Homotopy classification of braided graded categorical groups

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Abstract

For any group $G$, a certain cohomology theory of $G$-modules is developed. This cohomology arises from the homotopy theory of $G$-spaces and it is called the \textquoteleft abelian cohomology of $G$-modules\textquoteright. Then, as the main results of this paper, natural one-to-one correspondences between elements of the 3rd cohomology groups of $G$-modules, $G$-equivariant pointed simply-connected homotopy 3-types and equivalence classes of braided $G$-graded categorical groups are established. The relationship among all these objects with equivariant quadratic functions between $G$-modules is also discussed.

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1. Introduction and summary

For any two abelian groups $M$ and $N$, the abelian groups

$$H^n_{ab}(M, N) = \text{Hom}_{\text{HoS}_n}(K(M, 2), K(N, n+1)),$$

where $\text{HoS}_n$ is the homotopy category of pointed spaces (simplicial sets), define the (first level) cohomology theory of the abelian group $M$ with coefficients in the abelian group $N$ \cite{10,11,19}. It is a well-known fact that for each $k \in H^3_{ab}(M, N)$, $n \geq 3$, there exists a pointed space $X = (X, *)$, unique up to weak homotopy equivalence, such that $\pi_2X = M$, $\pi_nX = N$, $\pi_iX = 0$ for all $i \neq 2, n$ and $k$ is the (unique non-trivial) Postnikov invariant of $X$. On the other hand, Joyal and Street proved in \cite[Theorem 3.3]{18} that every $k \in H^3_{ab}(M, N)$ also determines a braided categorical group $G$, unique up to braided monoidal equivalence, such that $M$ is the abelian group of isomorphism classes of its objects, $N$ is the abelian group of automorphisms of its identity object, and $k$ is the cohomology class of those abelian 3-cocycles of $M$ with coefficients in $N$ canonically deduced from the coherence pentagons and hexagons in $G$. Hence, braided categorical groups arise as algebraic models for homotopy 3-types of simply connected pointed spaces \cite{17}.

The main purpose of this paper is to show how all the above facts are actually instances of what happens in a more general equivariant context. Indeed, our first motivation for carrying out this work was to state and prove a precise

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classification theorem for braided graded categorical groups; that is, for groupoids $G$ equipped with a grading functor to a group $G$, $gr : G \to G$, a graded monoidal structure by graded functors $\otimes : G \times G \to G$ and $I : G \to G$, corresponding coherent 1-graded associativity and unit constraints $X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, $X \otimes I \to X \leftarrow I \otimes X$ and natural 1-graded braiding morphisms $X \otimes Y \to Y \otimes X$ compatible with the graded monoidal structure, such that for any object $X$ there is an object $X'$ with an arrow $X \otimes X' \to I$ of grade 1 (the non-braided case was dealt with in [4]). These graded categorical groups were originally introduced by Fröhlich and Wall in [12], where they presented a suitable abstract setting to study Brauer groups in equivariant situations (cf. also [13,4] or [5] for more instances). For their classification, two braided $G$-graded categorical groups that are connected by a braided graded monoidal equivalence are considered the same. Therefore, the problem arises of giving a complete invariant of this equivalence relation, which we solve by means of triples $(M, N, k)$, consisting of $G$-modules $M$, $N$ and a cohomology class $k \in H^3_{G,ab}(M, N)$. Here, the $H^n_{G,ab}(M, -)$ are Quillen cohomology groups [21, II, Section 5] of the pointed $G$-space $K(M, 2)$ in the homotopy category of pointed $G$-spaces, $\text{Ho}S^G_*$, with respect to the closed model structure where weak equivalences are those $G$-equivariant pointed maps that are weak equivalences on the underlying spaces, that is,

$$H^n_{G,ab}(M, N) = \text{Hom}_{\text{Ho}S^G_*}(K(M, 2), K(N, n + 1)).$$

Since equivariant weak homotopy types of pointed $G$-spaces $X$ with only two non-trivial homotopy groups at dimensions 2 and $n$ are classified by the elements of the cohomology group $H^n_{G,ab}(\pi_2 X, \pi_n X)$ (see Theorem 4 in Section 1), we get our main result, namely: the homotopy category of $G$-equivariant pointed simply-connected 3-types is equivalent to the homotopy category of braided $G$-graded categorical groups.

We should stress that the equivariant weak equivalences we are using should not be confused with the stronger notion of weak equivariant-homotopy equivalences, which are $G$-equivariant maps inducing weak equivalences on the fixed point subspaces of all subgroups of $G$. Thus, whereas the Postnikov invariants of weak equivariant-homotopy types live in Bredon–Moerdijk–Svensson’s cohomology groups [1,20], the cohomology groups $H^n_{G,ab}(M, N)$ we use are isomorphic to ordinary reduced equivariant cohomology groups (see Proposition 3 in Section 1).

The plan of this paper, briefly, is as follows. The first section includes the definition of the cohomology groups of $G$-modules $H^*_G$, and it is mainly dedicated to stating some concepts, results and notations concerning the homotopy theory of $G$-spaces we are going to use. The material in Section 1 is quite standard, so an expert reader may skip most of the proofs here. Both for theoretical and computational interests, it is appropriate to have an explicit description of a manageable cochain complex $C^*_G(M, N)$ to compute the cohomology groups $H^n_{G,ab}(M, N)$, and this is the goal of the second section. Next, in Section 3, we describe a particular subcomplex $E^*_G(M, N) \subseteq C^*_G(M, N)$, whose cohomology groups are precisely the abelian groups $\text{Ext}^n_{ZG}(M, N)$, allowing us to explicitly show the relationship between the groups $\text{Ext}^n_{ZG}$ and $H^n_{G,ab}$ without any argument based on the universal coefficient or the Borel spectral sequences (see [2, Section 3]). Namely, we describe isomorphisms $\text{Ext}^n_{ZG}(M, N) \cong H^{n+1}_{G,ab}(M, N)$ for $n = 0, 1$, and, in Section 4, a natural exact sequence

$$0 \to \text{Ext}^2_{ZG}(M, N) \to H^2_{G,ab}(M, N) \to \text{Quad}_G(M, N) \to \text{Ext}^3_{ZG}(M, N),$$

where $\text{Quad}_G(M, N)$ is the abelian group of all $G$-equivariant quadratic functions from $M$ to $N$ (cf. [19, Theorem 3]), where it was stated that $H^3_{ab}(M, N)$ is isomorphic to the group of all quadratic functions from $M$ to $N$, for any two abelian groups $M$ and $N$). In the fifth and final section we include our theorems on the homotopy classification of braided $G$-graded categorical groups and their homomorphisms by means of the cohomology groups $H^3_{G,ab}(M, N)$ and $H^2_{G,ab}(M, N)$.

2. Cohomology of $G$-modules

The material of this section is fairly standard concerning basic facts of the homotopy theory of $G$-spaces. Our main goal is the introduction of cohomology groups of $G$-modules $H^*_G$. We refer the reader to the book by Goerss and Jardine [14] for background.

Throughout $G$ is a fixed group, $S$ denotes the category of simplicial sets, and $S^G_*$ is the category of all pointed simplicial sets $X = (X, *)$ with a (left) $G$-action by pointed automorphisms, hereafter referred to as pointed $G$-spaces.

There is a Quillen closed model category structure on $S^G_*$ such that a pointed $G$-map $f : X \to Y$ is

– a weak equivalence if and only if $f$ is a weak equivalence in $S$, that is, $\pi_i(f) : \pi_iX \to \pi_iY$ is an isomorphism for all $i \geq 0$;
– a fibration if and only if \( f \) is a (Kan) fibration in \( S \);
– a cofibration if and only if it is injective and \( Y \setminus f(X) \) is a free \( G \)-set.

Thus, in this homotopy theory a pointed \( G \)-space \( (X, *) \) is fibrant whenever \( X \) is a Kan simplicial set, while \( (X, *) \) is cofibrant if no nonidentity element of \( G \) fixes a simplex different from the base point.

For any \( G \)-module \( N \), the Eilenberg–MacLane minimal complexes \( K(N, n) \) have an evident structure of pointed \( G \)-spaces and the \( G \)-equivariant cohomology groups of a pointed \( G \)-space \( X \) with coefficients in \( N \), \( H^n_G(X, N) \), are defined by

\[
H^n_G(X, N) = \text{Hom}_{\text{Ho}\mathcal{S}^G} \left( \int_G X, K(N, n) \right), \quad n \geq 0,
\]

where \( \text{Ho}\mathcal{S}^G \) is the homotopy category associated to the closed model category \( \mathcal{S}^G \) described above.

For any \( G \)-group \( \Pi \), the equivariant cohomology groups

\[
H^n_G(\Pi, N) = \text{Hom}_{\text{Ho}\mathcal{S}^G} \left( \int_\Pi \Pi, K(\Pi, 1), N \right),
\]
even for twisted local coefficients \( N \), are treated in [3] and several algebraic applications of this cohomology theory of groups with operators are shown in [4,5] or [6]. In this paper we deal with the case when \( \Pi \) is abelian, that is, with cohomology groups of a \( G \)-module \( M \), \( H^n_{G, ab}(M, N) \), which are defined as follows.

**Definition 1.** The cohomology groups of a \( G \)-module \( M \) with coefficients in a \( G \)-module \( N \) are defined by

\[
H^n_{G, ab}(M, N) = H^{n+1}_G(K(M, 2), N), \quad n \geq 1.
\]

Note that when \( G = 1 \) is the trivial group, then the cohomology groups (2.1) are just those of Eilenberg–MacLane cohomology theory of abelian groups \( H^n_{ab}(M, N) \) [10,11,19] (see the next section).

Both for theoretical and computational interests, it is appropriate to have an interpretation of the cohomology groups \( H^n_{G, ab}(M, N) \) in terms of an ordinary singular cohomology with local coefficients. This is the aim of the definitions below.

Any \( G \)-space \( X \) can be regarded as a functor \( X : G \to \mathbb{S} \), that is, as a diagram of spaces with the shape of \( G \). Let

\[
\int_G X = \text{hocolim}_G X.
\]

Thus, \( \left( \int_G X \right)_n = \{ (\sigma_1, \ldots, \sigma_n, x) | \sigma_i \in G, x \in X_n \} \) and the face and degeneracy operators are given by

\[
\begin{align*}
d_0(\sigma_1, \ldots, \sigma_n, x) &= (\sigma_2, \ldots, \sigma_n, \sigma_1^{-1}d_0 x), \\
d_i(\sigma_1, \ldots, \sigma_n, x) &= (\sigma_1, \ldots, \sigma_i \sigma_{i+1}, \ldots, \sigma_n, d_i x), \quad 0 < i < n, \\
d_n(\sigma_1, \ldots, \sigma_n, x) &= (\sigma_1, \ldots, \sigma_{n-1}, d_n x), \\
\sigma_i(\sigma_1, \ldots, \sigma_n, x) &= (\sigma_1, \ldots, \sigma_i, 1, \sigma_{i+1}, \ldots, \sigma_n, s_i x), \quad 0 \leq i \leq n.
\end{align*}
\]

In particular, for \( X = * \) we have

\[
\int_G * = K(G, 1) = BG \text{ (the classifying space of } G),
\]

and therefore, a pointed \( G \)-space \( X = (X, *) \) gives rise to a retraction diagram

\[
\int_G X : \quad \int_G X \xrightarrow{\epsilon} \int_G * \xrightarrow{\rho} BG, \quad \rho \epsilon = id,
\]

where \( \rho(\sigma_1, \ldots, \sigma_n, x) = (\sigma_1, \ldots, \sigma_n) \) and \( \epsilon(\sigma_1, \ldots, \sigma_n) = (\sigma_1, \ldots, \sigma_n, *) \). Note that the retraction \( \rho \) is a fibration if and only if \( X \) is fibrant and also that the fibre space of \( \rho \) at the unique vertex of \( BG \) is isomorphic to \( X \) as simplicial set.

This construction defines a functor

\[
\int_G : \mathcal{S}^G \longrightarrow \mathcal{S}^G_{BG}, \quad X \longmapsto \int_G X,
\]

where \( \mathcal{S}^G \) is the category of \( G \)-spaces and \( \mathcal{S}^G_{BG} \) is the category of \( G \)-spaces with coefficients in \( BG \).
The functor is a weak equivalence, and whose morphisms are simplicial maps \( f : R \to R' \) such that \( r'f = r \) and \( fs = s' \).

The category \( S_{BG}^G \) has a closed model structure induced by the usual one of simplicial sets; that is, a map \( f \) in \( S_{BG}^G \) is a weak equivalence, cofibration or fibration if and only if \( f \) is a weak equivalence, cofibration or fibration of simplicial sets respectively. The following proposition states that the homotopy theory of pointed \( G \)-spaces is equivalent to the homotopy theory of spaces over and under \( BG \).

**Proposition 2.** The functor \( f_G : S_*^G \to S_{BG}^G \) is a right Quillen equivalence. Then it induces an equivalence

\[
\text{Ho} S^G_* \simeq \text{Ho} S_{BG}^G
\]

between the associated homotopy categories.

**Proof.** The proposition is known to a large extent so we omit its proof in full detail. To describe the left adjoint functor \( L_G : S_{BG}^G \to S_*^G \) to \( f_G \), let \( EG \) be the universal cover of \( G \), that is, \( EG \) is the \( G \)-space with \( (EG)_n = G^{n+1} \) and face maps given by \( d_i(\sigma_0, \ldots, \sigma_n) = (\sigma_0, \ldots, \sigma_i \sigma_{i+1}, \ldots, \sigma_n), i < n \), \( d_n(\sigma_0, \ldots, \sigma_n) = (\sigma_0, \ldots, \sigma_{n-1}) \), and with \( G \)-action \( l(\sigma_0, \ldots, \sigma_n) = (\sigma_0, \sigma_1, \ldots, \sigma_n) \). The space \( EG \) is contractile and \( EG/G \cong BG \) via the canonical projection \( EG \to BG \), \( (\sigma_0, \ldots, \sigma_n) \mapsto (1, \ldots, \sigma_n) \). Then, for any \( R \) in \( S_{BG}^G \) as in \( (2.3) \), the pointed \( G \)-space \( L_G R \) is defined by the cofibre product

\[
EG \to \pi \to EG \times_{BG} R
\]

where \( \bar{\pi}(\sigma_0, \ldots, \sigma_n) = (\sigma_0, \ldots, \sigma_n, s(\sigma_1, \ldots, \sigma_n)) \) and the \( G \)-action on it is induced by the one on \( EG \). The unit of the adjunction \( R \to f_G L_GR \) is given by \( x \mapsto (r(x), [1, r(x), x]) \) and, for any pointed \( G \)-space \( X \), the counit \( f_G L_G X \to X \) is given by

\[
[s_0, \ldots, \sigma_n, (\sigma_0, \ldots, \sigma_n, x)] \mapsto s_0x.
\]

By [14, VI, Lemma 4.2 and IV, Proposition 1.7] the functor \( f_G \) preserves both fibrations and weak equivalences. Then, by [16, Lemma 1.3.4] \( (L_G, f_G) \) is a Quillen adjunction. For any \( R \) in \( S_{BG}^G \), by [8, Proposition 2.3] the map \( R \to f_G EG \times_{BG} R, x \mapsto (r(x), (1, r(x), x)) \) is a weak equivalence. Since the map \( \pi \) in \( (2.4) \) is a weak equivalence, the induced one \( f_G EG \times_{BG} R \to f_G L_G R \) is also a weak equivalence. The unit of the adjunction is just the composition \( R \to f_G EG \times_{BG} R \to f_G L_G R \) and therefore it is also a weak equivalence. An analogous argument, using once again [8, Proposition 2.3], proves that the counit is a weak equivalence and then the proposition follows from [16, Proposition 1.3.13].

Since every object in \( S_{BG}^G \) is cofibrant and, for any \( G \)-module \( N \), the retractions \( f_G K(N, n) \cong BG \) are fibrant (indeed, they are split minimal fibrations), it is a consequence of the above Proposition 2 that for any pointed \( G \)-space \( X \) there are natural isomorphisms

\[
H^n_G(X, N) \cong \text{Hom}_{\text{Ho} S_{BG}^G} \left( f_G X, f_G K(N, n) \right)
\]

\[
\cong \left[ f_G X, f_G K(N, n) \right]_{S_{BG}^G}
\]

\[
\cong H^n \left( f_G X, BG; N \right).
\]
where \( \int_G K(G, n) \) is the abelian group of homotopy classes of maps and \( H^n(\int_G X, BG; N) \) is the ordinary cohomology of the simplicial set \( \int_G X \) relative to \( BG = \int_G \ast \) with local coefficients in the \( G \)-module \( N \). The last isomorphism sends a homotopy class represented by a simplicial map \( f : \int_G X \to \int_G K(N, n) \) to the cohomology class represented by the cocycle \( G^n \times X_n \to G^n \times N \). The classification of pointed minimal fibre sequences is necessarily isomorphic to the split minimal fibre sequence (2.5) with local coefficients in the \( G \)-module \( N \) (see [14, VI, Proposition 4.13]). In particular, we have

**Proposition 3.** For any \( G \)-modules \( M \) and \( N \) there are natural isomorphisms

\[
H^n_{ab}(M, N) \cong H^{n+1}(\int_G K(M, 2), BG; N).
\]

We shall finish this section by proving the following result.

**Theorem 4.** Every pointed \( G \)-space \( X \) with \( \pi_i X = 0 \) for all \( i \neq 2, n \), where \( n \geq 3 \), determines a cohomology class \( k(X) \in H^n_{ab}(\pi_2 X, \pi_n X) \). Two such \( G \)-spaces \( X, Y \) are in the same equivariant weak homotopy class (i.e., they are isomorphic in \( HoS^G_k \)) if and only if there are \( G \)-module isomorphisms \( p : \pi_2 X \cong \pi_2 Y \) and \( q : \pi_n X \cong \pi_n Y \) such that \( p^* q_* k(X) = k(Y) \).

We refer to \( k(X) \) as the *equivariant Postnikov invariant of \( X \).*

**Proof.** Let \( M \) and \( N \) be any two \( G \)-modules. From **Proposition 2**, the classification of pointed \( G \)-spaces \( X \) with \( \pi_2 X = M, \pi_n X = N \) and \( \pi_i X = 0 \) for all \( i \neq 2, n \), by its equivariant weak homotopy type, is equivalent to the classification of minimal fibre sequences

\[
F \leftarrow R \xleftarrow{s} BG
\]

such that \( \pi_2 F = M, \pi_n F = N \) and \( \pi_i F = 0 \) for all \( i \neq 2, n \), two such split minimal fibrations being equivalent if there exists an isomorphism \( f : R \cong R' \) such that \( r' f = r \) and \( f s' = s \).

In the case when \( N = 0 \), the split minimal fibre sequence (2.5) is necessarily isomorphic to the split minimal fibre sequence

\[
K(M, 2) \leftarrow \int_G K(M, 2) \xleftarrow{\epsilon} BG
\]

and, therefore, for \( N \) arbitrary, the natural Postnikov system of (2.5) is of the form

\[
K(N, n) \leftarrow R \xleftarrow{p} \int_G K(M, 2) \xleftarrow{s} BG
\]

where \( K(N, n) \leftarrow R \xleftarrow{p} \int_G K(M, 2) \) is a minimal fibre sequence. It follows that the classification of the split minimal fibre sequence (2.5) is equivalent to the classification of the diagrams

\[
K(N, n) \leftarrow R \xleftarrow{p} \int_G K(M, 2), \quad ps = \epsilon,
\]
that is, of all minimal fibre sequences \( K(N, n) \rightarrow R \xrightarrow{p} \int_G K(M, 2) \) with a crossed section from \( BG \). Since these fibre sequences are classified precisely by the relative cohomology group \( H^{n+1}(\int_G K(M, 2), BG; N) \), the assertion follows from Proposition 3. □

3. An equivariant Bar reduction. The complex \( C^\bullet_{G, ab}(M, N) \)

In \([10, 11, 19]\) Eilenberg and MacLane defined a chain complex, \( \mathcal{A}(M, 2) \), associated to any abelian group \( M \), to compute the (co)homology groups of a space \( K(M, 2) \). In fact, they proved the existence of a cochain equivalence

\[
c_M : C^\bullet(K(M, 2), N) \rightarrow \text{Hom}(\mathcal{A}(M, 2), N)
\]

for any abelian group \( N \) [11, I, Theorem 20.3]. Hence, the cohomology groups of an abelian group \( M \) with coefficients in an abelian group \( N \) can be computed as

\[
H^n_{ab}(M, N) = H^{n+1}(\text{Hom}(\mathcal{A}(M, 2), N)).
\]

The main result of this section is to show an explicit description of a cochain complex, \( C^\bullet_{G, ab}(M, N) \), to compute the cohomology \( H^n_{ab}(M, N) \), of a \( G \)-module \( M \) with coefficients in a \( G \)-module \( N \), given in Definition 1.

We shall use the following notations for \( X, Y \) any two groups:

\[\begin{align*}
&X^p|Y^q = \{ (x|y) = (x_1, \ldots, x_p|y_1, \ldots, y_q), x_i \in X, y_j \in Y \}, \ p, q \geq 0. \\
&X^0 = \{ () \} \text{ is the trivial group. Then, } X^0|Y^q = Y^q \text{ and } X^p|Y^0 = X^p. \\
&\text{Shuf}(p, q) \text{ is the set of all } (p, q)\text{-shuffles. Any } \pi \in \text{Shuf}(p, q) \text{ defines a map } \\
&\pi : X^p|X^q \longrightarrow X^{p+q},
\end{align*}\]

given by \( \pi(x_1, \ldots, x_p|x_{p+1}, \ldots, x_{p+q}) = (x_{\pi(1)}, \ldots, x_{\pi(p+q)}) \).

**Definition 5.** Let \( M, N \) be two \( G \)-modules. The complex \( C^\bullet_{G, ab}(M, N) \) is defined to be trivial in dimension zero, that is, \( C^0_{G, ab}(M, N) = 0 \) and for \( n \geq 1 \) the elements of \( C^n_{G, ab}(M, N) \), related as abelian \( n \)-cochains of the \( G \)-module \( M \) with coefficients in the \( G \)-module \( N \), are the maps

\[
f : \bigcup M^{p_1}|M^{p_2}|\cdots|M^{p_r}|G^q \rightarrow N,
\]

where the union is taken over all \( p_1, \ldots, p_r \geq 1, q \geq 0 \) with \( r + \sum_{i=1}^{r} p_i + q = n + 1 \), which are normalized in the sense that

\[
f(x_1^1, \ldots, x_{p_1}^1|\cdots|x_{p_r}^r|\sigma_1, \ldots, \sigma_q) = 0,
\]

whenever some \( x_j^i = 0 \) or some \( \sigma_k = 1 \).

Addition in \( C^n_{G, ab}(M, N) \) is given by adding pointwise in the abelian group \( N \). The coboundary homomorphism \( \partial : C^n_{G, ab}(M, N) \rightarrow C^{n+1}_{G, ab}(M, N) \) is defined by the formula

\[
(\partial f)(x_1^1|x_2^2|\cdots|x_{n}^n) = \sigma_1 f(x_1^1|\cdots|x_{p_1}^1|d_0\sigma) + \sum_{1 \leq i \leq q} (-1)^i \, f(x_1^1|\cdots|x_{p_i}^i|d_i\sigma)
\]
\[
+ (-1)^q \, f(\sigma_1 x_1^1|\cdots|\sigma_q x_1^1|d_q\sigma)
\]
\[
+ \sum_{1 \leq i, j \leq n} (-1)^{n+e_i+e_j} \, f(x_1^1|\cdots|d_j x_1^i|\cdots|x_{n}^n|\sigma)
\]
\[
+ \sum_{\pi \in \text{Shuf}(p_i,p_{i+1})} (-1)^{q+e_\pi} \, f(x_1^1|\cdots|\pi(x_1^{i+1})|\cdots|x_{n}^n|\sigma),
\]

where \( (x_1^1|x_2^2|\cdots|x_{n}^n) \in M^{p_1}|\cdots|M^{p_r}|G^q \), \( \sigma = (\sigma_1, \ldots, \sigma_q) \), \( d_i : G^q \rightarrow G^{q-1} \) are the face operators of \( BG = K(G, 1) \); \( d_i : M^{p_i} \rightarrow M^{p_i-1} \) are the face operators of \( BM = K(M, 1) \); \( e_i = p_1 + \cdots + p_i + i \) and \( \epsilon(\pi) \) is the parity of the shuffle \( \pi \).
Thus,
\[ C_{G,ab}^1(M, N) \text{ consists of all normalized maps } \]
\[ M \xrightarrow{f} N, \]
\[ C_{G,ab}^2(M, N) \text{ consists of all normalized maps } \]
\[ M^2 \cup M|G \xrightarrow{g} N, \]
\[ C_{G,ab}^3(M, N) \text{ consists of normalized maps } \]
\[ M^3 \cup M|M \cup M^2|G \cup M|G^2 \xrightarrow{h} N, \]
\[ C_{G,ab}^4(M, N) \text{ consists of normalized maps } \]
\[ M^4 \cup M^2|M \cup M^2|G \cup M|M |G \cup M^2|G^2 \cup M|G^3 \rightarrow N, \]
\[ \ldots \ldots \]

with the coboundary maps
\[
(\partial f)(x, y) = f(x) - f(x + y) + f(y), \quad (3.2) \\
(\partial f)(x|\sigma) = \sigma f(x) - f(\sigma x); \quad (3.3) \\
(\partial g)(x, y, z) = g(y, z) - g(x + y, z) + g(x, y + z) - g(x, y), \quad (3.4) \\
(\partial g)(x|y) = g(x, y) - g(y, x). \quad (3.5) \\
(\partial g)(x|\sigma) = \sigma g(x, y) - g(\sigma x, \sigma y) - g(y|\sigma) + g(x + y|\sigma) - g(x|\sigma), \quad (3.6) \\
(\partial g)(x|\sigma, \tau) = \sigma g(x|\tau) - g(x|\sigma \tau) + g(\tau x|\sigma); \quad (3.7) \\
(\partial h)(x, y, z, t) = h(y, z, t) - h(x + y, z, t) + h(x, y + z, t) - h(x, y, z + t) + h(x, y, z). \quad (3.8) \\
(\partial h)(x|y, z) = h(x|z) - h(x|y + z) + h(x|y) + h(x, y, z) - h(y, x, z) + h(y, z, x), \quad (3.9) \\
(\partial h)(x, y|z) = h(y|z) - h(x + y|z) + h(x|z) - h(x, y, z) + h(x, y, y), \quad (3.10) \\
(\partial h)(x|y, z|\sigma) = \sigma h(x, y, z) - h(\sigma x, \sigma y, \sigma z) - h(y, z|\sigma) + h(x + y, z|\sigma) - h(x, y + z|\sigma) + h(x, y|\sigma), \quad (3.11) \\
(\partial h)(x|y|\sigma) = \sigma h(x|y) - h(\sigma x|\sigma y) - h(x, y|\sigma) + h(y, x|\sigma), \quad (3.12) \\
(\partial h)(x|y|\sigma, \tau) = \sigma h(x|y, \tau) - h(x, y|\sigma \tau) + h(\tau x|\sigma y) + h(y|\sigma, \tau) - h(x + y|\sigma, \tau) + h(x|\sigma, \tau), \quad (3.13) \\
(\partial h)(x|\sigma, \tau, \gamma) = \sigma h(x|\tau, \gamma) - h(x|\sigma \tau, \gamma) + h(x|\sigma, \tau \gamma) - h(\gamma x|\sigma, \tau). \quad (3.14) \\
\ldots \ldots \]

Below is our main result in this section.

**Theorem 6.** For any $G$-modules $M$ and $N$, $C_{G,ab}^\bullet(M, N)$ is actually a cochain complex and there are natural isomorphisms
\[ H_{G,ab}^n(M, N) \cong H^n(C_{G,ab}^\bullet(M, N)), \quad n \geq 0. \]

**Proof.** Let $X_G(M)$ be the bisimplicial set whose $(p, q)$-simplices are the elements of the cartesian product $G^p \times K(M, 2)_q$. The vertical face and degeneracy maps are defined by those of the Eilenberg–MacLane simplicial set $K(M, 2)$, and the horizontal face and degeneracy maps are those of $BG = K(G, 1)$ except $d_0^h$, which is defined by
\[ d_0^h(\sigma_1, \ldots, \sigma_p, x) = (\sigma_2, \ldots, \sigma_p, \sigma_1^{-1} x). \] Hence
\[ \text{diag} X_G(M) = \int_G K(M, 2). \quad (3.15) \]
Then we obtain a double cosimplicial abelian group $C^\bullet(X_G(M), N)$ in which the group $C^{p,q}(X_G(M), N)$ consists of all maps $f: X_G(M)_{p,q} \to N$, the horizontal cofaces are defined by

$$
(d^h_i f)(\sigma_1, \ldots, \sigma_p, x) = \begin{cases} 
  f(\sigma_2, \ldots, \sigma_p, \sigma_i^{-1} x), & \text{if } i = 0, \\
  f(\sigma_1, \ldots, \sigma_i \sigma_{i+1}, \ldots, \sigma_p, x), & \text{if } 0 < i < p, \\
  \sigma_p f(\sigma_1, \ldots, \sigma_{p-1}, x), & \text{if } i = p,
\end{cases}
$$

and the vertical cofaces are defined by

$$
(d^v_j f)(\overline{\sigma}, x) = f(\overline{\sigma}, d_j x), \quad 0 \leq j \leq q.
$$

We denote again by $C^\bullet(X_G(M), N)$ the associated double complex of normalized cochains, where differentials are obtained from the faces by taking alternating sums, and by Tot $C^\bullet(X_G(M), N)$ the associated total complex. Observe that, by equality (3.15), we have $\text{diag } C^\bullet(X_G(M), N) \cong C^\bullet(\int_G K(M, 2), N)$. Then, as a result of Dold and Puppe [7], there is a natural equivalence of cochain complexes

$$
\varphi_M : \text{Tot } C^\bullet(X_G(M), N) \to C^\bullet \left( \int_G K(M, 2), N \right).
$$

Now, let us note that the map $c_M$ in (3.1) is an equivalence of cochain complexes of $G$-modules, if we consider the diagonal $G$-action on both complexes $C^\bullet(K(M, 2), N)$ and $\text{Hom}(\mathcal{A}(M, 2), N)$. Then, $c_M$ induces a homomorphism of bicomplexes

$$
c_M : C^\bullet(G, C^\bullet(K(M, 2), N)) \to C^\bullet(G, \text{Hom}(\mathcal{A}(M, 2), N)),
$$

where the cochain map $c_M : C^q(G, C^\bullet(K(M, 2), N)) \to C^q(G, \text{Hom}(\mathcal{A}(M, 2), N))$ is an equivalence for every $q$. Therefore, the induced cochain map on the total complexes

$$
c_M : \text{Tot } C^\bullet(G, C^\bullet(K(M, 2), N)) \to \text{Tot } C^\bullet(G, \text{Hom}(\mathcal{A}(M, 2), N))
$$

is also an equivalence. Since there is quite an obvious isomorphism of bicomplexes

$$
C^\bullet(G, C^\bullet(K(M, 2), N)) \cong C^\bullet(X_G(M), N),
$$

we deduce a natural equivalence

$$
c_M : \text{Tot } C^\bullet(X_G(M), N) \to \text{Tot } C^\bullet(G, \text{Hom}(\mathcal{A}(M, 2), N)).
$$

By combining the quasi-isomorphisms $\varphi_M$, $c_M$, $\varphi_0$ and $c_0$ (the corresponding ones for $M = 0$), we get the following commutative diagram of cochain complexes induced by the retraction $M \cong 0$:

$$
\begin{array}{cccc}
C^\bullet(\int_G K(M, 2), N) & \xrightarrow{r_1} & C^\bullet(\int_G K(0, 2), N) & \xrightarrow{\varphi_0} \\
\text{Tot } C^\bullet(X_G(M), N) & \xrightarrow{r_2} & \text{Tot } C^\bullet(X_G(0), N) & \xrightarrow{c_0} \\
\text{Tot } C^\bullet(G, \text{Hom}(\mathcal{A}(M, 2), N)) & \xrightarrow{r_3} & \text{Tot } C^\bullet(G, \text{Hom}(\mathcal{A}(0, 2), N)) & \\
\end{array}
$$

where $r_is_i = id$, $i = 1, 2, 3$.

Let us observe now that $K(0, 2) = *$ is the one point simplicial set, so that

$$
C^\bullet \left( \int_G K(0, 2), N \right) = C^\bullet(BG, N)
$$

and the top retraction in (3.16) is precisely the one induced by the split fibration $\int_G K(M, 2) \cong BG$ (see (2.2)). Therefore, the kernel of $r_1$ is the relative cochain complex $C^\bullet(\int_G K(M, 2), BG; N)$ whose homology groups are $H^*_G,\text{ab}(M, N)$ up to a shift dimension (see Proposition 3).
The bicomplexes $C^\bullet(\mathcal{X}_G(0), \mathcal{M})$ and $C^\bullet(G, \text{Hom}(\mathcal{A}(0, 2), \mathcal{N}))$ are both isomorphic to the double cochain complex which is the complex $C^\bullet(BG, N)$ constant in the vertical direction, and a straightforward identification shows that the complex $C^\bullet_{G,ab}(\mathcal{M}, \mathcal{N})$, with its dimension raised by 1, occurs in the diagram (3.16) as the kernel of $r_3$. It follows that $C^\bullet_{G,ab}(\mathcal{M}, \mathcal{N})$ is actually a cochain complex and the induced maps by $\varphi_M$ and $c_M$

$$C^\bullet \left( \bigwedge_G K(M, 2), BG; N \right) \xrightarrow{\varphi_M} \text{Ker}(r_2) \xrightarrow{c_M} C^\bullet_{G,ab}(\mathcal{M}, \mathcal{N})$$

are quasi-isomorphisms, whence $H^\bullet_{G,ab}(\mathcal{M}, \mathcal{N}) \cong H^\bullet(C^\bullet_{G,ab}(\mathcal{M}, \mathcal{N}))$ for all $n \geq 0$. □

4. Relationship between $\text{Ext}^\bullet_{G,ab}(\mathcal{M}, \mathcal{N})$ and $H^\bullet_{G,ab}(\mathcal{M}, \mathcal{N})$

In this section, for any group $G$ and $G$-modules $\mathcal{M}$, $\mathcal{N}$ we present a new description by cocycles of the abelian groups $\text{Ext}^\bullet_{G,ab}(\mathcal{M}, \mathcal{N})$. This description gives us the possibility to establish explicitly their relationship with $H^\bullet_{G,ab}(\mathcal{M}, \mathcal{N})$ groups in an elementary way, that is, without any argument based on the universal coefficient or the Borel spectral sequences, as was done by Breen in [2].

Let $\mathcal{M}$ be any fixed $G$-module. We begin by inductively constructing an exact sequence of $G$-modules

$$0 \rightarrow R_0(\mathcal{M}) \xrightarrow{i_n} P_{n-1}(\mathcal{M}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} P_1(\mathcal{M}) \xrightarrow{\partial_0} P_0(\mathcal{M}) \xrightarrow{\partial_0} \mathcal{M} \rightarrow 0,$$

in which all $P_i(\mathcal{M})$ are free $G$-modules, as follows. Denote by $\mathbb{Z}[M^*]$ the free abelian group on the set of elements $\{x\}, x \in \mathcal{M}$, with the only relation $[0] = 0$ (so that $\mathbb{Z}[M^*]$ is the free abelian group on $M^* = M \setminus \{0\}$) and let $\mathbb{Z}[G]$ be the free abelian group on the elements $[\sigma], \sigma \in G$ (so that $\mathbb{Z}[G]$ is the free $G$-module on the unitary set $\{[1]\}$). Then we define

$$P_0(\mathcal{M}) = \mathbb{Z}[M^*] \otimes \mathbb{Z}[G]$$

with $G$-action determined by $^\sigma(\omega \otimes [\tau]) = \omega \otimes [\sigma \tau], \omega \in \mathbb{Z}[M^*], \sigma, \tau \in G$. Hence, $P_0(\mathcal{M})$ is the free $G$-module on the set of elements $\{x\} \otimes [1], x \in \mathcal{M}, x \neq 0$. The $G$-module epimorphism $\partial_0$ is given on the free generators by $\partial_0([x] \otimes [1]) = x$ and we define $R_1(\mathcal{M})$ as the kernel of $\partial_0$, so that we have the short exact sequence

$$0 \rightarrow R_1(\mathcal{M}) \xrightarrow{i_1} P_0(\mathcal{M}) \xrightarrow{\partial_0} \mathcal{M} \rightarrow 0.$$

Proceeding by induction, suppose we have defined the sequence (4.1) for $n \geq 1$. Then we define

$$P_n(\mathcal{M}) = R_n(\mathcal{M}) \otimes \mathbb{Z}[G]$$

with $G$-action such that $^\sigma(\omega \otimes [\tau]) = \omega \otimes [\sigma \tau], \omega \in R_n(\mathcal{M}), \sigma, \tau \in G$. Note that $P_n(\mathcal{M})$ is a free $G$-module with basis $\{e_i \otimes [1]\}$ where $\{e_i\}$ is a basis of the free abelian group $R_n(\mathcal{M})$. The $G$-module epimorphism $\partial_n : P_n(\mathcal{M}) \rightarrow R_n(\mathcal{M})$ is given by $\partial(\omega \otimes [\sigma]) = ^\omega \omega$ and we define $R_{n+1}(\mathcal{M})$ as the kernel of $\partial_n$, so that we have the short exact sequence

$$0 \rightarrow R_{n+1}(\mathcal{M}) \xrightarrow{i_{n+1}} P_n(\mathcal{M}) \xrightarrow{\partial_n} R_n(\mathcal{M}) \rightarrow 0.$$ (4.2)

Next we show a description of the $G$-module $R_n(\mathcal{M})$ in (4.1) by a set of generators and relations.

**Proposition 7.** For any $n \geq 1$, the $G$-module $R_n(\mathcal{M})$ is defined by a set of generators

$$\{[x, y, \sigma_1, \ldots, \sigma_{n-1}], [x, \sigma_1, \ldots, \sigma_n]; x, y \in \mathcal{M}, \sigma_1, \ldots, \sigma_n \in G\}$$

with the relations

$$[x, y, \sigma_1, \ldots, \sigma_{n-1}] = 0 = [x, \sigma_1, \ldots, \sigma_n], \text{ if } x = 0 \text{ or some } \sigma_i = 1,$$

$$[x, y, \sigma_1, \ldots, \sigma_{n-1}] = [y, x, \sigma_1, \ldots, \sigma_{n-1}],$$

$$[y, z, \sigma_1, \ldots, \sigma_{n-1}] + [x, y + z, \sigma_1, \ldots, \sigma_{n-1}] = [x + y, z, \sigma_1, \ldots, \sigma_{n-1}] + [x, y, \sigma_1, \ldots, \sigma_{n-1}].$$
We proceed by induction. For $n = 1$, hold. In effect, and

\[
\sigma_1[x, y, \sigma_2, \ldots, \sigma_n] = \sum_{i=1}^{n-1} (-1)^{i+1}[x, y, \sigma_1, \ldots, \sigma_i\sigma_{i+1}, \ldots, \sigma_n] + (-1)^n [\sigma_n x, \sigma_n y, \sigma_1, \ldots, \sigma_{n-1} + [x, \sigma_1, \ldots, \sigma_n] - [x + y, \sigma_1, \ldots, \sigma_n]]. \tag{4.6}
\]

\[
\sigma_1[x, \sigma_2, \ldots, \sigma_{n+1}] = \sum_{i=1}^{n} (-1)^{i+1}[x, \sigma_1, \ldots, \sigma_i\sigma_{i+1}, \ldots, \sigma_{n+1}] + (-1)^n [\sigma_{n+1} x, \sigma_1, \ldots, \sigma_n]. \tag{4.7}
\]

Furthermore, as an abelian group, $R_n(M)$ is defined by the same set of generators and with the only relations (4.3), (4.4) and (4.5).

**Proof.** We proceed by induction. For $n = 1$, let $[x, y], [x, \sigma] \in R_1(M)$ be the elements defined by the equalities

\[
[x, y] = [x] \otimes [1] + [y] \otimes [1] - [x + y] \otimes [1], \quad x, y \in M, \tag{4.8}
\]

\[
[x, \sigma] = [x] \otimes [\sigma] - [\sigma^0 x] \otimes [1], \quad x \in M, \sigma \in G. \tag{4.9}
\]

To see that these elements generate $R_1(M)$, let us denote by $R'_1(M) \subseteq R_1(M)$ the subgroup generated by them. We claim that $\partial_0 : P_0(M) \to M$ is the cokernel of the inclusion of $R'_1(M)$ into $P_0(M)$. In effect, for any abelian group homomorphism $f : P_0(M) \to A$, which verifies $f(R'_1(M)) = 0$, the map $\tilde{f} : M \to A$ given by $\tilde{f}(x) = f([x] \otimes [1])$ is a group homomorphism satisfying $\tilde{f} \partial_0 = f$ since

\[
\tilde{f}(x + y) = f([x + y] \otimes [1]) = f([x] \otimes [1]) + f([y] \otimes [1]) - f[x, y] = \tilde{f}(x) + \tilde{f}(y),
\]

\[
\tilde{f} \partial_0 ([x] \otimes [\sigma]) = \tilde{f}(\sigma x) = f([\sigma^0 x] \otimes [1]) = f([x] \otimes [\sigma]) + f[x, \sigma] = f([x] \otimes [\sigma]).
\]

Therefore, $R'_1(M) = \text{Ker}(\partial_0) = R_1(M)$, and thus, $R_1(M)$ is generated (even as an abelian group) by the elements $[x, y], [x, \sigma], x, y \in M, \sigma \in G$. The elements (4.8) and (4.9), clearly verify the relations (4.3) and (4.4). Next we observe that the relations (4.5)–(4.7) hold. In effect,

\[
[y, z] + [x, y + z] = ([y] + [z] - [y + z]) \otimes [1] + ([x] + [y + z] - [x + y + z]) \otimes [1] = ([x] + [y] + [z] - [x + y + z]) \otimes [1]
\]

\[
\sigma_0[x, y] = [x] \otimes [\sigma] + [y] \otimes [\sigma] - [x + y] \otimes [\sigma] + [x + y] \otimes [\sigma] - [\sigma^0 x + \sigma^0 y] \otimes [1] = ([x] \otimes [\sigma] - [\sigma^0 x] \otimes [1]) + ([y] \otimes [\sigma] - [\sigma^0 y] \otimes [1]) + ([\sigma^0 x] + [\sigma^0 y] - [\sigma^0 x + \sigma^0 y]) \otimes [1] = [x, y] + [x, \sigma] + [y, \sigma] + [\sigma^0 x, \sigma^0 y],
\]

\[
\sigma_0[x, \tau] = [x] \otimes [\sigma \tau] - [\tau^0 x] \otimes [\sigma] = [x] \otimes [\sigma \tau] - [\sigma^0 \tau x] \otimes [1] + [\sigma^0 \tau x] \otimes [1] - [\tau^0 x] \otimes [\sigma] = [x, \sigma \tau] - [\tau^0 x, \sigma].
\]

Let us suppose now that $A$ is any abelian group and $a_{[x, y]}, a_{[x, \sigma]} \in A$ are elements satisfying (4.3)–(4.5) (with respect to the indexes). We shall prove the existence of a (necessarily unique) group homomorphism $f : R_1(M) \to A$ such that $f([x, y]) = a_{[x, y]}$ and $f([x, \sigma]) = a_{[x, \sigma]}$. For, we first build the abelian group extension

\[
0 \to A \xrightarrow{j} A \star M \xrightarrow{p} M \to 0,
\]

where $A \star M$ is the cartesian product set $A \times M$ with the addition

\[
(a, x) + (a', y) = (a + a' + a_{[x, y]}, x + y),
\]

$j$ and $p$ are given by $j(a) = (a, 0)$ and $p(a, x) = x$. Then, we define a group homomorphism $\varphi : P_0(M) \to A \star M$ by putting $\varphi([x] \otimes [\sigma]) = (a_{[x, y]}, x)$. Since $p \varphi = \partial_0 : P_0(M) \to M$, there is a unique homomorphism of abelian groups $f : R_1(M) \to A$ such that $(f(\omega), 0) = \varphi(\omega)$, for all $\omega \in R_1(M)$. In particular, we have

\[
(f([x, y]), 0) = \varphi([x, y]) = \varphi([x] \otimes [1]) + \varphi([y] \otimes [1]) - \varphi([x + y] \otimes [1]) = (0, x) + (0, y) - (0, x + y) = (a_{[x, y]}, 0),
\]
\[
(f([x, \sigma]), 0) = \varphi([x, \sigma]) = \varphi([x] \otimes [\sigma]) - \varphi([x] \otimes [1])
\]
\[
= (a[x, \sigma], \sigma x) - (0, \sigma x) = (a[x, \sigma], 0).
\]

Thus, \(f([x, y]) = a[x, y]\) and \(f([x, \sigma]) = a[x, \sigma]\), as required.

Here we observe, simply by checking on the generators, that if \(A\) is a \(G\)-module and the elements \(a[x, y], a[x, \sigma]\) also satisfy (4.6) and (4.7), then the above group homomorphism \(f : R_1(M) \to A\) is actually a \(G\)-modules. This completes the proof for \(n = 1\).

Next, we suppose that the proposition is true for \(n \geq 1\). The short exact sequence (4.2) is split by the group homomorphism \(s : R_n(M) \to P_n(M)\) given by \(s(\omega) = \omega \otimes [1]\). Therefore, we have an isomorphism of abelian groups \(P_n(M) \cong R_{n+1}(M) \times R_n(M)\), with projection \(q : P_n(M) \to R_{n+1}(M)\) defined by
\[
q(\omega \otimes [\sigma]) = \omega \otimes [\sigma] - \varphi(s(\omega \otimes [\sigma])) = \omega \otimes [\sigma] - \sigma \omega \otimes [1].
\]

Therefore, as an abelian group and so also as a \(G\)-module, \(R_{n+1}(M)\) is generated by the elements
\[
[x, y, \sigma_1, \ldots, \sigma_n] = q([x, y, \sigma_2, \ldots, \sigma_n] \otimes [\sigma_1])
\]
\[
= [x, y, \sigma_2, \ldots, \sigma_n] \otimes [\sigma_1] - \sigma_1 [x, y, \sigma_2, \ldots, \sigma_n] \otimes [1],
\]
\[
[x, \sigma_1, \ldots, \sigma_{n+1}] = q([x, \sigma_2, \ldots, \sigma_{n+1}] \otimes [\sigma_1])
\]
\[
= [x, \sigma_2, \ldots, \sigma_{n+1}] \otimes [\sigma_1] - \sigma_1 [x, \sigma_2, \ldots, \sigma_{n+1}] \otimes [1].
\]

It is routine and we leave it to the reader to check that these elements verify all the equations (4.3)–(4.7), since such equations are satisfied by the corresponding generators of \(R_n(M)\) due to the hypothesis of induction.

Let us suppose now that \(A\) is any abelian group and
\[
a[x, y, \sigma_1, \ldots, \sigma_n], a[x, \sigma_1, \ldots, \sigma_{n+1}] \in A
\]
are elements satisfying (4.3), (4.4), (4.6) and (4.7). We shall prove the existence of a (necessarily unique) group homomorphism \(\varphi : R_{n+1}(M) \to A\) such that \(\varphi([x, y, \sigma_1, \ldots, \sigma_n]) = a[x, y, \sigma_1, \ldots, \sigma_n]\) and \(\varphi([x, \sigma_1, \ldots, \sigma_{n+1}]) = a[x, \sigma_1, \ldots, \sigma_{n+1}]\). For, we first observe, by using inductive hypothesis on \(R_n(M)\), that for each fixed \(\sigma \in G\), the mapping
\[
[x, y, \sigma_2, \ldots, \sigma_n] \mapsto a[x, y, \sigma_1, \ldots, \sigma_n], [x, \sigma_2, \ldots, \sigma_{n+1}] \mapsto a[x, \sigma_1, \ldots, \sigma_{n+1}]
\]
extends to an abelian group homomorphism from \(R_n(M)\) to \(A\). Then, we can define a homomorphism of abelian groups \(\varphi : P_n(M) \to A\) by equalities
\[
\varphi([x, y, \sigma_2, \ldots, \sigma_n] \otimes [\sigma_1]) = a[x, y, \sigma_1, \ldots, \sigma_n],
\]
\[
\varphi([x, \sigma_2, \ldots, \sigma_{n+1}] \otimes [\sigma_1]) = a[x, \sigma_1, \ldots, \sigma_{n+1}].
\]

Clearly \(\varphi s = 0 : R_n(M) \to A\), thanks to condition (4.3). Therefore, \(\varphi = \varphi s \circ \varphi = \varphi id = \varphi(i_{n+1}q + s \partial_n) = \varphi(i_{n+1}q)\), that is, \(\varphi(\alpha) = \varphi(q(\alpha))\) for all \(\alpha \in P_n(M)\). In particular, the restriction of \(\varphi\) to \(R_{n+1}(M)\) satisfies
\[
\varphi([x, y, \sigma_1, \ldots, \sigma_n]) = \varphi q([x, y, \sigma_2, \ldots, \sigma_n] \otimes [\sigma_1]) = \varphi([x, y, \sigma_2, \ldots, \sigma_n] \otimes [\sigma_1]) = a[x, y, \sigma_1, \ldots, \sigma_n],
\]
\[
\varphi([x, \sigma_1, \ldots, \sigma_{n+1}]) = \varphi q([x, \sigma_2, \ldots, \sigma_{n+1}] \otimes [\sigma_1]) = \varphi([x, \sigma_2, \ldots, \sigma_{n+1}] \otimes [\sigma_1])
\]
\[
= a[x, \sigma_1, \ldots, \sigma_{n+1}],
\]
as required.

Finally observe, simply by checking on the generators, that if \(A\) is in addition a \(G\)-module and the elements \(a[x, y, \sigma_1, \ldots, \sigma_n], a[x, \sigma_1, \ldots, \sigma_{n+1}]\) also satisfy (4.6) and (4.7), then the above \(\varphi : R_{n+1}(M) \to A\) is actually a homomorphism of \(G\)-modules. Therefore, as an abelian group, \(R_{n+1}(M)\) is indeed defined by the generators \([x, y, \sigma_1, \ldots, \sigma_n], [x, \sigma_1, \ldots, \sigma_{n+1}]\) and the relations (4.3)–(4.5), while as a \(G\)-module it is generated by the same elements but with the relations (4.3)–(4.7).

\[\square\]

Now we are ready to describe the abelian groups \(\text{Ext}^n_{G\text{-}A}(M, N)\) as the cohomology groups of the cochain complex introduced below.

**Definition 8.** Let \(M, N\) be two \(G\)-modules. The cochain complex
\[
\mathcal{E}^0_G(M, N) : 0 \to \mathcal{E}^0_G(M, N) \xrightarrow{\partial_1^G} \mathcal{E}^1_G(M, N) \xrightarrow{\partial_2^G} \cdots
\]
For any two $G$-modules $M$ and $N$ there are natural isomorphisms
\[
\tilde{f} : M^2 | G^{n-1} \cup M | G^n \rightarrow N,
\]
satisfying
\[
\tilde{f}(x, y | \sigma_1, \ldots, \sigma_{n-1}) = \tilde{f}(y, x | \sigma_1, \ldots, \sigma_{n-1}), \quad (4.10)
\]
\[
\tilde{f}(y, z | \sigma_1, \ldots, \sigma_{n-1}) + \tilde{f}(x, y+z | \sigma_1, \ldots, \sigma_{n-1}) = \tilde{f}(x + y, z | \sigma_1, \ldots, \sigma_{n-1}) + \tilde{f}(x, y | \sigma_1, \ldots, \sigma_{n-1}). \quad (4.11)
\]

The coboundary homomorphism $\partial : E^n_G(M, N) \rightarrow E^{n+1}_G(M, N)$ is defined by the formulas
\[
(\partial \tilde{f})(x | \sigma_1, \ldots, \sigma_n) = \sigma_1 \tilde{f}(x, y | \sigma_2, \ldots, \sigma_n) + \sum_{i=1}^{n-1} (-1)^i \tilde{f}(x, y | \sigma_1, \ldots, \sigma_{i+1}, f, \ldots, \sigma_n)
\]
\[
+ (-1)^n \left[ \tilde{f}(\sigma_n x, \sigma_n y | \sigma_1, \ldots, \sigma_{n-1}) + \tilde{f}(x | \sigma_1, \ldots, \sigma_n)
\right.
\]
\[
- \tilde{f}(x + y | \sigma_1, \ldots, \sigma_n) + \tilde{f}(y | \sigma_1, \ldots, \sigma_n) \right],
\]
\[
(\partial \tilde{f})(x | \sigma_1, \ldots, \sigma_{n+1}) = \sigma_1 \tilde{f}(x, y | \sigma_2, \ldots, \sigma_{n+1}) + \sum_{i=1}^{n} (-1)^i \tilde{f}(x | \sigma_1, \ldots, \sigma_{i+1}, \ldots, \sigma_{n+1})
\]
\[
+ (-1)^{n+1} \tilde{f}(\sigma_{n+1} x | \sigma_1, \ldots, \sigma_n).
\]

Thus,
\[E^n_G(M, N)\] consists of all normalized maps $\tilde{f} : M \rightarrow N$,
\[E^n_G(M, N)\] consists of normalized maps $\tilde{g} : M^2 \cup M | G \rightarrow N$ such that
\[
\tilde{g}(x, y) = \tilde{g}(y, x),
\]
\[
\tilde{g}(y, z) + \tilde{g}(x, y + z) = \tilde{g}(x + y, z) + \tilde{g}(x, y),
\]
\[E^n_G(M, N)\] consists of normalized maps $\tilde{h} : M^2 | G \cup M | G^2 \rightarrow N$ satisfying
\[
\tilde{h}(x, y | \sigma) = \tilde{h}(y, x | \sigma),
\]
\[
\tilde{h}(y, z | \sigma) + \tilde{h}(x, y + z | \sigma) = \tilde{h}(x + y, z | \sigma) + \tilde{h}(x, y | \sigma),
\]
\[E^n_G(M, N)\] consists of normalized maps $\tilde{k} : M^2 | G^2 \cup M | G^3 \rightarrow N$ such that
\[
\tilde{k}(x, y | \sigma, \tau) = \tilde{k}(y, x | \sigma, \tau),
\]
\[
\tilde{k}(y, z | \sigma, \tau) + \tilde{k}(x, y + z | \sigma, \tau) = \tilde{k}(x + y, z | \sigma, \tau) + \tilde{k}(x, y | \sigma, \tau),
\]
\[\ldots \]

with the coboundary maps
\[
(\partial \tilde{f})(x, y) = \tilde{f}(y) - \tilde{f}(x + y) + \tilde{f}(x),
\]
\[
(\partial \tilde{f})(x | \sigma) = \sigma \tilde{f}(x) - \tilde{f}(\sigma x);
\]
\[
(\partial \tilde{g})(x, y | \sigma) = \sigma \tilde{g}(x, y) - \tilde{g}(\sigma x, \sigma y) - \tilde{g}(y | \sigma) + \tilde{g}(x + y | \sigma) - \tilde{g}(x | \sigma),
\]
\[
(\partial \tilde{g})(x | \sigma, \tau) = \sigma \tilde{g}(x | \tau) - \tilde{g}(x | \sigma \tau) + \tilde{g}(\tau x | \sigma);
\]
\[
(\partial \tilde{h})(x, y | \sigma, \tau) = \sigma \tilde{h}(x, y | \tau) - \tilde{h}(x, y | \sigma \tau) + \tilde{h}(\tau x, \tau y | \sigma) + \tilde{h}(y | \sigma, \tau) - \tilde{h}(x + y | \sigma, \tau) + \tilde{h}(x | \sigma, \tau);
\]
\[\ldots \]

**Theorem 9.** For any two $G$-modules $M$ and $N$ there are natural isomorphisms
\[
\text{Ext}^n_G(M, N) \cong H^n \text{E}_G^n(M, N).
\]
Proof. Let \( P_\ast (M) = \cdots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0 \) be the free resolution of the \( G \)-module \( M \) built using the sequences \((4.1)\). Since

\[
\text{Hom}_{\mathbb{Z}G}(P_n(M), N) = \text{Hom}_{\mathbb{Z}G}(R_n(M) \otimes \mathbb{Z}(G), N) \cong \text{Hom}_{\mathbb{Z}}(R_n(M), N),
\]

Proposition 7 determines isomorphisms \( \text{Hom}_{\mathbb{Z}G}(P_n(M), N) \cong \mathcal{E}_G^n(M, N) \), \( f \mapsto \bar{f} \), where

\[
\bar{f}(x, y | \sigma_1, \ldots, \sigma_{n-1}) = f ([x, y, \sigma_1, \ldots, \sigma_{n-1}] \otimes [1])
\]

\[
\bar{f}(x | \sigma_1, \ldots, \sigma_n) = f ([x, \sigma_1, \ldots, \sigma_n] \otimes [1]).
\]

These isomorphisms define a cochain complex isomorphism \( \text{Hom}_{\mathbb{Z}G}(P_\ast (M), N) \cong \mathcal{E}_G^\ast (M, N) \), whence the theorem follows. \( \square \)

There is a natural injective cochain map of degree +1,

\[
i : \mathcal{E}_G^\ast (M, N) \rightarrow C_{G,ab}^{\ast +1}(M, N),
\]

\[
i \colon \begin{array}{ccc}
\mathcal{E}_G^0(M, N) & \rightarrow & \mathcal{E}_G^1(M, N) \\
\downarrow & & \downarrow \\
C_{G,ab}^1(M, N) & \rightarrow & C_{G,ab}^2(M, N)
\end{array},
\]

\[
\rightarrow \cdots \rightarrow \mathcal{E}_G^{n-1}(M, N) \rightarrow \cdots
\]

\[
0 \rightarrow C_{G,ab}^0(M, N) \rightarrow C_{G,ab}^1(M, N) \rightarrow C_{G,ab}^2(M, N) \rightarrow \cdots
\]

\[
\rightarrow C_{G,ab}^n(M, N) \rightarrow \cdots
\]

which sends an \((n - 1)\)-cochain \( \bar{f} : M^2|G^{n-2} \cup M|G^{n-1} \rightarrow N \) of \( \mathcal{E}_G^\ast (M, N) \) to the \( n \)-cochain \( f : \bigcup M^{|M^p|} \rightarrow N \) of \( C_{G,ab}^\ast (M, N) \) such that

\[
f|_{M^2|G^{n-2} \cup M|G^{n-1}} = \bar{f},
\]

\[
f|_{M^{|M^p|} \rightarrow G^q} = 0 \quad \text{whenever } q < n - 2.
\]

Then, as a consequence, we have the following:

Theorem 10. For any \( G \)-modules \( M \) and \( N \) there is a natural homomorphism

\[
\text{Ext}^n_{\mathbb{Z}G}(M, N) \rightarrow H_{G,ab}^{n+1}(M, N)
\]

for every \( n \geq 0 \), induced by the above cochain map \((4.12)\). This homomorphism is actually an isomorphism for \( n = 0 \) and \( n = 1 \), so that

\[
\begin{align*}
H_{G,ab}^1(M, N) & \cong \text{Hom}_G(M, N), \\
H_{G,ab}^2(M, N) & \cong \text{Ext}_{\mathbb{Z}G}^2(M, N),
\end{align*}
\]

and for \( n = 2 \) it is a monomorphism

\[
\text{Ext}_{\mathbb{Z}G}^2(M, N) \rightarrow H_{G,ab}^3(M, N).
\]

5. \( H_{G,ab}^3(M, N) \) and quadratic maps

In \([10,11,19]\), Eilenberg and MacLane proved that, for any two abelian groups \( M, N \), the cohomology group \( H_{ab}^3(M, N) \) is isomorphic to the abelian group of quadratic functions from \( M \) to \( N \). This fact was an algebraic proof of a classic result by Whitehead \([22]\), which states that the quadratic map \( \eta^\ast : \pi_2 X \rightarrow \pi_3 X \), induced by the Hopf map \( \eta : S^3 \rightarrow S^2 \), completely determines the homotopy type of a path-connected \( CW \)-complex \( X \) with only non-trivial homotopy groups \( \pi_2 X \) and \( \pi_3 X \) (recall that \( H_{ab}^3(M, N) \cong H^3(K(M, 2), N) \)). When \( X \) is a pointed \( G \)-space, the quadratic map \( \eta^\ast : \pi_2 X \rightarrow \pi_3 X \) is a \( G \)-equivariant quadratic map and it is natural to enquire about the relationship between the \( G \)-equivariant homotopy type of \( X \), which is determined by its equivariant Postnikov invariant \( k(X) \in H_{G,ab}^2(\pi_2 X, \pi_3 X) \) (see Theorem 4), and the induced \( G \)-equivariant quadratic map \( \eta^\ast \). And also, to enquire whether any \( G \)-equivariant quadratic map \( q : M \rightarrow N \) is induced from an equivariant homotopy 3-type of a simply connected pointed \( G \)-space \( X \).
Let $M, N$ be two $G$-modules. Then the third Eilenberg–MacLane cohomology group of the abelian group $M$ with coefficients in the abelian group $N$, $H^3_{ab}(M, N) = H^3_{ab}(M, N)$ is a $G$-module, where the $G$-action on 3-cocycles is given diagonally. Let $H^3_{ab}(M, N)^G$ be the subgroup of $H^3_{ab}(M, N)$ of all $G$-invariant elements. We state below a main result in this section.

**Theorem 11.** For any $G$-modules $M$ and $N$, there is a natural exact sequence of abelian groups

$$0 \to \text{Ext}^2_{SG}(M, N) \xrightarrow{i} H^3_{G, ab}(M, N) \xrightarrow{j} H^3_{ab}(M, N)^G \xrightarrow{k} \text{Ext}^3_{SG}(M, N).$$

**Proof.** We begin by describing the homomorphisms $i, j$ and $k$. The monomorphism $i$ is exactly the one given in Theorem 10.

The “forgetting $G$” homomorphism $j$ is the cohomological expression of the obvious map $\text{Hom}_{\text{HoSet}}(K(M, 2), K(N, 3)) \to \text{Hom}_{\text{HoSet}}(K(M, 2), K(N, 3))$. That is, for any abelian 3-cocycle $h : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N$

$$j[h] = [h_{|M^3 \cup M|M}].$$

To describe the homomorphism $k$, take $[h] \in H^3_{ab}(M, N)^G$. Then

$$h : M^3 \cup M|M \to N$$

is a normalized map satisfying $\partial h = 0$ in Eqs. (3.8)–(3.10) and $\partial [h] = [h]$ for all $\sigma \in G$. This last condition means that there are normalized maps $g_\sigma : M^2 \to N$ such that

$$\sigma h(\sigma^{-1} x, \sigma^{-1} y, \sigma^{-1} z) = h(x, y, z) + g_\sigma(y, z) - g_\sigma(x + y, z) + g_\sigma(x, y + z) - g_\sigma(x, y),$$

$$\sigma h(\sigma^{-1} x | \sigma^{-1} y) = h(x|y) + g_\sigma(x, y) - g_\sigma(y, x).$$

Inserting $\sigma x$, $\sigma y$ and $\sigma z$ instead of $x$, $y$ and $z$ respectively in the above expressions and denoting again by $h$ the map

$$h : M^2|G \to N$$

defined by $h(x, y | \sigma) = g_\sigma(\sigma x, \sigma y)$, the equalities (5.2) and (5.3) become

$$\sigma h(x, y, z) - h(\sigma x, \sigma y, \sigma z) - h(y, z | \sigma) + h(x + y, z | \sigma) - h(x, y + z | \sigma) + h(x, y | \sigma) = 0,$$

$$\sigma h(x|y) - h(\sigma x|\sigma y) - h(x, y|\sigma) + h(y, x|\sigma) = 0.$$

Then, let

$$\tilde{\kappa}_h : M^2|G^2 \cup M|G^3 \to N$$

be the map defined by

$$\tilde{\kappa}_h(x, y | \sigma, \tau) = \sigma h(x, y | \sigma - h(x, y | \sigma \tau) + h(\tau x, \tau y | \sigma),$$

$$\tilde{\kappa}_h(x | \sigma, \tau, \gamma) = 0.$$

By using (5.4) and (5.5), it is routine and requires only care over the definitions to show that $\tilde{\kappa}_h$ is a 3-cocycle of the complex $\mathcal{E}_h^3(M, N)$. Thanks to the isomorphism in Theorem 9 for $n = 3$, we define the homomorphism $\kappa$ by

$$\kappa[h] = [\tilde{\kappa}_h] \in \text{Ext}^3_{SG}(M, N).$$

It is easy to verify that $\kappa : H^3_{ab}(M, N)^G \to \text{Ext}^3_{SG}(M, N)$ is a well-defined homomorphism of abelian groups.

We next prove the exactness of the sequence

$$\text{Ext}^2_{SG}(M, N) \xrightarrow{i} H^3_{G, ab}(M, N) \xrightarrow{j} H^3_{ab}(M, N)^G \xrightarrow{k} \text{Ext}^3_{SG}(M, N).$$

Obviously, $ji = 0$. To prove that $\text{Ker}(j) \subseteq \text{Im}(i)$, let $j[h] = [h_{|M^3 \cup M|M}] = 0$. Then, there exists an abelian 2-cochain $g : M^2 \to N$ such that

$$h(x, y, z) = g(y, z) - g(x + y, z) + g(x, y + z) - g(x, y),$$

$$h(x|y) = g(x, y) - g(y, x),$$

respectively in the above expressions and denoting again by $h$ the map

$$h : M^2|G \to N$$

defined by $h(x, y | \sigma) = g_\sigma(\sigma x, \sigma y)$, the equalities (5.2) and (5.3) become

$$\sigma h(x, y, z) - h(\sigma x, \sigma y, \sigma z) - h(y, z | \sigma) + h(x + y, z | \sigma) - h(x, y + z | \sigma) + h(x, y | \sigma) = 0,$$

$$\sigma h(x|y) - h(\sigma x|\sigma y) - h(x, y|\sigma) + h(y, x|\sigma) = 0.$$
and we define a map \( \tilde{h} : M^2 | G \cup M | G^2 \rightarrow N \) by taking

\[
\tilde{h}(x, y \mid \sigma, \tau) = h(x, y \mid \sigma) - \sigma \cdot g(x, y) + g(\sigma x, \sigma y),
\]

\[
\tilde{h}(x, y \mid \sigma, \tau) = h(x \mid \sigma, \tau).
\]

An easy direct calculation shows that \( \tilde{h} \in E^*_G(M, N) \), \( \partial \tilde{h} = 0 \) and \( i[\tilde{h}] = [h] \). Thus, \( \text{Im}(i) = \text{Ker}(j) \).

And finally, we prove the exactness of the sequence

\[
H^3_{ab}(M, N) \xrightarrow{j} H^3_{ab}(M, N)^G \xrightarrow{\kappa} \text{Ext}^3_G(M, N).
\]

For any \( [h] \in H^3_{ab}(M, N) \) we have \( \kappa j[h] = \kappa | h \big|_{M^3 \cup M | M} = [\kappa h] \), where

\[
\kappa_h(x \mid \sigma, \tau, y) = 0,
\]

\[
\kappa_h(x, y \mid \sigma, \tau) = \sigma \cdot h(x, y \mid \tau) - h(x, y \mid \sigma \tau) + h(\tau x, \tau y \mid \sigma)
\]

\[
= h(y \mid \sigma, \tau) + h(x | \sigma, \tau) - h(x + y | \sigma, \tau).
\]

Then the map \( \tilde{h} : M^2 | G \cup M | G^2 \rightarrow N \) given by the equalities

\[
\tilde{h}(x, y \mid \sigma, \tau) = -h(x | \sigma, \tau),
\]

\[
\tilde{h}(x, y \mid \sigma, \tau) = 0
\]

is a 2-cocycle of \( E^*_G(M, N) \) and clearly \( \partial \tilde{h} = \kappa_h \). Thus, \( \kappa j = 0 \).

Let us now consider any \( [h] \in H^3_{ab}(M, N) \) into the kernel of \( \kappa \), that is, \( \kappa[h] = [\tilde{\kappa}_h] = 0 \). Recall that \( \tilde{\kappa}_h \) is defined by equalities (5.6) and (5.7), via a map \( h : M^2 | G \rightarrow N \) such that equalities (5.4) and (5.5) hold. Since \( [\tilde{\kappa}_h] = 0 \), there exists \( \tilde{h} \in E^*_G(M, N) \) such that \( \partial \tilde{h} = \kappa_h \). Then, it is straightforward to verify that the map

\[
h' : M^3 \cup M | M \cup M^2 | G \cup M | G^2 \rightarrow N
\]

defined by

\[
h'(x, y, z) = h(x, y, z),
\]

\[
h'(x, y \mid y) = h(x | y),
\]

\[
h'(x, y \mid \sigma) = h(x, y | \sigma) - \tilde{h}(x, y | \sigma),
\]

\[
h'(x \mid \sigma, \tau) = \tilde{h}(x | \sigma, \tau)
\]

is actually an abelian 3-cocycle and clearly \( j[h'] = [h] \). This complete the proof. \( \square \)

Let us recall that a map \( q : M \rightarrow N \) between abelian groups is called \textit{quadratic} if the function \( M \times M \rightarrow N \), \( (x, y) \mapsto q(x) + q(y) - q(x + y) \), is bilinear and \( q(-x) = q(x) \). The Whitehead quadratic functor \( \Gamma \) [23] is characterized by the universal property:

\[
\text{Hom}(\Gamma(M), N) \cong \text{Quad}(M, N),
\]

where \( \text{Quad}(M, N) \) is the abelian group of quadratic functions from \( M \) to \( N \).

The \textit{trace} of an abelian 3-cocycle \( h : M^3 \cup M | M \rightarrow N \) is the map

\[
t_h : M \rightarrow N, \quad t_h(x) = h(x | x).
\]

An easy calculation shows that traces are quadratic maps, and Eilenberg and MacLane [10,11,19] proved that trace determines an isomorphism

\[
H^3_{ab}(M, N) \cong \text{Quad}(M, N), \quad [h] \mapsto t_h.
\]

(5.8)

Suppose now that \( M \) and \( N \) are two \( G \)-modules. Then, both \( H^3_{ab}(M, N) \) and \( \text{Quad}(M, N) \) are \( G \)-modules by the diagonal \( G \)-action, and (5.8) is actually a \( G \)-module isomorphism. Therefore, it restricts to the corresponding subgroups of all \( G \)-invariant elements

\[
H^3_{ab}(M, N)^G \cong \text{Quad}(M, N)^G = \text{Quad}_G(M, N),
\]
where \( \text{Quad}_G(M, N) \) is the abelian group of all quadratic \( G \)-maps from \( M \) to \( N \). Taking into account this isomorphism, the exact sequence (5.1) yields

**Theorem 12.** For any \( G \)-modules \( M \) and \( N \) there is an exact sequence

\[
0 \to \text{Ext}^2_{\Sigma G}(M, N) \xrightarrow{i} H^3_{G, ab}(M, N) \xrightarrow{j} \text{Quad}_G(M, N) \xrightarrow{\kappa} \text{Ext}^3_{\Sigma G}(M, N).
\]  

(5.9)

Theorem 11 has an interesting interpretation in the equivariant homotopy theory of pointed spaces:

For \( M \) and \( N \), two given \( G \)-modules, let

\[
S^G_*(M, N)
\]
denote the set of equivalence classes of triples \((X, \alpha, \beta)\), where \( X \) is a pointed \( G \)-space with \( \pi_i X = 0 \) for all \( i \neq 2, 3 \) and \( \alpha, \beta \) are isomorphisms of \( G \)-modules \( \alpha : \pi_2 X \cong M, \beta : \pi_3 X \cong N \). We say that \((X, \alpha, \beta)\) is congruent with \((X', \alpha', \beta')\) if there is \( G \)-equivariant pointed simplicial map \( f : X \to X' \) such that \( \alpha' \pi_2(f) = \alpha \) and \( \beta' \pi_3(f) = \beta \) and then, that \((X, \alpha, \beta)\) is equivalent to \((X', \alpha', \beta')\) if there is a zig-zag chain of congruences

\[
(X, \alpha, \beta) \leftarrow (X_1, \alpha_1, \beta_1) \to \cdots \to (X_n, \alpha_n, \beta_n) \to (X', \alpha', \beta').
\]

By Theorem 4, elements of \( S^G_*(M, N) \) are in one-to-one correspondence with the elements of the group \( H^3_{G, ab}(M, N) \); more precisely, we have:

**Proposition 13.** For any \( G \)-modules \( M \) and \( N \), there is a bijection

\[
S^G_*(M, N) \cong H^3_{G, ab}(M, N), [X, \alpha, \beta] \mapsto \beta_\alpha^* k(X),
\]

where \( k(X) \) is the equivariant Postnikov invariant of \( X \).

It follows from Theorem 12 that any \([X, \alpha, \beta] \in S^G_*(M, N)\) determines the \( G \)-equivariant quadratic map

\[
q_{[X, \alpha, \beta]} = j\beta_\alpha^* k(X) : M \to N.
\]

Therefore, we have a partition

\[
S^G_*(M, N) = \bigsqcup_q S^G_q(M, N, q),
\]

where, for each \( G \)-equivariant quadratic map \( q : M \to N \), \( S^G_q(M, N, q) \) is the set of classes \([X, \alpha, \beta] \in S^G_q(M, N)\) that fulfill \( q \) in the sense that \( q_{[X, \alpha, \beta]} = q \).

On the other hand, each equivariant quadratic map \( q : M \to N \) determines a cohomology class

\[
\text{Obs}(q) = \kappa(q) \in \text{Ext}^3_{\Sigma G}(M, N),
\]

which we refer to as the obstruction of \( q \). Then, the exactness of (5.9) in \( \text{Quad}_G(M, N) \) implies the following:

**Theorem 14.** A \( G \)-equivariant quadratic function \( q : M \to N \) is realizable, that is, \( S^G_*(M, N, q) \neq \emptyset \), if and only if its obstruction vanishes.

And the exactness of (5.9) in \( H^3_{G, ab}(M, N) \) means that:

**Theorem 15.** If the obstruction of a quadratic \( G \)-map \( q : M \to N \) vanishes, then there is a bijection

\[
S^G_q(M, N, q) \cong \text{Ext}^2_{\Sigma G}(M, N).
\]

At the end of this section we consider two particular cases of Theorem 11.

**Proposition 16.** Let \( G \) act trivially on \( M \) and \( N \), then there is an isomorphism

\[
H^3_{G, ab}(M, N) \cong \text{Ext}^2_{\Sigma G}(M, N) \oplus \text{Quad}(M, N).
\]
Proof. In the hypothesis of the theorem, we have $H^3_{ab}(M, N)^G = H^3_{ab}(M, N) \cong \text{Quad}(M, N)$. Moreover, any abelian 3-cocycle of the abelian group $M$ in the abelian group $N$, say

$$h : M^3 \cup M|M \to N,$$

defines an abelian 3-cocycle of the $G$-module $M$ in the $G$-module $N$

$$h' : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N,$$

simply by putting $h'|_{M^3UM|M} = h$ and $h'|_{M^2|GUM|G^2} = 0$. Then, the map

$$H^3_{ab}(M, N) \to H^3_{G, ab}(M, N), [h] \mapsto [h'],$$

is a group homomorphism that splits the homomorphism $j$ of the sequence (5.1), and the proposition follows. □

Proposition 17. Let $M$, $N$ be $G$-modules such that $H_2(M) = 0$ and $\text{Hom}_\mathbb{Z}(H_2(M), N) = 0$, where the $H_i(M) = H_i(M, \mathbb{Z})$ are the integral homology groups of the group $M$. Then the sequence

$$0 \to \text{Ext}^2_{\mathbb{Z}}(M, N) \xrightarrow{i} H^3_{G, ab}(M, N) \xrightarrow{j} \text{Quad}_G(M, N) \to 0.$$ 

is short exact.

Proof. Let us fix any quadratic $G$-map $q : M \to N$. By (5.8), there is an abelian 3-cocycle of the abelian group $M$ with coefficients in $N$, say $h : M^3 \cup M|M \to N$, with trace $q$, that is, such that $h(x|y) = q(x)$ for all $x \in M$. Since $h'|_{M^3} : M^3 \to N$ is an ordinary 3-cocycle of the group $M$ with coefficients in the (trivial) $M$-module $N$, and by the Universal Coefficient Theorem $H^3(M, N) = 0$, there is a normalized map $g : M^2 \to N$ such that

$$h(x, y, z) = g(y, z) - g(x + y, z) + g(x, y + z) - g(x, y).$$

Then, $h$ is cohomologous to the abelian 3-cocycle $h' : M^3 \cup M|M \to N$ given by

$$h'(x|y) = h(x|y) - g(x, y) + g(y, x).$$

Let us now define, for each $\sigma \in G$, the map $h_\sigma : M^3 \cup M|M \to N$ by

$$h_\sigma(x, y, z) = 0,$$

$$h_\sigma(x|y) = \sigma h'(x|y) - h'(|\sigma x| \sigma y).$$

It is easy to check that so defined, each $h_\sigma$ is an abelian 3-cocycle of the abelian group $M$ with coefficients in $N$ and, moreover, with zero trace:

$$t_{h_\sigma}(x) = h_\sigma(x| y) = \sigma h'(x| y) - h'(|\sigma x| \sigma y) = \sigma q(x) - q(|\sigma x) = 0,$$

since $q$ is $G$-equivariant. It follows that $0 = [h_\sigma] \in H^3_{ab}(M, N)$, that is, there is a map $g_\sigma : M^2 \to N$ such that

$$h_\sigma(x, y, z) = 0 = g_\sigma(y, z) - g_\sigma(x + y, z) + g_\sigma(x, y + z) - g_\sigma(x, y),$$

$$h_\sigma(x|y) = g_\sigma(x, y) - g_\sigma(y, x).$$

Since $H_2(M) = 0$, the Universal Coefficient Theorem implies that the canonical injection $\text{Ext}_{\mathbb{Z}}^1(M, N) \to H^2(M, N)$ is an isomorphism and, therefore, every group extension of $M$ by $N$ is abelian. This is equivalent to saying that every ordinary 2-cocycle of the group $M$ with coefficients in the (trivial) $M$-module $N$, say $f : M^2 \to N$, satisfies the symmetric condition $f(x, y) = f(y, x)$. In particular, we have $g_\sigma(x, y) = g_\sigma(y, x)$, whence $h_\sigma(x| y) = 0$, for all $\sigma \in G$ and $x, y \in M$. Therefore, we have the equality $\sigma h'(x|y) = h'(\sigma x| \sigma y)$. Then the map

$$h' : M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N,$$

defined by $h'(x, y, z) = 0$, $h'(x|y) = h(x|y) - g(x, y) + g(y, x)$, as above, and $h'(x, y|\sigma) = 0$ and $h'(x|\sigma, \tau) = 0$, is an abelian 3-cocycle of the $G$-module $M$ with coefficients in the $G$-module $N$. Clearly its trace is

$$t_{h'}(x) = h'(x| x) = h(x|x) = q(x),$$

that is $j[h'] = q$. Hence the homomorphism $j$ in the sequence (5.9) is surjective, whence the proposition follows. □
6. Braided graded categorical groups and classification results

As mentioned in the introduction, this last section is dedicated to showing precise theorems on the homotopy classification of braided $G$-graded categorical groups and their homomorphisms. The results are stated and proved by means of the cohomology theory of $G$-modules $H^n_{Gr}(M, N)$ studied throughout the previous sections.

We shall begin by recalling some needed terminology about graded monoidal categories and the definition of braided $G$-graded categorical groups, for $G$ a given group.

We regard the group $G$ as a category with one object, where the morphisms are elements of $G$ and the composition is the group operation. A grading on a category $\mathbb{G}$ is then a functor, say $gr : \mathbb{G} \to G$. For any morphism $f$ in $\mathbb{G}$, we refer to $gr(f) = \sigma$ as the grade of $f$ and say that $f$ is a $\sigma$-morphism. The grading is said to be stable if for any object $X$ of $\mathbb{G}$ and any $\sigma \in G$ there exists an isomorphism $X \cong Y$ with domain $X$ and grade $\sigma$; in other words, the grading is a cofibration in the sense of Grothendieck [15]. A functor $F : \mathbb{G} \to \mathbb{H}$ between graded categories is called a graded functor if it preserves grades of morphisms. From [15, Corollary 6.12], every graded functor between stably $G$-graded categories is cocartesian. Suppose $F' : \mathbb{G} \to \mathbb{K}$ is another graded functor. Then, a graded natural equivalence $\theta : F \to F'$ is a natural equivalence of functors such that all isomorphisms $\theta_X : FX \to F'X$ are of grade 1. If $G$ is a graded category, the category $\text{Ker} \mathbb{G}$ is the subcategory consisting of all morphisms of grade 1. A graded functor $F : \mathbb{G} \to \mathbb{H}$ between stable graded categories is an equivalence if and only if the induced functor $\text{Ker} \mathbb{G} \to \text{Ker} \mathbb{H}$ is an equivalence of categories [15, Proposition 6.5].

For a $G$-graded category $\mathbb{G}$, we denote by $\mathbb{G} \times_G \mathbb{G}$ the subcategory of the product category $\mathbb{G} \times \mathbb{G}$ whose morphisms are all pairs of morphisms of $\mathbb{G}$ with the same grade. This category $\mathbb{G} \times_G \mathbb{G}$ has an obvious grading, which is stable if and only if the grading of $\mathbb{G}$ is as well.

A braided $G$-graded monoidal category $\mathbb{G} := (\mathbb{G}, gr, \otimes, I, A, L, R, C)$ consists of a category $\mathbb{G}$, a stable grading $gr : \mathbb{G} \to G$, graded functors $\otimes : \mathbb{G} \times_G \mathbb{G} \to \mathbb{G}$ and $I : G \to \mathbb{G}$, and graded natural equivalences defined by isomorphisms of grade 1 $A_{\mathbb{G},\mathbb{G}}(X,Y,Z) : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $L_X : I \otimes X \to X$, $R_X : X \otimes I \to X$ and $C_{X,Y} : X \otimes Y \sim Y \otimes X$, such that for any objects $X, Y, Z, T$ of $\mathbb{G}$ the following four coherence conditions hold:

\begin{align*}
A_{X,Y,Z} & : (id_X \otimes A_{Y,Z}, T) A_{X,Y,Z,T} (A_{X,Y,Z} \otimes id_T), \quad (6.1) \\
(id_X \otimes L_T) A_{X,Y,Z,T} & = R_X \otimes id_Y, \quad (6.2) \\
(id_Y \otimes C_{X,Z}) A_{Y,X,Z} & = A_{Y,Z,X} C_{X,Y,Z} A_{X,Y,Z}, \quad (6.3) \\
(C_{X,Z} \otimes id_Y) A_{X,Y,Z}^{-1} & = A_{Z,X,Y}^{-1} (id_X \otimes C_{X,Z}) A_{X,Y,Z}^{-1}. \quad (6.4)
\end{align*}

If $\mathbb{G}, \mathbb{H}$ are braided $G$-graded monoidal categories, then a braided graded monoidal functor $F := (F, \Phi, \Phi_*) : \mathbb{G} \to \mathbb{H}$ consists of a graded functor $F : \mathbb{G} \to \mathbb{H}$, and natural graded equivalences defined by 1-graded isomorphisms $\Phi_{X,Y} : FX \otimes FY \sim F(X \otimes Y)$ and $\Phi_* : I \to FI$, such that, for all $X, Y, Z \in G$, the following coherence conditions hold:

\begin{align*}
\Phi_{X,Y,Z} (id_{FX} \otimes \Phi_{Y,Z}) A_{FX,FY,FZ} & = F(A_{X,Y,Z}) \Phi_{X,Y,Z} (\Phi_{X,Y} \otimes id_{FZ}), \quad (6.5) \\
F(R_X) \Phi_{X,1} (id_{FX} \otimes \Phi_*) & = R_{FX}, \quad F(L_X) \Phi_{I,X} (\Phi_* \otimes id_{FX}) = L_{FX}, \quad (6.6) \\
\Phi_{Y,X} C_{FX,FY} F(C_{X,Y}) & = F(C_{X,Y}) \Phi_{X,Y}. \quad (6.7)
\end{align*}

Suppose $F' : \mathbb{G} \to \mathbb{H}$ is another braided graded monoidal functor. A homotopy (or braided graded monoidal natural equivalence) $\theta : F \sim F'$ is a graded natural equivalence such that, for all objects $X, Y \in \mathbb{G}$, the following coherence conditions hold:

\begin{align*}
\Phi'_{X,Y} (\theta_X \otimes \theta_Y) = \theta_{X \otimes Y} \Phi_{X,Y}, \quad (6.8)
\end{align*}

For later use we state here the lemma below, whose proof is parallel to Lemma 1.1 in [4].

**Lemma 18.** Every braided graded monoidal functor $F = (F, \Phi, \Phi_*) : \mathbb{G} \to \mathbb{H}$ is homotopic to a braided graded monoidal functor $F' = (F', \Phi', \Phi'_*)$ with $F' I = I$ and $\Phi'_* = id$.

A braided categorical group [18] is a braided monoidal category in which every morphism is invertible and, for each object $X$, there is an object $X'$ with a morphism $X \otimes X' \to I$. If $\mathbb{G}$ is a braided $G$-graded monoidal category,
then the subcategory \( \mathrm{Ker} \mathcal{G} \) inherits a braided monoidal structure. When \( \mathrm{Ker} \mathcal{G} \) is a braided categorical group, then \( \mathcal{G} \) is said to be a braided \( G \)-graded categorical group. Since \( \mathrm{Ker} \mathcal{G} \) is a groupoid if and only if \( \mathcal{G} \) is, we make the following:

**Definition 19.** A braided \( G \)-graded categorical group is a braided \( G \)-graded monoidal groupoid such that, for any object \( X \), there is an object \( X' \) with an arrow \( X \otimes X' \to 1 \) of grade 1.

We denote by \( \mathcal{B} \mathcal{G} \mathcal{C} \mathcal{G} \) the category of braided \( G \)-graded categorical groups, whose morphisms are the braided graded monoidal functors between them. Homotopy is an equivalence relation among morphisms of \( \mathcal{B} \mathcal{G} \mathcal{C} \mathcal{G} \) and it is compatible with compositions in \( \mathcal{B} \mathcal{G} \mathcal{C} \mathcal{G} \). We can therefore define the homotopy category of braided graded categorical groups to be the quotient category with the same objects, but morphisms are homotopy classes of braided graded monoidal functors. A braided graded monoidal functor inducing an isomorphism in the homotopy category is said to be a braided graded monoidal equivalence and two braided graded categorical groups are equivalent if they are isomorphic in the homotopy category.

The homotopy classification of braided \( G \)-graded categorical groups is our major objective. For, we will associate to each braided \( G \)-graded categorical group \( \mathcal{G} \) the algebraic data \( \pi_0 \mathcal{G}, \pi_1 \mathcal{G} \) and \( k(\mathcal{G}) \), which are invariants under braided graded monoidal equivalence. We next introduce the first two.

- \( \pi_0 \mathcal{G} \) is the abelian group of 1-isomorphism classes of the objects in \( \mathcal{G} \) where multiplication is induced by the tensor product.
- \( \pi_1 \mathcal{G} \) is the abelian group of 1-automorphisms of the unit object 1 where the operation is composition.

Thus, \( \pi_i \mathcal{G} = \pi_i \mathrm{Ker} \mathcal{G}, i = 0, 1 \), the first invariants of the braided categorical group \( \mathrm{Ker} \mathcal{G} \) considered by Joyal and Street in [18]. Note that \( \pi_1 \mathcal{G} \) is abelian since the multiplication \( \pi_1 \mathcal{G} \times \pi_1 \mathcal{G} \to \pi_1 \mathcal{G}, (a, b) \mapsto L_1(a \otimes b) L_1^{-1} \), is a group homomorphism [9]. The group \( \pi_0 \mathcal{G} \) is also abelian because of the braiding. Next we observe that both \( \pi_0 \mathcal{G} \) and \( \pi_1 \mathcal{G} \) are \( G \)-modules.

If \( \sigma \in G \) and \([X] \in \pi_0 \mathcal{G}\), then we write

\[
\sigma [X] = [X'],
\]

whenever there exists a morphism of grade \( \sigma \), \( X \to X' \). Since the grading on \( \mathcal{G} \) is stable, \( \sigma [X] \) is defined for all \( \sigma \in G \) and \([X] \in \pi_0 \mathcal{G}\). The map \((\sigma, [X]) \mapsto \sigma [X]\), is well defined since every morphism in \( \mathcal{G} \) is invertible: if \( g : X \to X' \) and \( g' : Y \to Y' \) are both \( \sigma \)-morphisms and \( h : X \to Y \) is an 1-morphism, then \( h' = g'h g^{-1} \) is a \( \sigma \)-morphism, and \( X' = [Y'] \in \pi_0 \mathcal{G} \). If \( g \) and \( g' \) are any two \( \sigma \)-morphisms as before, then \( g \otimes g' : X \otimes Y \to X' \otimes Y' \) is also a \( \sigma \)-morphism; whence, \( \sigma [X \otimes Y] = [X'] \otimes [Y'] = \sigma [X] \otimes \sigma [Y] \). Furthermore, if \( \sigma, \tau \in G \), then for any \([X] \in \pi_0 \mathcal{G}\), any \( \tau \)-morphism \( h : X \to Y \) and any \( \sigma \)-morphism \( g : Y \to Z \), since the composition \( fg \) is a \( \sigma \tau \)-morphism, we have \( \sigma \tau [X] = [Z] = \sigma [Y] = \sigma ([\tau [X]]) \). Hence, \( \pi_0 \mathcal{G} \) is a \( G \)-module.

The \( G \)-module structure on \( \pi_1 \mathcal{G} \) is easier to explain. For any \( \sigma \in G \) and any arrow \( a : 1 \to 1 \) in \( \mathcal{G} \) of grade 1, it is

\[
\sigma a = I(\sigma) a I(\sigma)^{-1}.
\]

The mappings \( \mathcal{G} \mapsto \pi_i \mathcal{G}, i = 0, 1 \), are functorial from the category of braided \( G \)-graded categorical groups to the category of \( G \)-modules. Moreover, since any braided graded monoidal functor between braided graded categorical groups is a braided graded equivalence if and only if its restriction to kernels is a braided equivalence, then, by [18] and by a little modification of the proof of [4, Proposition 1.3] (see also [12]), the proposition below easily follows:

**Proposition 20.** (i) Every braided graded monoidal functor \( F : \mathcal{G} \to \mathcal{G}' \) between braided \( G \)-graded categorical groups induces homomorphisms of \( G \)-modules

\[
\pi_i F : \pi_i (\mathcal{G}) \to \pi_i (\mathcal{G}'), \quad i = 0, 1,
\]

given by \( \pi_0 F : [X] \mapsto [FX], \pi_1 F : a \mapsto \Phi_a^{-1} F(a) \Phi_a \).

(ii) Two homotopic braided graded monoidal functors induce the same homomorphisms of \( G \)-modules.

(iii) A braided graded monoidal functor is a braided graded equivalence if and only if the induced homomorphisms of \( G \)-modules, \( \pi_0 F \) and \( \pi_1 F \), are isomorphisms.

We shall now establish the following terminology.
**Definition 21.** A braided $G$-graded categorical group of type $(M, N)$, where $M$ and $N$ are $G$-modules, is a triple $(G, \alpha, \beta)$ in which $G$ is a braided $G$-graded categorical group, and $\alpha: \pi_0 G \cong M$ and $\beta: \pi_1 G \cong N$ are $G$-module isomorphisms.

If $(G', \alpha', \beta')$ is a braided $G'$-graded categorical group of type $(M', N')$, then a braided graded monoidal functor $F: G \to G'$ is said to be of type $(p, q)$, where $p: M \to M'$ and $q: N \to N'$ are $G$-module homomorphisms, whenever the two diagrams below commute:

$$
\begin{array}{ccc}
\pi_0 G & \xrightarrow{\alpha} & M \\
\pi_1 G & \xrightarrow{\beta} & N \\
\pi_0 G' & \xrightarrow{\alpha'} & M' \\
\pi_1 G' & \xrightarrow{\beta'} & N'.
\end{array}
$$

Two braided $G$-graded categorical groups of the same type $(M, N)$, say $(G, \alpha, \beta)$ and $(G', \alpha', \beta')$, are equivalent if there exists a braided graded monoidal functor (necessarily an equivalence by **Proposition 20**) $F: G \to G'$ of type $(id_M, id_N)$, that is, such that $\alpha' \pi_0 F = \alpha$ and $\beta' \pi_1 F = \beta$.

The set of equivalence classes of braided $G$-graded categorical groups of type $(M, N)$ is denoted by $BC\mathcal{G}_G[M, N]$.

Next, we shall prove that 3-cocycles in $C^3_{G,ab}(M, N)$ are the appropriate data to construct the manifold of all braided $G$-graded categorical groups of type $(M, N)$ up to equivalence.

Every abelian 3-cocycle $h \in Z^3_{G,ab}(M, N)$,

$$h: M^3 \cup M|M \cup M^2|G \cup M|G^2 \to N,$$

gives rise to a braided $G$-graded categorical group of type $(M, N)$,

$$\mathbb{G}(h) := (\mathbb{G}(h), \alpha_h, \beta_h),$$

which is defined as follows: the objects of $\mathbb{G}(h)$ are the elements of $M$; an arrow $x \to y$ in $\mathbb{G}(h)$ is a pair $(a, \sigma): x \to y$ where $a \in N$ and $\sigma \in G$ verifying $\sigma x = y$.

The composition of two morphisms $x \xrightarrow{(a, \sigma)} y \xrightarrow{(b, \tau)} z$ is defined by

$$(b, \tau)(a, \sigma) = (b + \tau a + h(x \mid \tau, \sigma), \tau \sigma).$$

This composition is unitary thanks to the normalization condition of $h$ and it is associative owing to the 3-cocycle condition $\partial h = 0$ in (3.14). Note that every morphism is invertible, indeed

$$(a, \sigma)^{-1} = (-\sigma^{-1} a - h(x \mid \sigma^{-1}, \sigma), \sigma^{-1}).$$

Hence, $\mathbb{G}(h)$ is a groupoid.

The stable $G$-grading is given by $gr(a, \sigma) = \sigma$.

The graded tensor product $\otimes: \mathbb{G}(h) \times_G \mathbb{G}(h) \to \mathbb{G}(h)$ is defined by

$$(x \xrightarrow{(a, \sigma)} y) \otimes (x' \xrightarrow{(b, \sigma')} y') = (x + x' \xrightarrow{(a+b+h(x', x', \sigma), \sigma)} y + y'),$$

which is a functor thanks to the 3-cocycle condition $\partial h = 0$ in (3.13) and the normalization condition of $h$.

The associativity isomorphisms are

$$A_{x,y,z} = (h(x, y, z), 1): (x + y) + z \to x + (y + z),$$

which satisfy the coherence condition (6.1) because of the 3-cocycle condition $\partial h = 0$ in (3.8). The naturalness here follows since $\partial h = 0$ in (3.11).

The unit graded functor $I: G \to \mathbb{G}(h)$ is defined by

$I(\sigma) = (0 \xrightarrow{(0, \sigma)} 0),$
and the unit constraints are identities: $L_x = (0, 1) = R_x : x \rightarrow x$.

The braiding for $\mathcal{G}(h)$ is given by

$$C_{x,y} = (h(x|y), 1) : x + y \rightarrow y + x.$$  \hfill (6.16)

The cocycle conditions $\partial h = 0$ in (3.9) and (3.10) amount precisely to the coherence conditions (6.3) and (6.4) respectively. The naturalness of the braiding follows since $\partial h = 0$ in (3.12).

Thus, $\mathcal{G}(h)$ is a braided $G$-graded monoidal groupoid, which is actually a braided $G$-graded categorical group since, for any object $x$ of $\mathcal{G}(h)$, we have $x \otimes (-x) = x + (-x) = 0 = I$.

Finally, we recognize that $\mathcal{G}(h)$ is of type $(M, N)$ by means of the obvious $G$-module isomorphisms

$$\alpha_h : \pi_0 \mathcal{G}(h) \cong M, \quad \beta_h : \pi_1 \mathcal{G}(h) \cong N,$$  \hfill (6.17)

which are defined by

$$\alpha_h(x) = x, \quad \beta_h(a, 1) = a.$$  

With the device $h \mapsto \mathcal{G}(h)$, $h \in Z^3(\mathcal{G}, \mathcal{M}, N)$, we are ready to prove the main results in this section. The next theorem deals with the classification of braided graded monoidal functors between braided $G$-graded categorical groups of the form $\mathcal{G}(h)$, and the following one shows that every braided $G$-graded categorical group is equivalent to $\mathcal{G}(h)$ for some $h$.

**Theorem 22.** Let $h \in Z^3(\mathcal{G}, \mathcal{M}, N)$, $h' \in Z^3(\mathcal{G}, \mathcal{M}', N')$ be abelian 3-cocycles, where $M, M', N', N'$ are $G$-modules, and suppose that $p : M \rightarrow M'$ and $q : N \rightarrow N'$ are any given $G$-module homomorphisms.

Then, there exists a braided graded monoidal functor $\mathcal{G}(h) \rightarrow \mathcal{G}(h')$ of type $(p, q)$ if and only if the abelian 3-cocycles $p^*h', q^*_a[h] \in Z^3(\mathcal{G}, \mathcal{M}, N')$ represent the same cohomology class, that is, if and only if

$$0 = p^*[h] - q^*_a[h] \in H^3(\mathcal{G}, \mathcal{M}, N').$$

Furthermore, when $p^*[h] = q^*_a[h]$, homotopy classes of braided monoidal functors $\mathcal{G}(h) \rightarrow \mathcal{G}(h')$ of type $(p, q)$ are in bijection with elements of the group $H^3(\mathcal{G}, \mathcal{M}, N')$.

**Proof.** We first assume that $p^*[h] = q^*_a[h] \in H^3(\mathcal{G}, \mathcal{M}, N')$. Then, we have $q^*_a[h] = p^*h' + \partial g$, for some $g \in C^2(\mathcal{G}, \mathcal{M}, N')$ which determines a braided graded monoidal functor of type $(p, q)$

$$F = F(g) : \mathcal{G}(h) \rightarrow \mathcal{G}(h'),$$  \hfill (6.18)

given by

$$F(x \xrightarrow{(a, \sigma)} y) = (p(x) \xrightarrow{(q(a) + g(x|a), \sigma)} p(y)),$$  \hfill (6.19)

together with the isomorphisms $\Phi$ of grade 1

$$\Phi_{x,y} = (g(x, y), 1) : p(x) + p(y) \rightarrow p(x + y)$$  \hfill (6.20)

and

$$\Phi_{a} = id = (0, 1) : p(0) \rightarrow 0.$$  

So defined, it is routine to check that $F$ is actually a functor because of the equality (3.7) and the normalization condition of $g$. The isomorphisms $\Phi_{x,y}$ define a graded natural equivalence $F(-) \otimes F(-) \cong F(- \otimes -)$ owing to the coboundary condition (3.6). The coherence conditions (6.5) and (6.7) hold thanks to (3.4) and (3.5) respectively, whilst (6.6) is trivially verified. Since $\alpha_{h'} \pi_0 F(x) = p(x) = p \alpha_h(x)$ and $\beta_{h'} \pi_1 F(a, 1) = q(a) = q \beta_h(a, 1)$, we see that $F$ is actually of type $(p, q)$.

Conversely, suppose that $F = (F, \Phi, \Phi_{a}) : \mathcal{G}(h) \rightarrow \mathcal{G}(h')$ is any braided graded monoidal functor of type $(p, q)$. By Lemma 18, there is no loss of generality in assuming that $F$ satisfies that $\Phi_{a} = id_0 = (0, 1)$. Then, $F$ acts as $p$ on objects, since $\alpha_{h'} \pi_0 F = p \alpha_h$, and

$$F(0 \xrightarrow{(a, 1)} 0) = 0 \xrightarrow{(q(a), 1)} 0,$$
for any \( a \in N \), since \( \beta h' \pi_1 F = q \beta h \). Furthermore, by coherence condition (6.6), one has \( \Phi_{x,0} = id_x = \Phi_{0,x} \) for all \( x \in M \) and then, since every morphism of grade 1, say \( x \xrightarrow{(a,1)} x \), can be expressed in the form \( x \xrightarrow{(a,1)} x = (0 \xrightarrow{(a,1)} 0) \otimes (x \xrightarrow{(0,1)} x) \), we deduce by naturalness that

\[
F(x \xrightarrow{(a,1)} x) = F(0 \xrightarrow{(a,1)} 0) \otimes F(x \xrightarrow{(0,1)} x) = (0 \xrightarrow{(q,a),1)} 0) \otimes \left( p(x) \xrightarrow{(0,1)} p(x) \right) = p(x) \xrightarrow{(q,a),1)} p(x).
\]

If we write for each \( \sigma \in G \) and \( x \in M \)

\[
F(x \xrightarrow{(0,\sigma)} x) = \left( p(x) \xrightarrow{(g(x|\sigma),\sigma)} p(x) \right), \quad g(x|\sigma) \in N',
\]

and

\[
\Phi_{x,y} = \left( x + y \xrightarrow{(g(x,y),1)} x + y \right), \quad g(x, y) \in N',
\]

for each \( x, y \in M \), we get an abelian 2-cochain \( g : M^2 \cup M|G \rightarrow N' \in C^2_{G,ab}(M, N') \), which determines \( F \) completely. Indeed, for any morphism in \( \mathcal{G}(h) \) say \( x \xrightarrow{(a,\sigma)} y \), we have

\[
F(x \xrightarrow{(a,\sigma)} y) = F(y \xrightarrow{(a,1)} y) \cdot F(x \xrightarrow{(0,\sigma)} y)
\]

\[
= \left( p(y) \xrightarrow{(a,1)} p(y) \right) \cdot \left( p(x) \xrightarrow{(g(x|\sigma),\sigma)} p(y) \right) = p(x) \xrightarrow{(a+g(x|\sigma),\sigma)} p(y).
\]

It is now straightforward to see that the equality \( q_s h = p^* h' + \partial g \) amounts to the conditions of \( F \) being a braided graded monoidal functor. More precisely, the equality \( q_s h(x, y, z) = p^* h'(x, y, z) + \partial g(x, y, z) \) follows from the coherence condition (6.5); \( q_s h(x|y) = p^* h'(x|y) + \partial g(x|y) \) is a consequence of (6.7); \( q_s h(x, y|\sigma) = p^* h'(x, y|\sigma) + \partial g(x, y|\sigma) \) owing to the naturalness of the isomorphisms \( \Phi_{x,y} \) and the equality \( q_s h(x|\sigma, \tau) = p^* h'(x|\sigma, \tau) + \partial g(x|\sigma, \tau) \) is a direct consequence of \( F \) being a functor. Therefore, \( q_s h \) and \( p^* h' \) are cohomologous abelian 3-cocycles of the \( G \)-module \( M \) with coefficients in \( N' \), as claimed.

To prove that homotopy classes of braided monoidal functors \( \mathcal{G}(h) \rightarrow \mathcal{G}(h') \) of type \( (p, q) \) are in bijection with elements of the group \( H^2_{G,ab}(M, N') \), we shall stress that we have actually proved before that the mapping \( g \mapsto F(g) \), given by construction (6.18), induces a surjection from the set of those abelian 2-cochains \( g \in C^2_{G,ab}(M, N') \) such that \( q_s h = p^* h' + \partial g \) onto the set of homotopy classes of braided graded monoidal functors \( \mathcal{G}(h) \rightarrow \mathcal{G}(h') \) of type \( (p, q) \) (compare formulas (6.21) and (6.22) with (6.19) and (6.20) respectively).

We now note that if we fix any \( g_0 \in C^2_{G,ab}(M, N') \) satisfying \( q_s h = p^* h' + \partial g_0 \), which exists under the hypothesis \( p^*[h'] = q_s[h] \), then any other such abelian 2-cochain \( g_1 \in C^2_{G,ab}(M, N') \) with \( q_s h = p^* h' + \partial g_1 \), is necessarily written in the form \( g_1 = g_0 + g \) with \( \partial g = 0 \), that is, where \( g \in Z^2_{G,ab}(M, N') \).

Then, to complete the proof of the theorem, it suffices to prove that two braided graded monoidal functors \( F(g_0 + g) \) and \( F(g_0 + g') \), where \( g, g' \in Z^2_{G,ab}(M, N') \), are homotopic if and only if the \( h \) and \( g' \) are cohomologous:

Let \( g' \sim g = \partial f \) for some \( f \in C^1_{G,ab}(M, N') \). Then the following family of isomorphisms of grade 1 in \( \mathcal{G}(h) \),

\[
\theta_x : p(x) \xrightarrow{(f(x),1)} p(x), \quad x \in M,
\]

defines a graded natural equivalence \( \theta : F(g_0 + g') \rightarrow F(g_0 + g) \) thanks to the condition (3.3), which also verifies the condition (6.8) due to the equality (3.2). That is, \( \theta \) is a homotopy of braided graded monoidal functors.

And conversely, if \( \theta : F(g_0 + g') \rightarrow F(g_0 + g) \) is any homotopy of braided graded monoidal functors and we write \( \theta_x = (f(x), 1) : p(x) \rightarrow p(x) \) for a map \( f : M \rightarrow N \), then one can easily check that \( f \in C^1_{G,ab}(M, N') \) and \( g' - g = \partial f \).

□
Corollary 23. Let \( h, h' \in \mathbb{Z}_G^3(M, N) \) be two abelian 3-cocycles of a \( G \)-module \( M \) with coefficients in a \( G \)-module \( N \). Then, \( G(h) \) and \( G(h') \) are equivalent, as braided \( G \)-graded categorical groups of type \((M, N)\), if and only if \( h \) and \( h' \) are cohomologous.

Proof. By definition, \( G(h) \) and \( G(h') \) are equivalent whenever there exists a braided monoidal functor \( G(h) \to G(h') \) of type \((\text{id}_M, \text{id}_N)\), which, by Theorem 22, occurs if and only if \( 0 = [h'] - [h] \in H^3_{G, \text{ab}}(M, N) \). \( \Box \)

Theorem 24. For any \( G \)-modules \( M \) and \( N \), the map \( h \mapsto \mathcal{G}(h) \), \( h \in \mathbb{Z}_G^3(M, N) \), induces a bijection between elements of the cohomology group \( H^3_{G, \text{ab}}(M, N) \) and equivalence classes of braided \( G \)-graded categorical groups of type \((M, N)\). Hence there is a bijection

\[
\mathcal{B}C\mathcal{G}_G[M, N] \cong H^3_{G, \text{ab}}(M, N), \quad [G, \alpha, \beta] \mapsto \alpha^*\beta_*k(G),
\]

where we refer to \( k(G) \in H^3_{G, \text{ab}}(\pi_0\mathcal{G}, \pi_1\mathcal{G}) \) as the Postnikov invariant of the braided \( G \)-graded categorical group \( \mathcal{G} \).

Proof. By the previous Corollary 23, the correspondence \([h] \mapsto [\mathcal{G}(h)]\) is a correctly defined injective map \( H^3_{G, \text{ab}}(M, N) \to \mathcal{B}C\mathcal{G}_G[M, N] \). Therefore it only remains to prove that this map is onto, that is, that every braided \( G \)-graded categorical group of type \((M, N)\), say \( \mathcal{G} = (\mathcal{G}, \alpha, \beta) \), is equivalent to one \( \mathcal{G}(h) \) for some \( h \in \mathbb{Z}_G^3(M, N) \).

To simplify the notation, we can assume, without loss of generality, that the \( G \)-module isomorphisms \( \alpha \) and \( \beta \) are identities, that is, \( \pi_0\mathcal{G} = M \) and \( \pi_1\mathcal{G} = N \).

Recall now that a (co)fibred category is skeletal when all its fibre categories are skeletal. In particular, a graded category is skeletal when any two objects isomorphic by an isomorphism of grade 1 are equal. The braided \( G \)-graded categorical group \( \mathcal{G} \) is equivalent to a skeletal one, say \( \hat{\mathcal{G}} \), which can be constructed as follows: for each \( x \in M \), let us choose an object \( O_x \in x \), with \( O_0 = I \), and for any other \( O \in x \), we fix a 1-morphism \( \Phi_O : O \to O_x \), with \( \Phi_{O_0} = \text{id}_{O_0}, \Phi_{O_0 \otimes I} = L_{O_x} \) and \( \Phi_{O \otimes O} = R_{O_x} \). Let \( \hat{\mathcal{G}} \) be the full subcategory of \( \mathcal{G} \) whose objects are all \( O_x, x \in M \). Then \( \hat{\mathcal{G}} \) is stably \( G \)-graded with grading \( \text{gr} \hat{\mathcal{G}} = \text{gr} \mathcal{G} : \hat{\mathcal{G}} \to G \); the inclusion functor \( \hat{\mathcal{G}} \hookrightarrow \mathcal{G} \) is a graded equivalence and clearly \( \hat{\mathcal{G}} \) is skeletal. Now, the braided graded categorical group structure of \( \mathcal{G} \) can be transported to \( \hat{\mathcal{G}} \), in a unique way such that the inclusion functor \( \hat{\mathcal{G}} \hookrightarrow \mathcal{G} \), together with the isomorphisms \( \Phi_{O_x \otimes O_y} = \Phi_{O_x \otimes O_y} : O_x \otimes O_y \to O_x \otimes O_y = O_{x+y} \) and \( \Phi_{O_0} = \text{id}_{O} \), turns out to be a braided graded monoidal equivalence (see (6.5) and (6.6)). Note that in the resulting skeletal braided \( G \)-graded categorical group \( \hat{\mathcal{G}} = \left( \hat{\mathcal{G}}, \text{gr} \hat{\mathcal{G}}, \otimes, I, \Lambda, \bar{L}, \bar{R}, \bar{C} \right) \), the unit \( I \) is strict in the sense that \( \bar{L} = \text{id} = \bar{R} \).

Hence it is no loss of generality if we suppose that \( \mathcal{G} \) is a skeletal braided \( G \)-graded categorical group in which the unit constraints are identities, and also \( \pi_0\mathcal{G} = M, \pi_1\mathcal{G} = N \).

Then, the following facts hold: \( \text{Ob}\hat{\mathcal{G}} = M \) and \( x \otimes y = x + y \) for all \( x, y \in M \). The (strict) unit object is \( I = 0 \), \( \text{Aut}_1(0) = N \) and \( a \otimes b = a + b = ab \) for all \( a, b \in N \). For any \( x \in M, N \cong \text{Aut}_1(x) \) by the isomorphism \( a \mapsto a \otimes \text{id}_x \); furthermore,

\[
a \otimes \text{id}_x = \text{id}_x \otimes a \quad (6.23)
\]

for any \( a \in N \) (by [18, Proposition 2.1] and the naturalness of the braiding). If \( x, y \in M \) and \( \sigma \in G \), then there exists a \( \sigma \)-morphism in \( \mathcal{G} \), \( u : x \to y \) if and only if \( \sigma x = y \) (according to (6.9)). For any such morphism \( u : x \to y \), of grade \( \sigma \), we have the composite bijection

\[
N \cong \text{Aut}_1(y) \cong \text{Hom}_G(x, y), \quad a \mapsto (a \otimes \text{id}_y)u, \quad (6.24)
\]

between elements of \( N \) and arrows of grade \( \sigma \) in \( \mathcal{G} \) from \( x \) to \( y = \sigma x \); further,

\[
I(\sigma) \otimes u = u = u \otimes I(\sigma) \quad (6.25)
\]

(due to the naturalness of the unit constraints) and, for all \( a \in N \),

\[
u (a \otimes \text{id}_x) = (\sigma a \otimes \text{id}_y) u. \quad (6.26)
\]

In effect,

\[
(\sigma a \otimes \text{id}_y) u \overset{(6.25)}{=} (\sigma a \otimes \text{id}_y) (I(\sigma) \otimes u) = (\sigma a I(\sigma) \otimes u) \overset{(6.10)}{=} (I(\sigma) a \otimes u) (\alpha \otimes \text{id}_y) \overset{(6.25)}{=} u (a \otimes \text{id}_y).
\]
We now choose, for each \( \sigma \in G \) and \( x \in M \), a morphism in \( \mathbb{G} \) with domain \( x \) and grade \( \sigma \), say
\[
u_{x,\sigma} : x \to y = \sigma x, \quad \text{with } \nu_{0,\sigma} = 1(\sigma), \nu_{x,1} = \text{id}_x
\]
and, for each \( a \in N \), we shall write
\[
x \xrightarrow{(a,\sigma)} y := x \xrightarrow{(a \otimes \text{id}_x)\nu_{x,\sigma}} y.
\]
That is, we are denoting by \((a, \sigma) : x \to y\) the \( \sigma \)-morphism from \( x \) to \( y \) that corresponds to \( a \) by the bijection (6.24) for \( u = \nu_{x,\sigma} \). Thus, for example, we have the equalities
\[
x \xrightarrow{(0,\sigma)} y = x \xrightarrow{\nu_{x,\sigma}} y, \quad x \xrightarrow{(a,1)} x = x \xrightarrow{a \otimes \text{id}_x} x,
\]
\[
x \xrightarrow{(0,1)} x = x \xrightarrow{\text{id}_x} x, \quad 0 \xrightarrow{(0,\sigma)} 0 = 0 \xrightarrow{1(\sigma)} 0,
\]
\[
x \xrightarrow{(a,\sigma)} y \overset{(6.27)}{=} (y \xrightarrow{(a,1)} y)(x \xrightarrow{(0,\sigma)} y),
\]
\[
x \xrightarrow{c(a,\sigma)} y \overset{(6.26)}{=} (x \xrightarrow{(0,\sigma)} y)(x \xrightarrow{(a,1)} y),
\]
\[
(x \xrightarrow{(a,1)} x)(x \xrightarrow{(b,1)} x) = (x \xrightarrow{(a+b,1)} x),
\]
since \((a \otimes \text{id}_x)(b \otimes \text{id}_x) = (a + b) \otimes \text{id}_x,\)
\[
(x \xrightarrow{(a,1)} x) \otimes (y \xrightarrow{(b,1)} y) = (x \xrightarrow{(a+b,1)} x + y),
\]
since \((a \otimes \text{id}_x) \otimes (b \otimes \text{id}_x) \overset{(6.23)}{=} (a + b) \otimes \text{id}_x + y.

All in all, we are now ready to build a 3-cocycle \( h = h^G \in \mathbb{Z}_3^{3,ab}(M, N) \) such that \( \mathbb{G}(h) \simeq \mathbb{G} \). For we begin by determining a 3-cochain
\[
h : M^3 \cup M \cup M^2 | G \cup M | G^2 \to N,
\]
by the four equations below.
\[
(y \xrightarrow{(0,\sigma)} z)(x \xrightarrow{(0,\tau)} y) = (x \xrightarrow{h(x|\sigma,\tau),\sigma \tau} z),
\]
for \( \sigma, \tau \in G, x \in M, \tau x = y, \sigma y = z;\)
\[
(x \xrightarrow{(0,\sigma)} y) \otimes (x' \xrightarrow{(0,\sigma)} y') = (x + x' \xrightarrow{h(x,x'|\sigma),\sigma} y + y'),
\]
for \( \sigma \in G, x, x' \in M, \sigma x = y, \sigma x' = y';\)
\[
(x + y \xrightarrow{c_{x,y}} y + x) = (x + y \xrightarrow{h(x|y),1} y + x),
\]
for \( x, y \in M;\)
\[
((x + y) + z \xrightarrow{A_{x,y,z}} x + (y + z)) = ((x + y) + z \xrightarrow{h(x,y,z),1} x + (y + z)),
\]
for \( x, y, z \in M.\)

So defined, this 3-cochain \( h \) completely determines the braided graded strictly unitary monoidal category structure of \( \mathbb{G} \), since the following two equalities hold:
\[
(y \xrightarrow{(b,\tau)} z)(x \xrightarrow{(a,\sigma)} y) = (x \xrightarrow{(b + \tau a + h(x|\tau,\sigma),\tau \sigma)} z),
\]
\[
(x \xrightarrow{(a,\sigma)} y) \otimes (x' \xrightarrow{(b,\sigma)} y') = (x + x' \xrightarrow{(a+b + h(x,x'|\sigma),\sigma)} y + y').
\]
for all \( \sigma, \tau \in G, x, x', z \in M, \sigma x = y, \sigma x' = y', \tau y = z \). In effect,

\[
\begin{align*}
(b, \tau) (a, \sigma) & \overset{(6.29)}{=} (b, 1) (0, \tau) (a, 1) (0, \sigma) \overset{(6.30)}{=} (b, 1) (\tau a, \tau) (0, \sigma) \\
& \overset{(6.29)}{=} (b, 1) (\tau a, 1) (0, \tau) (0, \sigma) \overset{(6.31),(6.33)}{=} (b + \tau a, 1) (h(x, x' | \sigma, \sigma)) \\
& \overset{(6.29)}{=} (a + b + h(x, x' | \sigma, \sigma)).
\end{align*}
\]

\( \text{Proposition 13} \)

\( \text{Theorem 24} \)

It is now easy to conclude from equalities (6.28), (6.37) and (6.38) that \( h \) is actually an abelian 3-cocycle of the \( G \)-module \( M \) with coefficients in \( N \). Since the composition and tensor in \( G \) are unitary and \( I \) is a functor, then the normalization of \( h \) follows from the equalities (6.25), (6.2) and (6.38). The cocycle condition \( \partial h = 0 \) in (3.14) follows from the associativity law for morphisms in \( G \). That \( \partial h = 0 \) in (3.13) is a consequence of the graded tensor product \( \otimes : G \times_G G \to G \) being functorial. The equality \( \partial h = 0 \) in (3.8) holds because of the coherence pentagons (6.1), and \( \partial h = 0 \) in (3.11) follows from the naturality of the associativity constraints. The cocycle conditions \( \partial h = 0 \) in (3.9) and (3.10) are verified owing to the coherence conditions (6.3) and (6.4) respectively. And, finally, the naturality of the braiding implies that \( \partial h = 0 \) in (3.12).

Hence \( h^G = h \in Z^3_{G,ab}(M, N) \) and, by comparison equalities (6.12) with (6.37), (6.13) with (6.38), (6.15) with (6.28), (6.14) with (6.36) and (6.16) with (6.35) respectively, it is obvious that \( G(h) \) and \( G \) are isomorphic braided \( G \)-graded categorical groups of type \( (M, N) \). This completes the proof of theorem. \( \square \)

The classifying results stated in this section can be summarized as follows (cf. [18, Theorem 3.3]).

Let \( \mathcal{H}^3_{G,ab} \) be the category whose objects are \( (M, N, k) \) consist of \( G \)-modules \( M, N \) and cohomology classes \( k \in H^3_{G,ab}(M, N) \). An arrow \( (M, N, k) \xrightarrow{(p,q)} (M', N', k') \) is a pair of \( G \)-module homomorphisms \( p : M \to M', q : N \to N' \) such that \( q_* k = p^* k' \),

\[
H^3_{G,ab}(M, N) \xrightarrow{q_*} H^3_{G,ab}(M, N') \xrightarrow{p^*} H^3_{G,ab}(M', N'),
\]

\[
k \mapsto q_* k = p^* k'.
\]

Then, we have the classifying functor

\[
cl : \mathcal{BC} G \mathcal{G} \longrightarrow \mathcal{H}^3_{G,ab},
\]

\[
\mathcal{G} \mapsto (\pi_0 \mathcal{G}, \pi_1 \mathcal{G}, k(\mathcal{G})),
\]

\[
F \mapsto (\pi_0 F, \pi_1 F),
\]

which has the following properties:

(i) For any object \( (M, N, k) \in \mathcal{H}^3_{G,ab} \), there exists a braided \( G \)-graded categorical group \( \mathcal{G} \) with an isomorphism \( \text{cl}(\mathcal{G}) \cong (M, N, k) \).

(ii) For any morphism \( (p, q) : \text{cl}(\mathcal{G}) \to \text{cl}(\mathcal{H}) \), there is a braided graded monoidal functor \( F : \mathcal{G} \to \mathcal{H} \) such that \( \text{cl}(F) = (p, q) \).

(iii) \( \text{cl}(F) \) is an isomorphism if and only if \( F \) is a braided graded monoidal equivalence.

(iv) For any arrow \( (p, q) : \text{cl}(\mathcal{G}) \to \text{cl}(\mathcal{H}) \), homotopy classes of braided graded monoidal functors \( F : \mathcal{G} \to \mathcal{H} \) such that \( \text{cl}(F) = (p, q) \) are in bijection with elements of the group \( H^3_{G,ab}(\pi_0 \mathcal{G}, \pi_1 \mathcal{H}) \).

As a final comment, we stress that, as a bonus from Proposition 13 and Theorem 24, for any two \( G \)-modules \( M, N \) we have a bijection

\[
\mathcal{S}^G_{\pi_1}[M, N] \cong \mathcal{BC} G \mathcal{G}[M, N],
\]

between the set of equivariant weak homotopy classes of pointed \( G \)-spaces \( X \) with \( \pi_i X = 0 \) for all \( i \not= 2, 3, \pi_2 X \cong M \) and \( \pi_3 X \cong N \) and the set of homotopy classes of braided \( G \)-graded categorical groups \( \mathcal{G} \) with \( \pi_0 \mathcal{G} \cong M \) and
\[ \pi_1 G \cong N. \] This bijection can be illustrated by the construction below of a braided \( G \)-graded categorical group \( G(X) \) associated to a pointed \( G \)-space \( X \), which represents its equivariant 3-type when \( X \) is 1-connected.

Let \( X = (X, \ast) \) be a (topological) space on which the (discrete) group \( G \) acts by pointed homeomorphisms. Then, the objects of \( G(X) \) are the double loops in \( X \) based on \( \ast \); that is, all the maps from the square \( I \times I \) into \( X \) which are constant along the edges, say \( \omega : (I^2, \partial I^2) \to (X, \ast) \). A morphism \( \omega \to \omega' \) of grade \( \sigma \in G \) is a pair \([ [h], \sigma ] \), where \([ h ] \) is the homotopy class (relative to \( \partial I \)) of a path between 2-loops \( h : \sigma \omega \to \omega' \). That is, a \( \sigma \)-morphism in \( G, ([h], \sigma) : \omega \to \omega' \), is represented by a relative map \( h : (I^2, \partial I^2 \times I) \to (X, \ast) \) with \( h(s, t, 0) = \sigma \omega(s, t) \) and \( h(s, t, 1) = \omega'(s, t) \); two such \( h, h' \) are equivalent whenever there exists a map \( H : ((I^4, \partial I^2 \times I^2) \to (X, \ast) \) such that \( H(s, t, u) = h(s, t, u), H(s, t, u, 1) = h'(s, t, u), H(s, t, 0, u) = \sigma \omega(s, t) \) and \( H(s, t, 1, u) = \omega'(s, t) \).

The composition is induced by the usual vertical composition of homotopies, according to the formula
\[ ([h'], \tau) [[h], \sigma] = ([h' \circ \tau h], \tau \sigma), \]
and on morphisms with the same grade by the horizontal composition of homotopies. The 1-graded associativity and unit constraints are defined to be the equivalence classes of the respective standard homotopies proving the associativity and unit of the loop composition, and the 1-graded braiding isomorphisms are the equivalence classes of the ordinary homotopies showing the commutativity of the second homotopy groups of spaces, namely

\[ \begin{array}{cccc}
\omega(s, t) & \otimes & \omega'(s, t) & = \\
\omega(2s, t) & \otimes & \omega'(2s-1, t)
\end{array} \]

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References