/\sqrt{2\pi}) \, \tilde{r}_\lambda * f(z) = R_\lambda f(z).$$

From this (b) follows for L^∞, C_b, UC_b, and C_0. Since the functions of rapid decrease are in L^∞ and dense in all other L^p-spaces the rest follows by the boundedness of R_λ (this is obtained from Lemma 4.3(a)) in combination with Lemma 4.4). ∎

The proof of Theorem 4.1 is completed by

LEMMA 4.6. *$A: D(A) \to E$ is the generator of $(S_t)_{t \geqslant 0}$ for all spaces E listed under* (L).

Proof. The assertion is shown by distribution techniques. Let $f \in E$ and let Ψ be a function of rapid decrease. In the following Ψ_- denotes the function given by $\Psi_-(x) := \Psi(-x)$. Then we have

$$\left\langle \left(\lambda - \sum_{j=0}^{k} a_j D^j\right) R_\lambda f, \Psi \right\rangle$$

$$= \int_{-\infty}^{\infty} (1/\sqrt{2\pi})(\tilde{r}_\lambda * f)(x)\left(\lambda \Psi(x) - \sum_{j=0}^{k} a_j (-1)^j \Psi^{(j)}(x)\right) dx$$

$$= \int_{-\infty}^{\infty} (1/\sqrt{2\pi})(\tilde{r}_\lambda * f)(x)\left(\lambda \Psi_- - \sum_{j=0}^{k} a_j \Psi_-^{(j)}\right)(-x)\, dx$$

$$= (1/\sqrt{2\pi})(\tilde{r}_\lambda * f) * \left(\lambda \Psi_- - \sum_{j=0}^{k} a_j \Psi_-^{(j)}\right)(0)$$

$$= f * \left((1/\sqrt{2\pi})\, \tilde{r}_\lambda * \left(\lambda \Psi_- - \sum_{j=0}^{k} a_j \Psi_-^{(j)}\right)\right)(0)$$

$$= f * (r_\lambda(\lambda - p(\cdot))(\Psi_-)^\wedge)^\sim (0)$$

$$= f * \Psi_-(0) = \int_{-\infty}^{\infty} f(x)\, \Psi(x)\, dx.$$

This implies the distributional equation

$$\sum_{j=0}^{k} a_j D^j R_\lambda f = (\lambda R_\lambda f - f).$$

The distribution on the right-hand side is in E. Therefore

$$R_\lambda f \in D(A) \qquad \text{and} \qquad (\lambda - AR)\, R_\lambda f = f.$$

In a similar argument one shows $R_\lambda(\lambda - A) f = f$ for $f \in D(A)$. This completes the proof. ∎

Remark 4.7. The natural question that arises here is whether Theorem 4.1 can be generalized for the case of $\mathbb{R}^n$. If we try to use the preceding proof for $\mathbb{R}^n$ an obstacle comes up: Polynomials in $\mathbb{R}^n$ need not grow as fast as it is needed in the proof. And even if we require strong enough growth of p we only obtain a weaker result:

Consider

$$A := \sum_{|\alpha| \leqslant k} a_\alpha D^\alpha, \text{ where } \alpha := (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n,$$

$$|\alpha| := \sum_{j=1}^{n} \alpha_j \quad \text{and} \quad D^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}. \text{ Define}$$

$$p(x) := \sum_{|\alpha| \leqslant k} a_\alpha i^{|\alpha|} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ and}$$

$$\operatorname{ord}(p) := \max\{|\alpha| : a_\alpha \neq 0\}.$$

$\sum_{|\alpha| = \operatorname{ord}(p)} a_\alpha(i)^{|\alpha|} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is called the principal part of $p(x)$.

The growth condition on p that we need is equivalent to "ellipticity": A polynomial $p(x)$ is called *elliptic* if its principal part vanishes only when $x = 0$.

LEMMA 4.8. *Equivalent are*:

(a) *$p(x)$ is elliptic.*

(b) *There are constants L_0, $c > 0$ such that*

$$|p(x)| \geqslant c\,|x|^{\operatorname{ord}(p)} \qquad \textit{for} \quad |x| \geqslant L_0.$$

Proof. We denote the principal part of $p(x)$ by $q(x)$ and abbreviate $m := \operatorname{ord}(p)$. The key property is $q(\lambda x) = \lambda^m q(x)$ for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

(a) $\Rightarrow$ (b) By continuity and compactness reasons we have $|q(y)| \geqslant c > 0$ for $|y| = 1$. This implies by the above property for arbitrary $x \in \mathbb{R}^n$

$$c \leqslant \left| q\left(\frac{x}{|x|}\right) \right| = \left(\frac{1}{|x|}\right)^m |q(x)|.$$

Therefore

$$\left(\frac{1}{|x|}\right)^m |p(x)| = \left(\frac{1}{|x|}\right)^m |q(x)| - \left(\frac{1}{|x|}\right)^m |p(x) - q(x)|$$

$$\geqslant c - \frac{c}{2} \qquad \text{for} \quad |x| \geqslant L_0$$

if L_0 is chosen large enough (since $\operatorname{ord}(p - q) < m$).

(b)$\Rightarrow$(a) Let (b) hold and let $q(x)=0$ for some $x \in \mathbb{R}^n$. Then $q(\lambda x)=\lambda^m q(x)=0$ for arbitrary $\lambda>0$. This implies

$$c\,|\lambda x|^m \leqslant |\,p(\lambda x)| = |\,p(\lambda x)-q(\lambda x)|.$$

Therefore

$$c\,|x|^m \leqslant |\,p(\lambda x)-q(\lambda x)|/\lambda^m.$$

Since $\operatorname{ord}(p-q)\subset m$ the right-hand term goes to zero as $\lambda \to \infty$. Thus $x=0$. ∎

The following theorem replaces Theorem 4.1 in the case of $\mathbb{R}^n$.

THEOREM 4.9. *Let* $p(x) := \sum_{|\alpha| \leqslant k} a_{\alpha_n} i^{|\alpha|} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *be an elliptic polynomial with* $\operatorname{ord}(p) > n/2$ *and* $\sup_{x \in \mathbb{R}^n} \operatorname{Re} p(x) < \infty$. *Then* A *is the generator of an* $[n/2]+2$*-times integrated semigroup on the spaces*

(L^n) $\qquad C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $UC_b(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$.

For the definition of k-times integrated semigroups see Arendt [4].
We only sketch the proof. Define $m := [n/2]+2$ and show that

$$S_t f := \left(\frac{1}{2\pi}\right)^{n/2} \tilde{\phi}_t * f,$$

where

$$\phi_t := \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}\, e^{p(x)s}\, ds$$
$$= e^{p(x)t}/p(x)^m - \sum_{j=0}^{m-1} \frac{1}{j!}\, t^j/p(x)^{m-j}$$

defines an integrated semigroup with pseudo-resolvent $(R_\lambda)_{\operatorname{Re}\lambda>\omega}$. Here $R_\lambda f := (1/2\pi)^{n/2}\, \tilde{r}_\lambda * f$, $r_\lambda(x) := 1/(\lambda - p(x))$, and $R_\lambda := (\lambda - A)^{-1}$. To this end one verifies by induction that (for any polynomial $p(x)$) the following hold.

(1) $D^\alpha(e^{pt}/p^k) = e^{pt} q/p^r$, where $r \in \mathbb{N}$ and q is some polynomial of $\operatorname{ord}(q) \leqslant rm - (k - |\alpha|)\, m - |\alpha|$.

(2) $D^\alpha(1/p^j) = q/p^r$, where $r \in \mathbb{N}$ and q is some polynomial of $\operatorname{ord}(q) \leqslant rm - jm - |\alpha|$.

Using this one shows

LEMMA 4.10. *Let $p(x)$ fulfill the assumptions of Theorem 4.9. Then there are constants L_0, $c>0$ such that*

$$|D^\alpha \phi_t(x)| \leqslant c/|x|^{\operatorname{ord}(p)}, \qquad |D^\alpha r_\lambda(x)| \leqslant c/|x|^{\operatorname{ord}(p)}$$

for $|x| \geqslant L_0$ and $|\alpha| \leqslant [n/2]+1$.

LEMMA 4.11. *Let $p(x)$ fulfill the assumtions of Theorem 4.9. Then for $t \geqslant 0$ and $\lambda > \omega$ the following hold*

(a) $\phi_t, r_\lambda \in H^{m-1}$,

(b) $t \to \phi_t$ *is continuous with respect to* $\| \ \|_{H^{m-1}}$,

(c) $\|\phi_t\|_{H^{m-1}} \leqslant (c + c't^m)\, e^{wt}$.

The proof of Theorem 4.1 can now be copied if we replace Lemma 4.4 by

LEMMA 4.12. *For $k > n/2$ the inverse Fourier transform maps H^k $(\mathbb{R}^n)$ boundedly into $L^1(\mathbb{R}^n)$ (i.e., $\|\check{f}\|_{L^1} \leqslant c\, \|f\|_{H^k}$).*

The proof is similar to the proof of Lemma 4.4 and therefore is omitted. A consequence of Theorem 4.9 is

For $n \leqslant 3$ the Schrödinger operator $i\Delta$ generates a three-times integrated semigroup on all spaces listed under (L^n).

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