NOTE

Linear Spaces and Partitioning the Projective Plane

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The aim of this paper is to settle a question about the partitioning of the projective plane by lines except for a small set. Suppose that \( Q \) is a set of points in the projective plane of order \( n \) and \( H \) is a set of lines that partitions the complement of \( Q \). If \( Q \) has at most \( 2n-1 \) points and \( P \) has less than \( n+1+\sqrt{n} \) lines, then these lines are concurrent. An example is given which shows that the condition on the number of points of \( Q \) is sharp. However, it turns out that this is a 'pathological' example and if we exclude this case, then the statement can be improved.

1. INTRODUCTION

A linear space is a pair \( \mathcal{F} = (P, L) \) consisting of a set \( P \) of elements called points and a set \( L \) of distinguished subsets of points, called lines satisfying the following axioms:

1. Any two distinct points of \( \mathcal{F} \) belong to exactly one line of \( \mathcal{F} \).
2. Any line of \( \mathcal{F} \) has at least two points of \( \mathcal{F} \).
3. There are three points of \( \mathcal{F} \) not on a common line.

We shall restrict ourselves to finite linear spaces. We use \( v \) and \( b \) to denote the number of points and lines of \( \mathcal{F} \). For any point \( p \), \( r_p \) denotes the number of lines on \( p \). For any line \( L \), \( k_L \) denotes the number of points on \( L \). We also refer to \( r_p \) as the degree of the point \( p \) and to \( k_L \) as the size of the line \( L \).

Suppose that \( n = m^2 \) is a perfect square and that \( \mathcal{P} \) is a projective plane of order \( n \) with a Baer-subplane \( \mathcal{B} \), i.e. \( \mathcal{B} \) is a projective plane of order \( m \) embedded in \( \mathcal{P} \).

The problem originates from a conjecture of de Witte [3], Erdős, Mullin, Sós and Stinson [2]. They conjectured that linear spaces with
\( v \geq n^2 - n + 2 \) points, \( b = n^2 + n + 1 \) lines having at least one point of degree \( n + 2 \) do not exist. It turned out that, on the contrary, the closed complement of a Baer-subplane has the desired properties. Moreover, this is the only example for \( v > n^2 - \frac{2}{3}n + 1 \).

Suppose that \( S \) is a linear space with a point \( q \) of degree at least \( n + 2 \) and that \( S - q \) can be embedded into a projective plane \( P \) of order \( n \). Then the lines passing through \( q \) form a parallel class \( \Pi \) of \( S - q \). In order to determine \( S \) we need information about the structure of \( \Pi \) in \( P \). We shall obtain this information from Lemma 1, which we shall try to improve.

**Lemma 1** ([1], Lemma 7.1). Let \( P \) be a finite projective plane of order \( n \). Suppose that \( Q \) is a set of points and that \( \Pi \) is a set of lines with the property that no line of \( \Pi \) is contained in \( Q \) and that every point outside \( Q \) lies on a unique line of \( \Pi \). If \( |Q| \leq 2n - 1 \) and \( |\Pi| < 1 + n + \sqrt{n} \), then the lines of \( \Pi \) are concurrent.

**Example 1.** [1] Let \( P \) be a projective plane of order \( n = m^2 \) which contains a Baer-subplane \((B, G)\). \( G \) is the set of lines of \( P \) whose intersection with the subplane are lines of the subplane. Let \( L \) be a line of \( G \), denote by \( Q \) the set of points lying in \( B \) or on \( L \), and set \( \Pi = G - \{L\} \). Then \( |Q| = 2n + 1 \) and \( |\Pi| = n + m < n + 1 + \sqrt{n} \). Furthermore, every point outside of \( Q \) lies on a unique line of \( \Pi \) and no line of \( \Pi \) is contained in \( Q \).

This example shows that the bound for \( |Q| \) in Lemma 1 is almost the best possible. But, is it the best possible?

**2. RESULTS**

The following lemma considers the case when \( |Q| = 2n \). To state the lemma we need the following configuration.

**Example 2.** There is a line \( L \) with two points of \( Q \) on it, say \( q_1 \) and \( q_2 \). Through \( q_1 \) there are \( n - 1 \) lines of \( \Pi \), and through \( q_2 \) there is one \( X \). On \( X \) there are \( n \) points of \( Q \), and on the lines through \( q_1 \) there are at least one different from \( q_1 \). Hence, there is a line \( G \) through \( q_1 \), not in \( \Pi \). There are \( n - 1 \) points on it, so which necessarily are not on the lines \( \Pi \). These must be in \( Q \). Altogether we have \( 2n \) points.

**Lemma 2.** Let \( P \) be a projective plane of order \( n \). Suppose that \( Q \) is a set of points and that \( \Pi \) is a set of lines with the properties that no line of \( \Pi \) is entirely contained in \( Q \) and that every point outside \( Q \) lies on a unique line of \( \Pi \) and that is not the configuration of Example 2. If \( |Q| = 2n \) and \( |\Pi| < n + 1 + \sqrt{n} \), then the lines of \( \Pi \) are concurrent.
Proof. Let $P$ be the set of points outside $Q$, and set $v = |P|$. For every line $L$, we define $k_L = |L \cap P|$, and call $k_L$ the degree of $L$. We have
\[ n^2 - n + 1 = n^2 + n + 1 - |Q| = |P| = v = \sum_{L \in \Pi} k_L. \tag{1} \]

If $\Pi$ contains a line $L$ of degree $n + 1$, then $\Pi = \{L\}$, since every other line meets $L$ in a point $p$, which is a point of $P$. If $\Pi$ has a line $N$ of degree $n$, then every line of $\Pi$ has to contain the unique point of $N \cap Q$. W.l.o.g. we may therefore assume that every line of $\Pi$ has degree at most $n - 1$.

Choose a line $L$ of $\Pi$, with $L$ having maximal degree. Put $d = n + 1 - k_L$ and $L \cap Q = \{q_1, \ldots, q_d\}$. Furthermore, denote by $M_j$, the set of lines other than $L$ of $\Pi$ which contain $q_j$, and set $m_j = |M_j|$, $j = 1, \ldots, d$. We may assume w.l.o.g. that $m_j \geq m_k$ for $j < k$. Finally, set $M = M_2 \cup \cdots \cup M_d$, $m = |M|$ and $a = |\Pi| - n - 1$. Since every line of $\Pi$ has degree at most $n - 1$, (1) implies that $a \geq 0$. Furthermore, our definitions yield
\[ m_1 + m = \sum_{j=1}^d m_j = |\Pi| - 1 = n + a. \tag{2} \]

In order to prove our lemma we have to show that $M = \emptyset$. We shall do this in several steps.

Step 1. If $j, k \in \{1, \ldots, d\}$ with $j \neq k$, then every line of $M_j$ has degree at most $n - m_j$.

This is true, since a line $X$ of $M_j$ intersects each of the lines of $M_k \cup \{L\}$ in a point of $Q$.

Step 2. $M_1$ contains a line of degree $n - 1$.

In view of $v > k_L + n(n - 2)$, $q_1$ is contained in a line $G$ with $G \neq L$ and $k_G \geq n - 1$. If $G$ is in $M_1$, then $G$ has degree $n - 1$, since every line of $\Pi$ has degree at most $n - 1$. It suffices to show that $G$ is in $M_1$.

Assume to the contrary that $G \notin M_1$. Then each point of $G \cap P$ lies on a line of $M$. Consequently $m \geq k_G \geq n - 1$. In view of $m_j \geq m_k$ for $j < k$, we obtain
\[ m_1 \geq m_2 \geq \frac{m}{d - 1} \geq \frac{n - 1}{d - 1}. \]

It follows that
\[ n + a = m_1 + m \geq \frac{n - 1}{d - 1} + n - 1, \]

i.e.
\[ n - 1 \leq (a + 1)(d - 1). \tag{3} \]
Using (1), we conclude that
\[ n^2 - (a + 1)(d - 1) \leq n^2 - n + 1 = \nu \leq |\Pi| k_L \]
\[ = (n + 1 + a)(n + 1 - d) \]
\[ = n^2 + n(a + 2 - d) - (a + 1)(d - 1). \quad (4) \]

Now, there are two possibilities.

**Case 1.** \( d < a + 2. \)

Our hypothesis \( n > a^2 \), and (3) imply that \( d = a + 1 \). Because of
\[ a^2 + a < n + a = \sum_{j=1}^{d} m_j \leq dm_1, \]
it follows that \( m_1 > a \). Now, \( n + a = m_1 + m \) and \( m \geq n - 1 \) imply \( m_1 = a + 1 \) and \( m = n - 1 \). Since \( k_X \leq n - m_1 \), for every line \( X \) of \( M \) and \( k_X \leq k_L \) for every line \( X \) of \( M_1 \), (1) shows that
\[ n^2 - n + 1 = \nu = \sum_{X \in M_1} k_X \leq (m_1 + 1) k_L + m(n - m_1) = n^2 + 1 - a(a + 1). \quad (5) \]
We obtain \( a(a + 1) \leq n \), and therefore
\[ a(a + 1) \leq n = m + 1 = \sum_{j=2}^{d} m_j + 1 \leq (d - 1) m_2 + 1 \]
\[ \leq (d - 1) m_1 + 1 = a(a + 1) + 1. \]

If \( d > 2 \), then \( m_1 = m_2 = a + 1 \), and \( a(a + 1) = n \) or \( a(a + 1) = n - 1 \). Substituting \( a(a + 1) \) into (5), it follows that there exists a line \( X \) in \( M_1 \) such that \( k_X = k_L = n - a \). This is a contradiction, since by Step 1 \( k_X \leq n - m_2 \) for all \( X \in M_1 \).

If \( d = 2 \), then \( a = 1 \) so there are the following two possibilities:
1. \( n = 2, m_1 = 2 \) and \( m_2 = 1 \);
2. \( n = 3, m_1 = m_2 = 2 \).

It can also be easily checked that such configurations do not exist.

**Step 3 for Case 1.** We have shown that \( M_1 \) contains a line \( G \) of degree \( n - 1 \). Since \( L \) is a line of maximal degree of \( \Pi \), \( L \) has also degree \( n - 1 \), i.e. \( d = 2 \). In particular, \( \Pi = \{ L \} \cup M_1 \cup M_2 \) so that \( n + a = |\Pi| - 1 = m_1 + m_2 \).
Let \( q \) be the unique point other than \( q_1 \) on \( G \cap Q \). Since every line of \( M_2 \) contains \( q \), we have \( m_2 \leq 1 \) and hence \( m_1 \geq n + a - 1 \geq n - 1 \).
If $m_1 = n$ then $k_X$ is zero for every line $X$ of $M$. This is a contradiction, so $M = \emptyset$.

If $m_1 = n - 1$, then $m_2 = 1$ and the only possibility is the configuration described in Example 2.

**Case 2.** $d = a + 2$.

Here, $v = |\Pi| k_L$ and so the degree of the lines of $\Pi$ is constant. $|\Pi| = n + 1 + a |n^2 - n + 1|.$

From (3) $n - 1 = (a + 1)^2$, hence $n - 1$ must be a square. This implies that $k_L = n - a + 1$ for each line $L$ in $\Pi$.

We have

$$n + a = (a + 1)^2 + 1 + a \leq dm_1 = (a + 2) m_1.$$

Then $m_1 \geq (a + 1)$. If $m_1 > a + 1$ then by Step 1 $k_X \leq n - a - 2$ for every line $X$ not in $M$, which is a contradiction. Hence, $m_1 = a + 1$. Moreover, $n + a = m_1 + m$ implies that $m = n - 1$.

Now, we can write

$$(a + 1)^2 = n - 1 = m \leq (d - 1) m_2 \leq (d - 1) m_1 = (a + 1)(a + 1).$$

From this $m_1 = m_2 = a + 1$.

The number of points of $Q$ on the lines of $M_1$ is $(a + 1)^2 + a + 1 + 1$. From this $m_i = a + 1$ for all $i$. On each line $X$ of $M_1$ there are precisely $a + 1$ points of $Q$ different from $q_1$. Any line $Y \in M$ must meet every $X$ in one of these points. The number of lines in $\Pi$ is $n + a + 1 = (a + 1)^2 + a + 1 + 1$. It can be easily checked that these points form a projective plane of order $a + 1$.

There is one remaining point on each line through $q_1$ which is not on the lines of $\Pi$. Add these points to those on $M_1$. Altogether, these give the desired $2n$ points. However, such a subplane does not exist in a projective plane of order $n$.

The lines of $\Pi$ cannot be concurrent ones, because $n^2 - n + 1$ is not divisible by $n + 1$. Thus, we have finished the proof.

References