On Certain Graded $S_n$-Modules and
the $q$-Kostka Polynomials

A. M. Garsia* and C. Procesi

Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0119

We derive here a number of properties of the $q$-Kostka polynomials $K_{\lambda\mu}(q)$. In particular we obtain a very accessible proof that these polynomials have non-negative integer coefficients. Other monotonicity properties are also derived. These results are obtained by studying certain graded $S_n$-modules $R_\mu$ which afford a character that may be expressed in terms of the $K_{\lambda\mu}(q)$. Certain nesting properties of the $R_\mu$ which correspond to the dominance order of partitions then translate themselves into combinatorial inequalities involving the $K_{\lambda\mu}(q)$. The modules $R_\mu$ have been given an elementary presentation by DeConcini and Procesi (Invent. Math. 64 (1981), 203–219), as rather simple quotients of the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]$. We show here that their basic properties may also be derived in an entirely elementary manner.

INTRODUCTION

The $q$-Kosta polynomials can be defined as the coefficients $K_{\lambda\mu}(q)$ in the expansion

$$S_\lambda(X) = \sum_{\mu} K_{\lambda\mu}(q) P_\mu(X; q), \quad (I.1)$$

where $P_\mu(x; q)$ is the classical Hall–Littlewood polynomial (see [17, Chap. III]). It may be shown that as $q \to 1$ the polynomial $P_\mu(x; q)$ tends to the monomial symmetric function $m_\mu(X)$ and Eq. (I.1) reduces then to the classical expansion

$$S_\lambda(X) = \sum_{\mu} K_{\lambda\mu}(1) m_\mu(X), \quad (I.2)$$

giving the Schur function in terms of the monomial symmetric functions. This identifies the values $K_{\lambda\mu}(1)$ with the classical Kostka numbers $K_{\lambda\mu}$.

* Work carried out under NSF grant support.
These integers appeared in the representation theory of the symmetric group in Young's QSA VI (see [24]) in the equation

\[ p^\mu = \sum_\lambda \chi^\lambda K_{\lambda \mu}, \]  

which is commonly referred to as Young's rule. Here, \( p^\mu \) denotes the character of the permutation representation resulting from the action of \( S_n \) on the left cosets of the Young subgroup indexed by \( \mu \), and \( \chi^\lambda \) denotes the character of the irreducible representation of \( S_n \) indexed by \( \lambda \). Thus the integers \( K_{\lambda \mu} \) in this context appear as multiplicities of the irreducible constituents in this permutation representation. Young's rule states (see [24] or [17, I, §6, p. 56]) that \( K_{\lambda \mu} \) is equal to the number of column-strict tableaux of shape \( \lambda \) and type \( \mu \); let us denote this class of tableau by \( CS(\lambda, \mu) \). The coefficients \( K_{\lambda \mu}(q) \) have been shown to be polynomials in \( q \) with non-negative integer coefficients, (see [17, II §3 ex. 1, p. 92, and III, §7, ex. 9, p. 136]). This result derives from a long series of developments. References start as early as 1963 in a pioneering paper of Kostant [9]; the crucial papers here are those of Steinberg [22], Hotta and Springer [7], Kraft [10], and DeConcini and Procesi [3]. Unfortunately, the proofs of the results concerning the q-Kostka polynomials obtained by this approach require deep tools of Algebraic Geometry and are accessible only to a handful of experts. The polynomials \( K_{\lambda \mu}(q) \) have attracted a great deal of attention in the combinatorics audience since Foulkes [12] conjectured that it should be possible to obtain them by q-counting column-strict tableaux of suitable shape and type. This conjecture was proved by Lascoux and Schützenberger [12] in a deep combinatorial work where they introduced a statistic \( c(T) \) called charge (see [17, p. 129]) which gives

\[ K_{\lambda \mu}(q) = \sum_{T \in CS(\lambda, \mu)} q^{c(T)}. \]  

The charge interpretation has brought to light a number of interesting facts about the q-Kostka polynomials, as can be glimpsed from some past and recent announcements of Lascoux and Schützenberger (see [11], [13, 14]). In particular, it yields an elementary, though quite difficult, proof of the positivity of these polynomials. Unfortunately, these authors have not provided many details in their publications, which often turn out to be short lists of facts with only hints of how they may be proved. For these reasons a great deal remains to be done towards assembling a unified, comprehensive, and self-contained development of the theory of the q-Kostka polynomials. A helpful step in this connection is the work of L. Butler (1985 MIT Thesis), where the Lascoux–Schützenberger proof of (I.4) is given a fully detailed exposition. Moreover, a perusal of the quoted
literature reveals that the Algebraic Geometrical models have a great deal to offer in enlightening and guiding our understanding of the polynomials $K_{\lambda\mu}(q)$. We see there a beautiful interplay of Geometry, Commutative Algebra, and Representation Theory which is gravid with combinatorial implications. In view of the ubiquity of the $K_{\lambda\mu}(q)$, and the wide mathematical interest which is necessarily associated with them, it seems imperative that their study be carried out with tools accessible to the wider audience. The purpose of this paper is to initiate such a study.

Perhaps the most remarkable mathematical structure which gives rise to the $K_{\lambda\mu}(q)$ is the variety of flags fixed by a unipotent matrix. In a work of DeConcini and Procesi [3] it is shown that the cohomology ring of this variety may be given an elementary setting. This setting has been further simplified by Tanisaki in [23]. However, to this date, the connection between this ring and the polynomial $K_{\lambda\mu}(q)$ still required tools such Sheaves, Schemes, Cohomology theory, etc... Nevertheless, it is not difficult to see that once this connection is made, all the stated properties of the $K_{\lambda\mu}(q)$ follow with the greatest of ease. Moreover, this connection reveals a number of new and tantalizing properties of these polynomials. The main contribution of this paper is to show that also this connection can, in fact, be derived in a completely elementary manner.

To be more explicit, in the quoted work of DeConcini and Procesi, it is shown that the ring corresponding to a unipoint matrix of shape $\mu$ may be presented as a graded quotient $R_{\mu}$ of the polynomial ring $Q[x_1, x_2, \ldots, x_n]$. Combining this with some results of Kraft [10] it then follows that $R_{\mu}$ is an $S_\mu$-module with character given by (1.3). Moreover, since the $S_n$ action preserves the natural grading of $R_{\mu}$, there is also a graded version of $p^n$. The crucial fact is that the results in [20, 7, 10, 3] imply that the expansion of this graded version in terms of the irreducible characters of $S_n$ involves coefficients that are very closely related to the polynomials $K_{\lambda\mu}(q)$. We show here that this expansion may be obtained using no more than the tools of elementary linear algebra.

For a more precise description of the contents of this paper we need to introduce some notation. Let $X_n = \{x_1, x_2, \ldots, x_n\}$ be an ordered set of commuting variables. For any subset $S \subseteq X_n$ and $r \leq |S|$ we let $e_r(S)$ denote the $r$th elementary symmetric function of the variables in $S$. In this writing, partitions of $n$ are represented by $n$-vectors

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n) \quad (0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n).$$

The number of positive components of $\mu$ is denoted by $h(\mu)$ and is referred to as the height of $\mu$. The partition conjugate of $\mu$ is denoted by

$$\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_n).$$
We may sometimes, when convenient, drop the zero components from a partition. Setting
\[ d_k(\mu) = \mu'_1 + \mu'_2 + \cdots + \mu'_k \quad (1 \leq k \leq n) \]
we denote by \( I_\mu \) the ideal (in the polynomial ring \( \mathfrak{Z}[x_1, x_2, \ldots, x_n] \)) generated by the collection of partial elementary symmetric functions
\[ \mathcal{E}_\mu = \{ e_r(S) : k \geq r > k - d_k(\mu), |S| = k, S \subseteq X_n \} \quad (I.5) \]
This given, the ring \( R_\mu \) is defined as the quotient
\[ R_\mu = \mathbb{Q}[x_1, x_2, \ldots, x_n]/I_\mu. \quad (I.6) \]
For example, when \( \mu = (0, 0, 1, 2, 2) \) then \( \mu' = (0, 0, 0, 2, 3) \),
\[ (1 - d_1(\mu), 2 - d_2(\mu), \ldots, n - d_n(\mu)) = (1, 2, 3, 2, 0), \]
and \( \mathcal{E}_{122} \) consists of
\[ e_1(x_1, x_2, x_3, x_4, x_5), e_2(x_1, x_2, x_3, x_4, x_5), \ldots, e_5(x_1, x_2, x_3, x_4, x_5), \]
\[ e_3(x_1, x_2, x_3, x_4), e_3(x_1, x_2, x_3, x_5), e_3(x_1, x_2, x_4, x_5), \]
\[ e_3(x_1, x_3, x_4, x_5), e_3(x_2, x_3, x_4, x_5); \]
\[ x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_4x_5, x_1x_3x_4x_5, x_2x_3x_4x_5. \]

We should point out that for \( \mu = 1^n \), \( R_\mu \) reduces to the quotient ring
\[ \mathbb{Q}[x_1, x_2, \ldots, x_n]/(e_1, e_2, \ldots, e_n), \]
where \( (e_1, e_2, \ldots, e_n) \) denotes the ideal generated by the ordinary elementary symmetric functions in the variables \( x_1, x_2, \ldots, x_n \). All the other \( R_\mu \) are quotients of \( R_{1^n} \).

Note now that \( S_n \) acts naturally on \( \mathbb{Q}[x_1, x_2, \ldots, x_n] \) by permuting the variables. To be precise, we set for each \( \sigma \in S_n \) and \( P \in \mathbb{Q}[x_1, x_2, \ldots, x_n] \)
\[ \sigma P(x_1, x_2, \ldots, x_n) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}). \]
Clearly, the collections \( \mathcal{E}_\mu \) remain unchanged under this action, and thus all the ideals \( I_\mu \) are invariant subspaces of \( \mathbb{Q}[x_1, x_2, \ldots, x_n] \). This implies that we can define an action of \( S_n \) on each of the resulting quotients \( R_\mu \). Moreover, since the ideals \( I_\mu \) are generated by homogeneous polynomials,

\[ ^1 \text{It should be mentioned here that this particular presentation of the ring } R_\mu \text{ was introduced by Tanisaki (in [23]) precisely for the purpose of simplifying some of the proofs in the DeConcini–Procesi paper [3].} \]
each ring $R_\mu$ is graded. Let then $\mathcal{H}_m(R_\mu)$ denote the subspace consisting of the elements of $R_\mu$ which are homogeneous of degree $m$. The spaces $\mathcal{H}_m(R_\mu)$ decompose the action of $S_n$ on $R_\mu$ into a direct sum of representations. This given, let $p^\mu_m$ denote the character of the representation corresponding to $\mathcal{H}_m(R_\mu)$ and set

$$p^\mu(q) = \sum_{m \gg 0} q^m p^\mu_m.$$  

In the present context, the most remarkable fact concerning $R_\mu$ is that when we decompose $p^\mu(q)$ in terms of the irreducible characters $\{\chi^\lambda\}_{\lambda \vdash n}$ we discover that the coefficients $c_{\lambda \mu}(q)$ in the expansion

$$p^\mu(q) = \sum_{\lambda} \chi^\lambda c_{\lambda \mu}(q)$$  

are essentially the q-Kostka polynomials. The precise relation is simply

$$c_{\lambda \mu}(q) = K_{\lambda \mu}(1/q) q^{n(\mu)},$$  

where

$$n(\mu) = \sum_{i=1}^{h(\mu)} (i-1) \mu_{n+1-i}.$$  

An immediate consequence of (1.8) is that the coefficient of $q^k$ in the polynomial $K_{\lambda \mu}(q)$ is equal to the multiplicity of $\chi^\lambda$ in $p^\mu_{n(\mu)-k}$. In particular the establishment of (1.8) yields a representation theoretical proof of the positivity of the q-Kostka polynomials. We see then that a combinatorial decomposition of these rings should lead to interesting insights concerning the charge statistic.

We have endeavoured to make our presentation entirely self contained and our proofs accessible to a wider audience than those that can be found in the original manuscripts. We apologize if the original experts find our treatment unduly pedantic. We hope that this is compensated by the additional audience that will be able to appreciate their contributions.

The contents are divided into 5 sections. In Section 1 we give a recursive construction of a remarkable set of monomials $B(\mu)$ which forms a basis for $R_\mu$. In Section 2 we use an argument introduced by Tanisaki [23] to show that they span $R_\mu$. In Section 3 we present an elementary version of an argument due to Kraft [10] which calculates the dimension of $R_\mu$. The developments in this section also yield the result of DeConcini and Procesi identifying $p^\mu$ as the ungraded character of $R_\mu$. Another byproduct of this section is the identification of the restriction to $S_{n-1}$ of the graded character. In Section 4 we derive a number of containment properties of the rings
These are not only crucial for our later developments but also eventually yield the desired monotonicity properties of the polynomials $K_{i,j}(q)$. In Section 5 we work out some identities involving the Hall-Littlewood polynomials which are basic in the final determination of the graded character of $R_\mu$. In Section 6 we prove that the graded character is indeed given by (1.7) and (1.8).

The first named author is indebted to J. Macdonald for his strong support and advice in carrying out this project and to N. Bergeron for invaluable help and in putting the finishing touches in many of the proofs presented here.

1. A Monomial Basis for $R_\mu$

Let now $\mu$ be a fixed partition and $\mu'$ be its conjugate. For a given $1 \leq i \leq h(\mu) = \mu'_n$, define $a_i(\mu)$ by the condition

$$\mu'_i < i \leq \mu'_{i+1}.$$  \hfill (1.1)

This given we let $\mu^{(i)}$ be the partition obtained by removing the square at the end of column $n - a_i(\mu)$ ($= \mu_{n+1-i}'$) from the (French) Ferrers diagram of $\mu$. An example will make things clear here. We illustrate in Fig. 1 the construction of $\mu^{(2)}$ for $\mu = (1, 1, 1, 2, 2, 4)$ and all the partitions $\mu^{(i)}$ for $\mu = (1, 2, 2, 3)$. From the work of DeConcini and Procesi [3] it follows that the ring $R_\mu$ has a monomial basis $B(\mu)$ which can be constructed from the recursion

$$B(\mu) = \sum_{i=1}^{h(\mu)} x_{n-1} B(\mu^{(i)})$$  \hfill (1.2)

and the initial condition $B(\mu) = \{1\}$ for $\mu = (1)$. In (1.2) the "\(\sum\)" means...
disjoint union and the expression $x_n^{i-1} \mathcal{B}(\mu^{(i)})$ represents the set of monomials obtained by multiplying each monomial in $\mathcal{B}(\mu^{(i)})$ by $x_n^{i-1}$. The recursion in (1.2) leads to an algorithm, which we illustrate in Fig. 2 in the case of $\mathcal{B}(1, 1, 2)$. According to (1.2) each path, from the root of the three down to a leaf, yields a monomial upon multiplication of the terms labelling the edges of the path. The collection of the monomials thus obtained gives $\mathcal{B}(\mu)$. From the tree above, reading in preorder (from left to right), we get that

$$\mathcal{B}(112) = \{1, x_2, x_3, x_2, x_3, x_2^2, x_2x_3^2, x_4, x_4x_2, x_4x_3, x_4^2, x_4^2x_2, x_4^2x_3\}.$$  

The recursion in (1.2) is but one of the consequences of a beautiful relation between $R_\mu$ and the rings $R_\mu^{(i)}$, which may symbolically be written as

$$R_\mu \cong \bigoplus_{i=1}^{h(\mu)} x_n^{i-1} R_\mu^{(i)}. \quad (1.3)$$

We should interpret this here as a direct sum decomposition of $R_\mu$ as a vector space, and not as an $S_n$-module. Nevertheless, as we shall see, (1.3) does yield an important fact about the restriction of the graded character $p^{\mu}(q)$ to $S_{n-1}$. Clearly we can always write

$$\rho^{(i)}(p) \cong \bigoplus_{i=1}^{h(\mu)} x_n^{i-1} R_\mu^{(i)}.$$ \quad (1.4)

Thus (1.3) follows if we show that for each $i = 1, 2, \ldots, h(\mu)$ we have

$$x_n^{i-1} R_\mu^{(i)} \cong x_n^{i-1} \cdot R_\mu^{(i)}.$$ \quad (1.5)

Care must be taken in interpreting this relation properly. What we mean
here is that the natural map $\phi^{(i)}$ from $\mathbb{Q}[x_1, \ldots, x_{n-1}]$ to $x_{n-1}^l R_\mu$ which sends a polynomial $p(x_1, \ldots, x_{n-1})$ into the polynomial

$$\phi^{(i)} p(x_1, \ldots, x_{n-1}) = x_{n-1}^l p(x_1, \ldots, x_{n-1})$$

induces an isomorphism of $R_\mu^{(i)}$ onto $x_{n-1}^l R_\mu / x_{n-1}^l R_\mu$ as $S_{n-1}$-modules. This isomorphism is established here by combining the results of the next two sections.

**Remark 1.1.** We should mention that the surjectivity of $\phi^{(i)}$ was proved by Tanisaki in [23] in an entirely elementary way. This given, to prove injectivity, one needs only show that the dimensions of the spaces $R_\mu^{(i)}$ add up to the dimension of $R_\mu$. This particular fact follows from the developments in a paper of Kraft [10], and indeed Tanisaki in [23] refers to Kraft to complete his proof. We show in Section 3 that what is needed from Kraft's arguments to establish injectivity can be made completely elementary. The argument will also show that (1.5) is an isomorphism of $S_{n-1}$-modules.

**Remark 1.2.** It is convenient to denote the Hilbert series (or Poincaré polynomial) of $R_\mu$ by one of the symbols

$$\{n\} = \left\{ \begin{array}{c} n \\ \mu \end{array} \right\} = \left\{ \begin{array}{c} n \\ \mu_1 \mu_2 \cdots \mu_k \end{array} \right\}.

Clearly, if $B$ is any homogeneous basis for $R_\mu$, we must have

$$\{n\} = \sum_{\beta \in B} q^{\text{degree}(\beta)}. \quad (1.6)$$

Thus once we show that $B(\mu)$ is a basis, then from (1.2) we derive the recursion

$$\{n\} = \sum_{i=1}^{ht(\mu)} q^{i-1} \left\{ \begin{array}{c} n \\ \mu^{(i)} \end{array} \right\}. \quad (1.7)$$

Since the multinomial coefficients

$$\left( \begin{array}{c} n \\ \mu \end{array} \right) = \left( \begin{array}{c} n \\ \mu_1 \mu_2 \cdots \mu_k \end{array} \right)$$

satisfy the recursion

$$\left( \begin{array}{c} n \\ \mu \end{array} \right) = \sum_{i=1}^{ht(\mu)} \left( \begin{array}{c} n \\ \mu^{(i)} \end{array} \right),$$
we deduce that we must have

$$\left\{ \binom{n}{\mu_1 \mu_2 \cdots \mu_k} \right\}_{q=1} = \frac{n!}{\mu_1! \mu_2! \cdots \mu_n!}. \quad (1.8)$$

Thus the symbol

$$\left\{ \binom{n}{\mu_1 \mu_2 \cdots \mu_k} \right\}$$

may be viewed as another $q$-analogue of the multinomial coefficient. Examples for small values of $n$ show that this $q$-multinomial is different from the commonly encountered

$$\left[ \binom{n}{\mu_1 \mu_2 \cdots \mu_k} \right]_q.$$

This should not be too surprising. Indeed, assuming (1.8) to be true, from (1.7) we derive that

$$\left\{ \binom{n}{\mu} \right\} = \sum_{\lambda} n_{\lambda} K_{\lambda \mu} (1/q) q^{n(\mu)}, \quad (1.9)$$

where $n_\lambda$ denotes the number of standard tableaux of shape $\lambda$. Combining this with (1.4) and the fact that the charge is constant on Knuth equivalence classes (see [12, 13]), it is not difficult to conclude that we must have

$$\left\{ \binom{n}{\mu_1 \mu_2 \cdots \mu_k} \right\} = \sum_{\lambda \in \text{Re}(1^{\mu_1}2^{\mu_2} \cdots n^{\mu_n})} q^{\text{cocharge}(\lambda)}, \quad (1.10)$$

where $\text{Re}(1^{\mu_1}2^{\mu_2} \cdots n^{\mu_n})$ denotes the collection of words which are rearrangements of the word $1^{\mu_1}2^{\mu_2} \cdots n^{\mu_n}$ and $\text{cocharge}(\lambda) = n(\mu) - \text{charge}(\lambda)$.  

**Remark 1.3.** A further interesting identity for the cocharge statistic may be obtained by carrying out the algorithm for the construction of $A(\mu)$ all the way through. Note that if we replace in each of the monomials obtained from the tree all the variables $x_i$ by $q$ then (if $A(\mu)$ is indeed a basis) the resulting powers of $q$ must sum to the Hilbert series of $R_\mu$. On the other hand, the algorithm for constructing the tree allows us to express this polynomial as a sum of weights of standard tableaux of shape $\mu$. This is better understood if we modify a bit the tree produced by the algorithm. We see from the example we have given that different children of the same node are not necessarily labelled by different Ferrers' diagrams. Let us then collapse the tree, by identifying siblings labelled by the same diagram. To
keep track of this identification, we allow multiple edges from a node to a child. Since, we are only interested here in the Hilbert series of the ring, the label $x_i$ will be replaced by $q^i$. In Fig. 3 we have the resulting labelled tree for the partition $\mu = (0, 1, 1, 2)$.

We see that in the collapsed tree the paths from the leaves can be identified with standard tableaux. The contribution to the Hilbert series from each of these leaves can then be obtained by multiplying the contributions corresponding to each of the multiple edges encountered along the path. For instance the contribution corresponding to the double edge joining shape $(1, 1, 2)$ to shape $(1, 2)$ is $(q + q^2)$; that corresponding to the triple edge joining $(1, 1, 1)$ to $(1, 1)$ is $(1 + q + q^2)$. It is not difficult to conclude from this example that in general the Hilbert series of $R_\mu$ is given by the formula

$$\binom{n}{\mu} = \sum_{S \in ST(\mu)} W(S), \quad (1.11)$$

where $ST(\mu)$ denotes the collection of standard tableaux of shape $\mu$ and $W(S)$ is a polynomial which can be defined as follows. Given a partition $\mu$ let us assign a weight $w(c)$ to each corner square $c$ according to the following rule. If the coordinates of $c$ are $(i, \mu_i)$ and $m$ is the multiplicity of $\mu_i$ in $\mu$ then $w(c) = q^{i-m} + q^{i-m+1} + \cdots + q^{i-2} + q^{i-1}$. Finally, for a standard tableau $S$ we set $W(S)$ equal to the product of the weights of each of its entries. Where the weight of entry $s$ in $S$ is taken to be the weight of the corner square containing $s$ in the partition obtained from the shape of $S$ by removing all the squares containing entries bigger than $s$. This given combining (1.10) and (1.11) we derive that

$$\sum_{w \in Re(1^22^1 \ldots n^m)} q^{cocharge(w)} = \sum_{S \in ST(\mu)} W(S). \quad (1.12)$$

\begin{itemize}
  \item $w(\begin{array}{c} 3 \\ 1 \end{array}) = (1 + q + q^2)(1 + q)$
  \item $w(\begin{array}{c} 2 \\ 2 \end{array}) = (q + q^2)(1 + q)$
  \item $w(\begin{array}{c} 1 \\ 3 \end{array}) = (q + q^2)(q)$
\end{itemize}
2. An Algorithm for Expanding in Terms of the Monomials in $B(\mu)$

Our basic goal in this section is to establish the following:

**Proposition 2.1.**  
(a) The monomials in $B(\mu)$ span $R_{\mu}$.  
(b) The vector space dimension of $R_{\mu}$ satisfies the inequality

$$\dim R_{\mu} \leq \frac{n!}{\mu_1! \mu_2! \cdots \mu_n!}.$$  \hspace{1cm} (2.1)

**Proof.** We prove (a) by giving a recursive algorithm for expressing an element of $R_{\mu}$ as a linear combination of the monomials in $B(\mu)$. Note that then (b) follows immediately since the recursion in (1.2) (as was observed in Remark 1.2) implies that the cardinality of $B(\mu)$ is given by the multinomial coefficient in (2.1).

Clearly, in view of (1.4), to show (a) we need only to show how to express an arbitrary element of the quotient $x_n^{i-1}R_{\mu}/x_n^iR_{\mu}$ in terms of the monomials in $x_n^{i-1}B(\mu^{(i)})$. We thus proceed by induction on the number of squares of the Ferrer's diagram of the partition $\mu$. This given, we start by noting that any element of $x_n^{i-1}R_{\mu}/x_n^iR_{\mu}$ has a representative of the form $x_n^{i-1}p(x_1, \ldots, x_{n-1})$ with $p(x_1, \ldots, x_{n-1}) \in \mathbb{Q}[x_1, \ldots, x_{n-1}]$. Now $p(x_1, \ldots, x_{n-1})$, interpreted as a representative of an element of $R(\mu^{(i)})$, (by induction) can be expanded in terms of the monomials in $B(\mu^{(i)})$. Let the resulting expansion be

$$p(X_{n-1}) \cong_{\mu^{(i)}} \sum_{b \in B(\mu^{(i)})} c_b b(X_{n-1}),$$ \hspace{1cm} (2.2)

where by "$\cong_{\mu^{(i)}}$" we mean the congruence modulo the ideal $I_{\mu^{(i)}}$. We show then that (2.2) implies that

$$x_n^{i-1}p(X_{n-1}) \cong_{\mu} \sum_{b \in B(\mu^{(i)})} c_b x_n^{i-1}b(X_{n-1}) + E,$$ \hspace{1cm} (2.3)

where the error term $E$ lies in $x_n^iR_{\mu}$. To show this (in view of the definition of $I_{\mu^{(i)}}$), we need only establish that if a polynomial $p(X_{n-1}) \in \mathbb{Q}[x_1, \ldots, x_{n-1}]$ has an expansion of the form

$$p(X_{n-1}) = \sum_{r > k - d_i(\mu^{(i)})} A(r, S) e_r(S)$$ \hspace{1cm} (2.4)

(with $A(r, S) \in \mathbb{Q}[x_1, \ldots, x_n]$), then $x_n^{i-1}p(X_{n-1})$ is necessarily congruent, modulo the ideal $I_{\mu}$, to a polynomial $A \in \mathbb{Q}[x_1, \ldots, x_n]$ times $x_n^i$. To prove this we borrow an argument given by Tanisaki in [23]. We
deal with each of the terms in (2.4) individually. Note first that from (1.1) and the definition of the partition \( \mu^{(i)} \) it immediately follows that

\[
d_k(\mu^{(i)}) = \begin{cases} 
   d_{k+1}(\mu) & \text{if } k < a_i(\mu) \\
   d_{k+1}(\mu) - 1 & \text{if } k \geq a_i(\mu).
\end{cases}
\]  

(2.5)

Note next that

\[
e_r(S) = e_r(S \cup \{x_n\}) - x_n e_{r-1}(S).
\]  

(2.6)

This given, we see from (2.5) that when \( |S| = k \) and \( k \geq a_i(\mu) \) then \( r > k - d_k(\mu^{(i)}) \) implies \( r > k + 1 - d_{k+1}(\mu) \), which gives that

\[
e_r(S) \cong \mu - x_n e_{r-1}(S),
\]

and thus we must have

\[
x_n^{i-1} e_r(S) \cong \mu - x_n^{i-1} e_{r-1}(S).
\]  

(2.7)

On the other hand, even when \( k < a_i(\mu) \), if \( r > k + 1 - d_k(\mu^{(i)}) \) then we still have \( r > k + 1 - d_{k+1}(\mu) \) and (2.6) yields that (2.7) must still hold in this case. Thus the only terms in (2.4) we need to be careful about are those for which \( k < a_i(\mu) \) and \( r = k + 1 - d_{k+1}(\mu^{(i)}) = k + 1 - d_{k+1}(\mu) \). But now there is another trick at our disposal. Note first that when \( k < a_i(\mu) \), (1.1) gives

\[
d_{k+1}(\mu) = d_k(\mu) + \mu_{k+1}^{'} \leq d_k(\mu) + \mu_{a_i(\mu)}^{'} = d_k(\mu) + i.
\]

Thus if \( r = k + 1 - d_{k+1}(\mu) \), then \( k < a_i(\mu) \) implies that

\[
r + i - 1 = k + i - d_{k+1}(\mu) > k - d_k(\mu).
\]  

(2.8)

Now (2.6) with \( r \) replaced by \( r + 1 \) gives

\[
x_n e_r(S) = e_{r+1}(S \cup \{x_n\}) - e_{r+1}(S).
\]

Thus, when \( r \geq k + 1 - d_{k+1}(\mu) \) and \( |S| = k \) we have

\[
x_n e_r(S) \cong \mu - e_{r+1}(S),
\]

and by iteration we get

\[
x_n^{i-1} e_r(S) \cong \mu (-1)^{i-1} e_{r+i-1}(S),
\]

and from (2.8) we finally derive that

\[
x_n^{i-1} e_r(S) \cong 0
\]

as desired. This completes the proof of (2.3).
Remark 2.1. Note that if we assume that $\mathcal{B}(\mu^{(i)})$ is a basis for $R_{\mu^{(i)}}$ then the action of a permutation $\sigma \in S_{n-1}$ on a monomial $\mathcal{B}(x_1, \ldots, x_{n-1}) \in \mathcal{B}(\mu^{(i)})$ may be expressed in the form

$$\sigma b(x_1, \ldots, x_{n-1}) = \sum_{b' \in \mathcal{B}(\mu^{(i)})} b'(x_1, \ldots, x_{n-1}) a_{b', b}(\sigma) + E,$$

(2.9)

with $E \in I_{\mu^{(i)}}$, and this gives that

$$\sigma x_n^{i-1} b(x_1, \ldots, x_{n-1}) = \sum_{b' \in \mathcal{B}(\mu^{(i)})} x_n^{i-1} b'(x_1, \ldots, x_{n-1}) a_{b', b}(\sigma) + x_n^{i-1} E.$$

However, from what we have just shown, it follows that we must have

$$x_n^{i-1} E \cong \mu x_n^i A$$

for some polynomial $A \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$. In other words

$$\sigma x_n^{i-1} b(x_1, \ldots, x_{n-1}) \cong \sum_{b' \in \mathcal{B}(\mu^{(i)})} x_n^{i-1} b'(x_1, \ldots, x_{n-1}) a_{b', b}(\sigma)$$

(modulo $x_n^i R_{\mu}$). (2.10)

This is all that is needed to derive that $R_{\mu^{(i)}}$ and $x_n^{i-1} R_{\mu} / x_n^i R_{\mu}$ are equivalent as $S_{n-1}$-modules.

3. Identification of the Character

Throughout this section we work with a fixed partition $\mu = (\mu_1, \ldots, \mu_m)$ of height $m$. It is convenient here to change slightly our notation, drop the zero components of $\mu$, and let

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_m \quad \left( \sum_{i=1}^{m} \mu_i = n \right).$$

However, we keep our initial conventions with regards to the partition $\mu'$ conjugate of $\mu$.

This given, let $P_i = (b_1, b_2, \ldots, b_n)$ be a point with distinct coordinates $a_1, a_2, \ldots, a_m$ appearing with multiplicities $\mu_1, \mu_2, \ldots, \mu_m$ respectively and let

$$W = (P_1, P_2, \ldots, P_N) \quad \left( N = \frac{n!}{u_1! u_2! \cdots u_m!} \right)$$

be the orbit of $P_1$ under the action

$$\sigma: (y_1, y_2, \ldots, y_n) \rightarrow (y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n}) \quad (\sigma \in S_n).$$
Finally, let $I_W$ be the ideal of polynomials in $Q[x_1, x_2, ..., x_n]$ which vanish on $W$ and set

$$\mathcal{A} = Q[x_1, x_2, ..., x_n]/I_W.$$ 

We can see that $\mathcal{A}$ may be identified with the algebra of $Q$-valued functions on $W$ and as such it has a vector space basis consisting of the indicator functions of the points of $W$. More precisely, we can express the indicator function of a point $P = (y_1, y_2, ..., y_n) \in W$ in the form

$$F_P(x_1, x_2, ..., x_n) = \prod_{s=1}^{n} f(x_s, y_s),$$

where for each $i = 1, ..., m$

$$f(x, a_i) = \prod_{j=1, j \neq i}^{m} (x - a_j)/(a_i - a_j).$$

This in particular implies that each element of $\mathcal{A}$ may be represented by a polynomial of degree at most $m - 1$ in each of the variables $x_i$.

In view of the definition of $W$, the ideal $I_W$ is invariant under the action of $S_n$ on $Q[x_1, x_2, ..., x_n]$, thus we may define an action of $S_n$ on $\mathcal{A}$. Note that, since this action in terms of the basis $\{F_P\}_{P \in W}$ may be simply expressed as

$$\sigma F_P(x_1, x_2, ..., x_n) = F_{\sigma(P)}(x_1, x_2, ..., x_n),$$

we see that it is equivalent to the action of $S_n$ on the left cosets of the stabilizer of $P_i$. In other words, this action makes $\mathcal{A}$ into an $S_n$-module whose character is given precisely by $p^\mu$.

It develops that $\mathcal{A}$ and the ring $R_\mu$ are very closely connected. The crucial link between them is given by the following remarkable fact.

**Proposition 3.1.** Each of the elements in the collection $\{\phi_\mu\}$ is the leading homogeneous component of a polynomial which vanishes on $W$.

**Proof.** To get an idea of what is going on here we first deal with the special case of the elementary symmetric functions in all of the variables. To this end note that the definition of the point $P_1 = (b_1, b_2, ..., b_n)$ immediately yields the identity

$$\prod_{i=1}^{n} (t - b_i) = \prod_{i=1}^{m} (t - a_i)^{\mu_i}.$$
The same will of course hold true for any point in the orbit of $P_1$. That is we have

$$\prod_{i=1}^{n} (t - x_i) = \prod_{i=1}^{n} (t - b_i)$$

whenever $X = (x_1, x_2, \ldots, x_n)$ is one of the points $P_i$. Equating coefficients of $t$ in both sides we derive that the polynomials

$$e_s(x_1, x_2, \ldots, x_n) - e_s(b_1, b_2, \ldots, b_n) \quad (s = 1, \ldots, n) \quad (3.1)$$

vanish on $W$. This proves our assertion in the case of $e_s(x_1, x_2, \ldots, x_n)$. The proof for the general $e_s(S) \in \mathcal{C}_\mu$ needs a slightly more refined argument. Since $\mathcal{C}_\mu$ is $S_\mu$-invariant, we need only work with the case when $S$ consists of the first $k$ variables. Note then that, for $X \in W$ and a given $i \in [1, m]$, at most $n - \mu_i$ of $x_1, x_2, \ldots, x_k$ can be different from $a_i$. Thus when $k > n - \mu_i$ at least $k - (n - \mu_i)$ of these coordinates must be equal to $a_i$. The inevitable conclusion is that for any $X \in W$ the polynomial

$$P_{X_k}(t) = \prod_{i=1}^{k} (t + x_i)$$

must be divisible by

$$D(t) = \prod_{i=1}^{m} (t + a_i)^{(n - n + k)^+}, \quad (3.2)$$

where we use the symbol $(a)^+$ to denote $a$ if $a > 0$ and zero otherwise. Now this divisibility property can be translated into a set of equalities expressing the vanishing of the remainder of the division of $P_{X_k}(t)$ by $D(t)$. To see what this involves let us set

$$P_{Y_d}(t) = \prod_{j=1}^{d} (t + y_j)$$

and let $Q_{X_k, Y_d}(t)$ and $R_{X_k, Y_d}(t)$ denote the quotient and remainder of the long division of $P_{X_k}(t)$ by $P_{Y_d}(t)$. That is

$$P_{X_k}(t) = Q_{X_k, Y_d}(t) P_{Y_d}(t) + R_{X_k, Y_d}(t) \quad \text{(with degree $R_{X_k, Y_d}(t) \leq d - 1$).}$$

For convenience set

$$R_{X_k, Y_d}(t) = \sum_{s=0}^{d-1} t^s c_s(X_k, Y_d).$$
Clearly, each coefficient \( c_s(X_k, Y_d) \) is a polynomial which is homogeneous in the two sets of variables \( x_i, y_j \). Thus, if we set all the variables \( y_j \) equal to zero, \( c_s(X_k, Y_d) \) reduces to the sum of its terms which are of highest degree in the \( x_i \)'s. Let us refer to the latter as the \textit{leading} component of \( c_s(X_k, Y_d) \). Note now that when the \( y_j \)'s are set to zero, the polynomial \( P_{y_d}(t) \) reduces to \( t^d \) and thus we must have

\[
R_{x_k, y_d}(t) \big|_{y_d=0} = \sum_{s=0}^{d-1} t^s e_k(X_k).
\]

This gives that the leading components of the coefficients \( c_s(X_k, Y_d) \) are the elementary symmetric functions

\[
e_k(x_1, x_2, \ldots, x_k), e_{k-1}(x_1, x_2, \ldots, x_k), \ldots, e_{d-1}(x_1, x_2, \ldots, x_k).
\]  

(3.3)

Let us now apply this result when \( P_{y_d}(t) = D(t) = \prod_{i=1}^m (t + a_i)^{\mu_i - n + k} \) and \( x_1, x_2, \ldots, x_k \) are the first \( k \) coordinates of a point in \( W \). In this case, as we have observed, \( P_{y_k}(t) \) is divisible by \( D(t) \). This means that if we replace, in the coefficients \( c_s(X_k, Y_d) \), the variables \( y_j \) by the elements \( a_1, a_2, \ldots, a_m \) with respective multiplicities

\[
(a_1 - n + k)^+, (a_2 - n + k)^+, \ldots, (a_m - n + k)^+
\]

then the resulting polynomials must vanish identically on \( W \). But, from what we have just said, we can easily see that replacement of the \( y_j \)'s by constants, in the \( c_s(X_k, Y_d) \)'s, yields polynomials in the \( x_i \)'s whose leading homogeneous components are the elementary symmetric functions given in (3.3). Moreover, in this case we have

\[
d = (\mu_1 - n + k)^+ + (\mu_2 - n + k)^+ + \cdots + (\mu_m - n + k)^+
\]

\[
= \sum_{\mu_i > n - k} \mu_i - (n - k),
\]

and this when expressed in terms of the partition \( \mu' \) conjugate to \( \mu \) is none other than

\[
\mu_1' + \mu_2' + \cdots + \mu_k' = d_k(\mu).
\]

This shows that the assertion of the proposition must hold true for the elements \( e_r(S) \in \mathcal{E}_\mu \) when \( S = \{x_1, x_2, \ldots, x_k\} \). Clearly the proof could have been carried out verbatim for any other subset of the alphabet. At any rate,
we may also use the $S_n$-invariance of $G_\mu$ to deduce the result in full
generality from the special case just considered. Thus our proof is complete.

Let now $A_{\leq k}$ denote the linear span in $A$ of the images of the
monomials $x^p = x_1^{p_1}x_2^{p_2} \cdots x_n^{p_n}$ of degree less or equal to $k$. Note that in
view of the fact that we have already a basis for $A$ consisting of polynomials of degree at most $n(m - 1)$ we have that the integer

$$k_o = \inf\{k : A_{\leq k} = A\}$$

must necessarily be $\leq n(m - 1)$. This integer will soon be identified. Let us
then use Gauss elimination (in $A$) on the monomials $x^p$ arranged by
increasing degree (breaking equal degree ties with the lexicographic order
of their exponents.) Denote by $M_k = \{m_{jk} : 1 \leq j \leq m_k\}$ the collection of monomials of degree $k$ which survive the process, set $M = \bigcup_{k=0}^{k_o} M_k$ and let $M_k$ denote the subspace of the polynomial ring $Q[x_1, x_2, \ldots, x_n]$ which is
spanned by the elements in $M_k$. Finally, since we are to consider the same
polynomial, once as representing an element of $A$ and once as representing
an element of $R_\mu$, it will be convenient here to make use of the homomorphisms

$$\phi : Q[x_1, x_2, \ldots, x_n] \to Q[x_1, x_2, \ldots, x_n]/I_W = A$$
$$\psi : Q[x_1, x_2, \ldots, x_n] \to Q[x_1, x_2, \ldots, x_n]/I_\mu = R_\mu.$$ 

This given, we have

**Proposition 3.2.** The collections $\phi M$ and $\psi M$ are bases for $A$ and $R_\mu$, respectively. Thus we have the direct sum decompositions

$$A = \bigoplus_{k=1}^{k_o} \phi M_k, \quad R_\mu = \bigoplus_{k=1}^{k_o} \psi M_k.$$  \hspace{1cm} (3.5)

In particular

$$\dim R_\mu = \frac{n!}{\mu_1! \mu_2! \cdots \mu_m!}.$$  \hspace{1cm} (3.6)

**Proof.** It is clear, by the manner in which the monomials $m_{jk}$ have been
selected, that $\phi M = \{\phi m_{jk} : 1 \leq j \leq m_k, 0 \leq k \leq k_0\}$ is a basis for $A$. This gives that

$$\text{card } M = \dim A = \frac{n!}{\mu_1! \mu_2! \cdots \mu_m!}.$$
Combining this with (2.1) gives
\[ \dim R_\mu \leq \text{card } M. \]

Thus, to show (3.6) and derive that \( \psi M \) is a basis for \( R_\mu \) we need only establish that the \( \psi m_{jk} \)'s are independent in \( R_\mu \). To this end, let us assume if possible that for some constants \( c_{ik} \), not all zero, we have
\[ \sum_{k=0}^{k_0} \sum_{i=1}^{m_k} c_{ik} \psi m_{ik} \equiv 0 \pmod{I_\mu}, \quad (3.7) \]
and set
\[ b_k = \sum_{i=1}^{m_k} c_{ik} m_{ik}. \]

Now (3.7) may be rewritten in the form
\[ \sum_{k=0}^{k_0} b_k = \sum_{e_r(S) \in \mathcal{A}_\mu} P_{r,s} e_r(S), \]
for some suitable polynomials \( P_{r,s} \). Thus we must also have that
\[ b_k = \sum_{e_r(S) \in \mathcal{A}_\mu} Q_{r,s} e_r(S), \quad (3.8) \]
where \( Q_{r,s} \) is the homogeneous component of degree \( k - r \) in \( P_{r,s} \). On the other hand Proposition 3.1 tells us that each \( e_r(S) \in \mathcal{A}_\mu \) can be written in the form
\[ e_r(S) = E_r(S) - e_{\leq r-1}(S), \quad (3.9) \]
with \( E_r(S) \in I_\mu \) and degree \( e_{\leq r-1}(S) \leq r - 1 \). Substituting (3.9) in (3.8) gives
\[ \phi b_k \equiv - \sum_{e_r(S) \in \mathcal{A}_\mu} \phi Q_{r,s} e_{\leq r-1}(S) \pmod{I_\mu}, \]
but this implies that
\[ \sum_{i=1}^{m_k} c_{ik} \phi m_{ik} \in \mathcal{A}_{\leq k-1}, \]
which is plainly in conflict with the way the \( m_{ik} \)'s have been selected, unless all of the \( c_{ik} \) are equal to zero. Thus the independence of the \( \psi m_{ik} \)'s in \( R_\mu \) must necessarily hold true and our proof is complete.
This enables us to reach one of the main goals of this section.

**Theorem 3.1.** The collection of monomials $\mathcal{B}(\mu)$ defined recursively in (1.2) gives a basis for $R_\mu$. Moreover,

$$ k_o = n(\mu) = \sum_{i=1}^{h(\mu)} (i-1) \mu_i. \quad (3.10) $$

**Proof.** We have known in Section 2 that the images by $\psi$ of the monomials $\mathcal{B}(\mu)$ span $R_\mu$. Since now we know, from (3.6), that the dimension of $R_\mu$ is equal to the cardinality of $\mathcal{B}(\mu)$, we see that $\psi \mathcal{B}(\mu)$ must necessarily be a basis. In particular, since all the monomials in $\mathcal{B}(\mu)$ have degree $\leq n(\mu)$ we deduce that the largest $k$ for which $\mathcal{H}_k(R_\mu) \neq \{0\}$ must be equal to $n(\mu)$. On the other hand, (3.5) gives that

$$ \mathcal{H}_k(R_\mu) = \psi \mathcal{M}_k. $$

Combining this fact with the definition (3.4) of $k_o$, we see that (3.10) must hold true as asserted.

We can next derive the identification of the ungraded character. This may be stated as follows:

**Theorem 3.2.** The graded character of $R_\mu$ can be written in the form

$$ p^\nu(q) = q^{n(\mu)} p^\nu + (1-q) \sum_{k=0}^{n(\mu)-1} q^k \text{char } \mathcal{A}_\leq k. \quad (3.11) $$

**Proof.** To establish this identity, it is most convenient at this point to use the monomials $m_{jk}$ we dealt with in Proposition 3.2. Note first that since $\mathcal{H}_k(R_\mu)$ is $S_n$-invariant, for any $a \in S_n$ we must have

$$ \psi a m_{jk} \equiv \sum_{i=1}^{m_k} \psi m_{jk} a_i^{(k)}(\sigma) \pmod{I_\mu}, \quad (3.12) $$

where $A^{(k)}(\sigma) = \|a_i^{(k)}(\sigma)\|$ is a suitable matrix of coefficients. In particular, (dropping the dependence on $\sigma$) we have

$$ \sum_{i=1}^{m_k} a_i^{(k)} = \text{trace } A^{(k)} = \text{char } \mathcal{H}_k(R_\mu). \quad (3.13) $$

Now (3.12) can be rewritten as

$$ a m_{jk} - \sum_{i=1}^{m_k} m_{ik} a_i^{(k)}(\sigma) = \sum_{(S) \in \Psi(\mu)} P_{r,S} e_r(S), \quad (3.14) $$
with $P_{r,s}$ a homogeneous polynomial of degree $k - r$. But (as was the case for $b_k$ in (3.8)) we know that a relation such as (3.14) implies that

$$\phi m_{jk} - \sum_{i=1}^{m_h} \phi m_{ik} a^{(k)}_{ij} (\sigma) \in S_{\leq k-1}.$$ 

Thus the coefficient of $\phi m_{jk}$ in $\phi m_{jk}$ must be given precisely by $a^{(k)}_{ij} (\sigma)$. Since the elements $\phi m_{jh}$ with $1 \leq j \leq m_h$ and $0 \leq h \leq k$ form a basis for $S_{\leq k}$, we can conclude that

$$\text{char } \mathcal{A}_{\leq k} = \sum_{h=0}^{k} \sum_{j=1}^{m_h} a^{(k)}_{ij}.$$ 

Combining this with (3.13) gives

$$\text{char } \mathcal{A}_{\leq k} = \sum_{h=0}^{k} \text{char } \mathcal{H}_h (R_{\mu}).$$ 

In other words (for $k \geq 1$)

$$\text{char } \mathcal{H}_k (R_{\mu}) = \text{char } \mathcal{A}_{\leq k} - \text{char } \mathcal{A}_{\leq k-1}.$$ 

Going back to the definition (1.7) of the graded character $p^n(q)$ we can now write

$$p^n(q) = \text{char } \mathcal{A}_{\leq 0} + \sum_{k=1}^{n(\mu)} q^k (\text{char } \mathcal{A}_{\leq k} - \text{char } \mathcal{A}_{\leq k-1})$$

$$= q^{n(\mu)} \text{char } \mathcal{A}_{\leq n(\mu)} + (1 - q) \sum_{k=1}^{n(\mu)} q^k \text{char } \mathcal{A}_{\leq k},$$

and since $\text{char } \mathcal{A}_{\leq n(\mu)} = \text{char } \mathcal{A} = p^n$, the identity in (3.11) follows as asserted. This completes the proof.

An immediate corollary of Theorem 3.1 is the decomposition of $R_{\mu}$ as an $S_{n-1}$-module we announced in Section 1. More precisely, we have

**Theorem 3.3.** The restriction of the graded character of $R_{\mu}$ to $S_{n-1}$ decomposes into the sum

$$p^n(q)|_{S_{n-1}} = \sum_{i=1}^{\ell(\mu)} q^{i-1} p^{\ell(\mu)} (q).$$  \hspace{1cm} (3.15)

**Proof.** Now that we know that the monomials $^2$ in $\mathcal{B}(\mu)$ give a basis for

$^2$ We may omit the map $\phi$ here and in the rest of the paper since there is no more danger of confusion between elements of $R_{\mu}$ and elements of $\mathcal{A}$.
we may express the graded character of \( R_\mu \) at an element \( \sigma \in S_n \) in the form

\[
p^n(\sigma, q) = \sum_{b \in \mathcal{B}(\mu)} q^{\text{degree } b} \sigma b|_b.
\]

Moreover, since the arguments given in Remark 2.1 may now be applied, we can use formulas (2.9) and (2.10) and deduce that, for any \( \sigma \in S_{n-1} \), the contribution to this sum coming from the portion \( x_n^{i-1} \mathcal{B}(\mu^{(i)}) \) of the basis \( \mathcal{B}(\mu) \) is

\[
\sum_{b \in \mathcal{B}(\mu^{(i)})} q^{-1} q^{\text{degree } b} a_{b, b}(\sigma) = q^{-1} p^{\mu^{(i)}}(q, \sigma).
\]

This proves (3.15).

**Remark 3.1.** We should point out that Proposition 3.1 may be used to show that the ring \( R_\mu \) is none other than the graded algebra

\[
\text{gr}(\mathcal{A}) = \bigoplus_{k=0}^{m(\mu)} \mathcal{A}_{\leq k} / \mathcal{A}_{\leq k-1}.
\]

In fact, since we have

\[
\mathcal{A}_{\leq k} \subseteq \mathcal{A}_{\leq h+k}
\]

we see that \( \mathcal{A}_{\leq k} \) is spanned by monomials of degree \( \leq k \) in the elements of \( \mathcal{A}_{\leq 1} \). Thus it follows that \( \text{gr}(\mathcal{A}) \) is generated by its elements of degree 1 and is therefore a homomorphic image of the polynomial ring \( Q[x_1, x_2, \ldots, x_n] \). Let then \( J \) be the ideal defining \( \text{gr}(\mathcal{A}) \). That is

\[
\text{gr}(\mathcal{A}) \cong Q[x_1, x_2, \ldots, x_n]/J.
\]

Now it can be shown that if \( P \) is any polynomial which vanishes on \( W \) its homogeneous part of highest degree must lie in \( J \). From Proposition 3.1 we then get that \( I_\mu \subseteq J \). This gives the inequality

\[
\dim \mathcal{A} = \dim Q[x_1, x_2, \ldots, x_n]/J \leq \dim Q[x_1, x_2, \ldots, x_n]/I_\mu = \dim R_\mu.
\]

Thus equality must hold true throughout. In particular we must have that \( \dim I_\mu = \dim J \), which forces \( I_\mu = J \). So we must have \( \text{gr}(\mathcal{A}) = R_\mu \) as asserted.
It follows almost immediately from the definition of the ideals $I_\mu$ that they are nested into each other according to the dominance order of their partition indexing. To be precise, let us write $v \geq \mu$ if and only if

$$v_n + v_{n-1} + \cdots + v_{n-i} \geq \mu_n + \mu_{n-1} + \cdots + \mu_{n-i} \quad (\forall 0 \leq i < n).$$

This is consistent with the idea that higher in dominance means a flatter partition. This given we see that $v \geq \mu$ is equivalent to

$$v'_1 + v'_2 + \cdots + v'_k \geq \mu'_1 + \mu'_2 + \cdots + \mu'_k \quad (\forall 1 \leq k \leq n).$$

This implies that

$$k - d_k(v) \leq k - d_k(\mu) \quad (\forall 1 \leq k \leq n).$$

and from (1.1) we derive that $C_\mu \subseteq C_v$. In summary we must have

$$v \geq \mu \rightarrow I_v \supseteq I_\mu. \quad (4.1)$$

This in turn implies that

$$v \geq \mu \rightarrow R_v \cong R_\mu/I_v. \quad (4.2)$$

But this means that when the partition $v$ decreases in dominance then the number of occurrences of $x^1$ must increase in each $p^{n}_\nu(m)$. This fact translates into the inequality

$$K_{\lambda v}(q) \leq K_{\lambda \mu}(q) \quad (\forall v \geq \mu), \quad (4.3)$$

where the sign $\leq$ is to represent that the simple inequality $\leq$ holds coefficient-wise. So we see that this beautiful and deep extension of the so called monotonicity of the Kostka numbers follows from this approach with the greatest of ease. We should point out that Lascoux and Schützenberger in [13] announced that (4.3) can be verified through the charge interpretation of the polynomials $K_{\lambda \mu}(q)$.

There are two further combinatorial properties of the rings $R_\mu$ that we mention before closing this section.

**Proposition 4.1.**

$$v \geq \mu \rightarrow \mathcal{P}(v) \subseteq \mathcal{P}(\mu). \quad (4.4)$$

**Proof.** Clearly, this inequality need only be shown when $v$ is an immediate successor of $\mu$ in the dominance order. There are two cases to
be considered. The first case occurs when $v$ is obtained from $\mu$ by lowering an outer corner square $s$ to the inner corner in the column immediately to the right. The second case occurs when $v$ is obtained from $\mu$ by lowering an outer corner square $s$ to the inner corner in the row immediately below. We treat only the first case since the argument is the same in both cases and in the second case the geometry is even simpler. Let us say then that the lowered square is in position $(j, b)$ in the diagram of $\mu$ and that it lands in position $(j + 1, a)$ in the diagram of $v$, with $a < b$ (note that we use cartesian coordinates here with $(0, 0)$ the lowest lefthand position in the Ferrers' diagram). A look at Fig. 4 should clarify the situation. Note that, proceeding by induction, we need only show that, under these circumstances, we necessarily have $\mu^{(i)} < v^{(i)}$, for all $i < h(v)$. Clearly this inequality is trivial for $i > b$. Since then $\mu^{(i)}$ and $v^{(i)}$ are in the same relationship as $\mu$ and $v$. Now for $i < a$ we have the same situation if the corner square of $\mu$ in row $a - 1$ is not in column $j + 1$. However, if it is, for some values of $i$, $\mu^{(i)}$ is obtained from $\mu$ by removing the square in position $(j + 1, a - 1)$, while $v^{(i)}$ is obtained from $v$ by removing the square in position $(j + 1, a)$. Since the latter is higher than the former the inequality $\mu^{(i)} < v^{(i)}$ necessarily follows. In all other cases, for $i < a$ we are removing the same square from both $\mu$ and $v$, and the inequality is trivial. For $i = a$ we remove the square in $(j, b)$ from $\mu$ and that in $(j + 1, a)$ from $v$, that makes $\mu^{(i)} = v^{(i)}$. For $a < i < b$ we remove the square in $(j, b)$ from $\mu$ and that in $(j, b - 1)$ from $v$. In this case, $v^{(i)}$ is obtained from $\mu^{(i)}$ by lowering the square in $(j, b - 1)$ to position $(j + 1, a)$ and so again we have $\mu^{(i)} < v^{(i)}$. Finally for $i = b$ we remove the square in $(j, b)$ from $\mu$ and a higher square from $v$, so the desired inequality holds also in this case. This completes our proof in the case that $\mu$ and $v$ have the same height. In the case that the lowered square, is the highest in the diagram of $\mu$, then the root of the tree for $B(\mu)$ has one more child than that for $B(v)$, but this only strengthens the inequality in (4.1).

\[
\begin{align*}
\mu & \rightarrow \\
(0, 0) & \rightarrow \\
(j, b) & \\
\nu & \rightarrow \\
(0, 0) & \rightarrow \\
(j, 1, a) & 
\end{align*}
\]
For a given standard tableau $S$ let $h(i, S)$ denote the height of the letter $i$ in $S$. Let us then set

$$m(S) = \prod_{i \in S} x_i^{h(i, S) - 1}$$

and refer to it as the monomial of $S$. It develops that Proposition 4.1 has the following beautiful corollary.

**Proposition 4.2.** For each partition $\mu$ the basis $\mathcal{B}(\mu)$ is a lower-order ideal of monomials, whose maximal elements are the monomials of the standard tableaux of shape $\mu$.

**Proof.** We proceed by induction on the number of squares of the diagram of $\mu$. Since the elements of highest degree in of $\mathcal{B}(\mu)$ are the standard tableau monomials defined above, we only need to show that $\mathcal{B}(\mu)$ is a lower ideal of monomials. We recall that the natural partial order of monomials is defined by setting

$$x_1 p_1 x_2 p_2 \cdots x_n p_n \leq x_1 q_1 x_2 q_2 \cdots x_n q_n$$

if and only if $p_i \leq q_i$ for all $1 \leq i \leq n$.

This given, the induction hypothesis reduces us to show that if the monomial $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ occurs in $\mathcal{B}(\mu)$ then also the monomial $x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ can be found in $\mathcal{B}(\mu)$ for any $1 \leq j < i$. By our construction, the latter monomial must come out of $x_1^{p_1} \mathcal{B}(\mu^{(j)})$. In other words the monomial $m(x_1, ..., x_{n-1})$ itself should be in $\mathcal{B}(\mu^{(j)})$. Since this monomial must have come out of $\mathcal{B}(\mu^{(j)})$, we are left to show that $\mathcal{B}(\mu^{(i)}) \subseteq \mathcal{B}(\mu^{(j)})$.

However this follows immediately from Proposition 4.1, since it is easy to see that for $j < i$ we have $\mu^{(j)} \leq \mu^{(i)}$. This completes the proof.

For the shape 112 the standard tableaux monomials are

$$m \begin{pmatrix} 4 \\ 3 \\ 1 \\ 2 \end{pmatrix} = x_3 x_4^2, \quad m \begin{pmatrix} 4 \\ 2 \\ 1 \\ 3 \end{pmatrix} = x_2 x_4^2, \quad m \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix} = x_2 x_3^2.$$

Applying Proposition 4.2 we get to see where the monomials produced in Section 1 were coming from.

It is a well known fact that the set of monomials

$$\{x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} : 0 \leq p_i \leq i - 1 \}$$
is a basis for the ring $R_{1r}$. Thus we see that Proposition 4.2 essentially says that the same type of basis is available for each of the rings $R_{\mu}$.

**Remark 4.1.** During the preparation of this manuscript, Mark Haiman asked us if the present setting yields that the $K_{r\mu}(q)$, for any fixed $\lambda$, are modular with respect to $\mu$. Apparently, this question arose in connection with his recent work [6] on the immanant conjectures. What we can prove is the following property, which is clearly stronger than mere positivity.

**Theorem 4.1.** Let $\lambda$ be fixed and $\rho, \mu, v$ be partitions related by the conditions

$$\mu \geq \rho \quad \text{and} \quad v \geq \rho.$$  \hspace{1cm} (4.5)

Then the polynomial

$$c_{\lambda, \rho, \mu, v}(q) = c_{\lambda, \rho}(q) + c_{\lambda, \mu \lor v}(q) - c_{\lambda, \mu}(q) - c_{\lambda, v}(q)$$

has nonnegative integer coefficients.

**Proof.** Let us first review the meaning of $\rho \lor v$, that is the join of $\rho$ and $v$ in dominance order. To this end, for two partitions $\rho$ and $v$ let us set

$$d_k = \max\{d_k(\rho), d_k(v)\}.$$  

We see that since the parts of $\mu'$ and $v'$ are weakly increasing we must have

$$2d_k(\mu) \leq d_{k-1}(\mu) + d_{k+1}(\mu) \leq d_{k-1} + d_{k+1}.$$  

Similarly we deduce that

$$2d_k(v) \leq d_{k-1} + d_{k+1},$$

and thus we must also have

$$2d_k \leq d_{k-1} + d_{k+1}.$$  

Clearly, this last inequality, valid for all $k$, is necessary and sufficient to guarantee that the sequence of differences

$$\tau_k = d_k - d_{k-1}$$

are the successive parts of the conjugate of a partition $\tau$. This is the partition we are referring to as $\mu \lor v$.

Note then that from the definition (1.5) it immediately follows that

$$\mathcal{C}_{\mu \lor v} = \mathcal{C}_\mu \cup \mathcal{C}_v.$$
from which we deduce that

\[ I_{\mu \vee \nu} = I_{\mu} \vee I_{\nu}, \]

where the symbol \( \vee \) denotes here the join of two vector spaces. Interpreting \( I_{\mu} \), \( I_{\nu} \), and \( I_{\mu \vee \nu} \) as \( S_n \)-invariant subspaces of the ring \( R = \mathbb{Q}[x_1, x_2, ..., x_n] \), let \( M_{\mu} \) and \( M_{\nu} \) respectively be \( S_n \)-invariant complements of \( I_{\mu} \cap I_{\nu} \) in \( I_{\mu} \) and \( I_{\nu} \), respectively, yielding the direct sum decompositions

\[
I_{\mu} = I_{\mu} \cap I_{\nu} \oplus M_{\mu}, \quad I_{\nu} = I_{\mu} \cap I_{\nu} \oplus M_{\nu}, \quad I_{\mu \vee \nu} = I_{\mu} \cap I_{\nu} \oplus M_{\mu} \oplus M_{\nu}.
\]

Now this implies the character identities

\[
\text{char } I_{\mu} + \text{char } I_{\nu} = \text{char } I_{\mu \vee \nu} + \text{char } I_{\mu} \cap I_{\nu}.
\] (4.6)

But now the inequalities in (4.5) assure us that

\[
I_{\mu} \cap I_{\nu} \supseteq I_{\rho}.
\]

Combining this with the identities in (4.6) we deduce that the expression

\[
\text{char } I_{\mu} + \text{char } I_{\nu} - \text{char } I_{\mu \vee \nu} - \text{char } I_{\rho}
\]
is an actual character. So, by complementation, the same must be true for

\[
p_{\mu}(q) + p_{\nu}(q) - p_{\mu \vee \nu}(q) - p_{\rho}(q),
\]

and the assertion of the theorem immediately follows by taking coefficients of the irreducible character \( \chi^A \).

Lascoux and Schützenberger in [15] announce an even stronger property than modularity. In the present notation, their assertion may be stated as follows:

**Theorem 4.2.** For any partition \( \lambda \), the expression

\[
\Phi^\lambda(q) = \sum_{y \trianglerighteq x} p_y(q) \mu(y, x)
\]

is also an actual character. Note that \( \mu(y, x) \) denotes here the Moebius function of the lattice of partitions under dominance.

Unfortunately, here again, very little in the form of proof can be found on this matter in the published work of Lascoux and Schützenberger. Nevertheless, Theorem 4.2 can also be derived from the present setting. However, this derivation is a bit more elaborate than that of Theorem 4.1.
A better understanding of what is going on here is obtained by means of a new basis for $R_+$ recently constructed by E. Allen in [1]. This given, we refer the reader to [1] for this further material.

5. IDENTITIES WHICH CHARACTERIZE THE HL-POLYNOMIALS

In this section we put together the ingredients needed for the identification of the graded character $p^\mu(q)$. The crucial clue for carrying this out is given by the identity in (3.15). To see how this comes about, let us expand both sides of (3.15) in terms of the irreducible characters of $S_{n-1}$. Using the notation introduced in (1.7), we can rewrite (3.15) as

$$
\sum_{\lambda \vdash n} \chi^\lambda \mid_{S_{n-1}} c_{\lambda \mu}(q) = \sum_{i=1}^{h(\mu)} q^{i-1} \sum_{\rho \vdash n-1} \chi^\rho c_{\rho \mu}(i).
$$

(5.1)

It is well known (see for instance [17]) that the character $\chi^\lambda$ restricts to $S_{n-1}$ according to the formula

$$
\chi^\lambda \mid_{S_{n-1}} = \sum_{\rho} \chi^{\mu} \chi(\rho \to \lambda),
$$

(5.2)

where the symbol $\rho \to \lambda$ is to represent that $\rho$ is an immediate predecessor of $\lambda$ in Young's lattice. Substituting (5.2) in (5.1) and equating coefficients of $\chi^{\mu}$ give the recursions

$$
\sum_{\lambda \vdash n} c_{\lambda \mu}(q) \chi(\rho \to \lambda) = \sum_{i=1}^{h(\mu)} q^{i-1} c_{\rho \mu}(i)(q).
$$

(5.3)

Now, if the character $p^\mu(q)$ is what we say it is, then the polynomials $K_{\lambda \mu}(1/q) q^{|\mu|}$ must satisfy the same set of recursions. And indeed, setting (as in [17, p. 132])

$$
K_{\lambda \mu}(1/q) q^{|\mu|} = \overline{K}_{\lambda \mu}(q),
$$

(5.4)

we see that, for all $\rho \vdash n-1$ and $\mu \vdash n$ we also have

$$
\sum_{\lambda \vdash n} \overline{K}_{\lambda \mu}(q) \chi(\rho \to \lambda) = \sum_{i=1}^{h(\mu)} q^{i-1} \overline{K}_{\rho \mu}(i).
$$

(5.5)

Moreover it can be shown that the $c_{\lambda \mu}(q)$'s and the $\overline{K}_{\lambda \mu}(q)$'s satisfy the same boundary conditions. That is, we have

$$
c_{\mu \mu}(q) = q^{|\mu|} = \overline{K}_{\mu \mu}(q), \quad c_{n \mu}(q) = 1 = \overline{K}_{n \mu}(q).
$$

(5.6)
Thus we might be tempted to identify the graded character, that is, establish the identities
\[ c_{\lambda \mu}(q) = \overline{K}_{\lambda \mu}(q), \quad (5.7) \]
by an induction argument based on the recursions (5.3) and (5.5). Unfortunately, this idea works only on partitions of 5. In fact, a simple equation count shows that already for \( n = 6 \), Eqs. (5.3) and (5.5) (even with (5.6)) fail to have a unique solution. Nevertheless, it pays off to have a close look at this near miss. Indeed, a simple restatement of (5.5) in terms of Hall–Littlewood polynomials reveals what else must be shown about our coefficients \( c_{\lambda \mu}(q) \) to establish the identities in (5.7) in full generality.

To proceed we need some notation. First of all, we deal with a number of alphabets \( X = X_n = \{x_1, x_2, \ldots, x_n\} \), \( Y = Y_m = \{y_1, y_2, \ldots, y_m\} \), \( Z = Z_k = \{z_1, z_2, \ldots, z_k\} \), using subscripts only when necessary for clarity. The default number of variables will always be \( n > k \). We start by defining the basic kernel \( \Omega(A) \) for a given multiset of monomials by setting
\[ \Omega(A) = \prod_{a \in A} \frac{1}{1 - a}. \]
This expression, when \( A \) is a product of the two alphabets \( X_n = \{x_1, x_2, \ldots, x_n\} \) and \( Y_m = \{y_1, y_2, \ldots, y_m\} \), becomes
\[ \Omega(X_n Y_m) = \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j}. \]
This is usually referred to as the Cauchy kernel. In particular (for \( m = 1 \)) one has
\[ \Omega(X_n Y) = \prod_{i=1}^{n} \frac{1}{1 - x_i y} = \sum_{k \geq 0} y^k h_k(X_n), \quad (5.8) \]
which is the generating function of the so-called homogeneous symmetric functions.

Since we can write
\[ \frac{1}{1 - a} = \exp \left( \sum_{k \geq 1} \frac{a^k}{k} \right), \]
we also have
\[ \Omega(A) = \exp \left( \sum_{k \geq 1} \frac{p_k(A)}{k} \right), \quad (5.9) \]
where

\[ p_k(A) = \sum_{a \in A} a^k. \]

It is convenient when dealing with Hall–Littlewood functions to use \( \lambda \)-ring notation. We refer to [17] for the precise definitions and only give here what is needed to understand our presentation. Roughly speaking, we need to deal with symmetric functions evaluated at a difference of two multisets of monomials. Now, since every symmetric function can be expressed as a polynomial in the power symmetric functions, we need only extend the definition of \( p_k(A) \). This is simply achieved by considering \( p_k(A) \) as a linear function of its argument. For instance we set

\[ p_k(X_n - Y_m) = \sum_{i=1}^{n} x_i^k - \sum_{j=1}^{m} y_j^k, \quad p_k(1 - q) = 1 - q^k, \]

and when \( A = X_n Y_m(1 - q) \), then

\[ p_k(A) = p_k(X_n, Y_m) p_k(1 - q) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^k y_j^k - \sum_{i=1}^{n} \sum_{j=1}^{m} q^k x_i^k y_j^k. \]

Thus substituting in (5.9) we get

\[ \Omega(XY(1 - q)) = \prod_{i,j} \frac{1 - q x_i y_j}{1 - x_i y_j}, \quad (5.10) \]

which is the standard Hall–Littlewood kernel.

We denote here by \( \langle , \rangle_H \) the Hall scalar product of symmetric functions. We recall (see [17]) that two bases \( \{\xi_\lambda\} \) and \( \{\eta_\lambda\} \) are dual with respect to the Hall scalar product if and only if

\[ \sum_\lambda \xi_\lambda(X) \eta_\lambda(Y) = \Omega(XY). \]

Thus the classical Cauchy formula

\[ \sum_\lambda S_\lambda(X) S_\lambda(Y) = \Omega(XY) \quad (5.11) \]

expresses the orthonormality of the Schur functions. It is good to introduce here the linear operator \( \Gamma_v \), defined by setting

\[ \Gamma_v S_\lambda(X) = S_{\lambda, v}(X). \quad (5.12) \]
This is sometimes referred to as skewing by $v$. Because of the well known identity (see [17])

$$\langle S_\mu, S_{\lambda/v} \rangle_H = \langle S_\mu S_v, S_\lambda \rangle_H = g_{\mu v}^\lambda,$$  \hspace{1cm} (5.13)

we see that the adjoint of $\Gamma$, with respect to the Hall scalar product is none other than multiplication by $S_v$. Using (5.13) and the orthonormality of Schur functions we can write

$$\Gamma_v S_\lambda(X) = \sum \mu g_{\mu v}^\lambda S_\mu(X).$$  \hspace{1cm} (5.14)

In particular, we see from the classical Pieri formula [17] that when $v$ reduces to the partition consisting of a single part equal to 1, then

$$\Gamma_1 S_\lambda(X) = \sum \mu S_\mu(X) \chi(\mu \to \lambda).$$  \hspace{1cm} (5.15)

We also note that, using (5.14), the identification of the adjoint of $\Gamma_v$ can also be expressed in the form

$$\Gamma_v \Omega(XY) = \Omega(XY) S_v(Y).$$  \hspace{1cm} (5.16)

Finally, we should recall that the so-called Frobenius image $F_\chi$ of a class function $\chi = \sum \lambda c_\lambda \chi^\lambda$ is the symmetric polynomial

$$F_\chi = \sum \lambda c_\lambda S_\lambda(X).$$

This given, let us set

$$H_\mu(X; q) = \sum \lambda S_\lambda(X) K_{\lambda\mu}(q), \hspace{1cm} B_\mu(X; q) = \sum \lambda S_\lambda(X) K_{\lambda\mu}(q),$$  \hspace{1cm} (5.17)

and

$$C_\mu(X; q) = \sum \lambda S_\lambda(X) c_{\lambda\mu}(q).$$  \hspace{1cm} (5.18)

In view of (1.7), we may also write

$$C_\mu(X; q) = \# p^\mu(q).$$  \hspace{1cm} (5.19)

Since the Frobenius image of the ungraded character $p^\mu$ is known to be the
complete homogeneous symmetry function \( h_\mu \), we see that the Frobenius image of (1.3) is the classical expansion

\[
h_\mu = \sum_\lambda S_\lambda(X) K_{\lambda\mu}.
\]  

(5.20)

Thus we may view \( H_\mu(X; q) \) as a q-analogue of \( h_\mu \).

Taking account of (5.15) and (5.2), we see that the Frobenius image of (3.15) is the identity

\[
\Gamma_1 C_\mu(X; q) = \sum_{i=1}^{n(\mu)} q^{i-1} C_{\mu^{(i)}}(X; q).
\]  

(5.21)

Similarly, multiplying the identity in (5.5) by \( S_\lambda(X) \) and summing we get

\[
\Gamma_1 \bar{H}_\mu(X; q) = \sum_{i=1}^{n(\mu)} q^{i-1} \bar{H}_{\mu^{(i)}}(X; q).
\]  

(5.22)

Now, a simple calculation, which uses (5.4) and (1.9), shows that (5.22) is equivalent to the identity

\[
\Gamma_1 H_\mu(X; q) = \sum_{i=1}^{n(\mu)} q^{i-1} H_{\mu^{(i)}}(X; q),
\]  

(5.23)

where \( s_i \) here denotes the height of the square we must remove from \( \mu \) to get \( \mu^{(i)} \).

Note further that if we substitute (1.1) in the Cauchy formula (5.11), change the order of summation, and use (5.8) we get

\[
\Omega(XY) = \sum_\mu H_\mu(X; q) P_\mu(Y; q).
\]  

(5.24)

In other words, \( H_\mu(X; q) \) is dual to \( P_\mu(Y; q) \) with respect to the Hall scalar product. From (5.16) we then get that

\[
\Gamma_1 \Omega(XY) = \sum_\rho \Gamma_1 H_\rho(X; q) P_\rho(Y; q)
\]

\[
= \sum_\rho H_\rho(X; q) P_\rho(Y; q) e_1(Y).
\]

Setting as in [17]

\[
P_\rho e_1 = \sum_\mu g_\mu P_\mu,
\]  

(5.25)
we derive that (for $m = 1$)

$$
\sum_\rho \Gamma_1 H_\rho(X; q) P_\rho(Y; q) = \sum_\rho H_\rho(X; q) \sum_\mu g^\mu_\rho(1) P_\mu(Y)
$$

$$
= \sum_\mu \left( \sum_\rho g^\mu_\rho(1) H_\rho(X; q) \right) P_\mu(Y),
$$

which gives

$$
\Gamma_1 H_\mu(X; q) = \sum_\rho g^\mu_\rho(1) H_\rho(X; q).
$$

The coefficients $g^\mu_\rho(1)$ are given quite explicitly in [17, formula (3.2), p. 111]). Using Macdonald formula for $m = 1$ with a little manipulation we can show that (5.26) reduces to (5.23). We do not carry this out here partly because we shortly have to derive a more general result and partly because we want to illustrate a $\lambda$-ring approach which deals directly with the polynomials $H_\rho(X; q)$.

In summary, we can conclude that the recursions in (5.3) and (5.5) express that the two families of polynomials $C_\rho(X; q)$ and $\tilde{H}_\rho(X; q)$ behave in the same manner with respect to the dual of multiplication by the elementary symmetric function $e_1$. Although this by itself is not sufficient to imply that the two families are the same, we see that the identity of the two families must hold true if in addition we show that they behave in the same manner with respect to the dual of multiplication by an arbitrary symmetric function. Now, to derive this, by the fundamental theorem of the theory of symmetric functions, we need only show that two families behave in the same manner with respect to the dual of multiplication by any of the elementary symmetric functions $e_k$.

This given, we begin by working out the action of $\Gamma_1$ on $H_\mu(X; q)$. Our point of departure is the Raising operator formula for the Hall–Littlewood polynomials $Q_\mu(X; q)$, (see (2.15) of [17, p. 107]). For our purposes it is most convenient to write this formula in the form

$$
Q_\mu(X_n; q) = \prod_{1 < i < j < k} \frac{1-z_j/z_i}{1-qz_j/z_i} \Omega((1-q) X_n Z_k)|_{x^\mu},
$$

(5.27)

where here we assume that $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$ and $x^\mu = z_1^{\mu_1} z_2^{\mu_2} \cdots z_k^{\mu_k}$. It is also shown in [17] (see (6.3), p. 126) that this formula is equivalent to the identity

$$
Q_\mu(X_n; q) = \sum_\lambda S_\lambda((1-q) X_n) K_{x^\mu}(q).
$$

(5.28)
Comparing with our definition (5.17) of $H\mu(X; q)$ we see that

$$H\mu(X_n; q) = Q\mu(X_n/(1 - q); q).$$

Thus making the replacement $X_n \rightarrow X_n/(1 - q)$ in (5.28) we get

$$H\mu(X_n; q) = \prod_{1 \leq i < j \leq k} \frac{1 - z_j/z_i}{1 - qz_j/z_i} \Omega(X_nZ_k)|_{z^n}. \quad (5.29)$$

To proceed further, we need a λ-ring reformulation of this identity. This may be stated as follows:

**Proposition 5.1.** For a given variable $z$ and symmetric polynomial $P(X_n)$ set

$$H(z)P(X_n) = \Omega(X_nz)P\left(X_n - \frac{1 - q}{z}\right), \quad (5.30)$$

and let $H_m$ be the linear operator on symmetric polynomials defined by setting

$$H_mP(X_n) = H(z)P(X_n)|_{z^m}. \quad (5.31)$$

Then, for $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$

$$H\mu(X_n; q) = H_{\mu_1}H_{\mu_2}\cdots H_{\mu_k}1. \quad (5.32)$$

**Proof.** Note that applying $H(z_2)$ and $H(z_1)$ in succession we get

$$H(z_1)H(z_2)P(X_n) = H(z_1)\Omega(X_nz_2)P\left(X_n - \frac{1 - q}{z_2}\right)$$

$$= \Omega(X_nz_1)\Omega\left(\left(X_n - \frac{1 - q}{z_1}\right)z_2\right)P \left(X_n - \frac{1 - q}{z_1} - \frac{1 - q}{z_2}\right).$$

Using the multiplicativity property of the kernel $\Omega$ we can rewrite this as

$$H(z_1)H(z_2)P(X_n) = \Omega(X_nZ_2) \frac{1 - z_2/z_1}{1 - q^{z_2/z_1}}P \left(X_n - \frac{1 - q}{z_1} - \frac{1 - q}{z_2}\right). \quad (5.33)$$

An easy induction argument then yields that

$$H(z_1)H(z_2)\cdots H(z_k)P(X_n)$$

$$= \Omega(X_nZ_k) \prod_{1 \leq i < j \leq k} \frac{1 - z_j/z_i}{1 - qz_j/z_i}P \left(X_n - \frac{1 - q}{z_1} - \frac{1 - q}{z_2} - \cdots - \frac{1 - q}{z_k}\right).$$
In particular, for $P = 1$ we get

$$H(z_1) H(z_2) \cdots H(z_k) 1 = Q(X, Z_k) \prod_{1 \leq i < j \leq k} \frac{1 - z_j/z_i}{1 - q z_j/z_i}.$$ 

Equating the coefficients of $z^n$ on both sides of this identity, we see that formula (5.32) is just another way of writing (5.29).

Remark 5.1. We should note that the raising operator formula (5.29) was also given by Milne in [20]. Moreover, our introduction of the operator $H(z)$ was inspired by a recent work of N. Jing [8], where a formula for $Q_\mu(X; q)$, analogous to (5.32), is given in terms of the so-called vertex operators.

Remark 5.2. We should also note that formula (5.32) enables us to extend the definition of $H_\mu(X; q)$ to the case in which $\mu$ is only a composition. That is, when its parts are not necessarily weakly decreasing.

Before stating the basic identity involving $\Gamma_{1^k}$ we need one further property of the operators $H_m$. Namely,

**Proposition 5.2.** For any pair of indices $a$ and $b$ we have

$$H_{b-1}H_a - qH_bH_{a-1} + H_{a-1}H_b - qH_aH_{b-1} = 0. \tag{5.34}$$

In particular setting $a = b = m$ we obtain

$$H_{m-1}H_m = qH_mH_{m-1}. \tag{5.35}$$

**Proof.** From (5.33) we easily derive that

$$(z_2 - qz_1) H(z_2) H(z_1) + (z_1 - qz_2) H(z_1) H(z_2) = 0,$$

and formula (5.34) follows immediately by taking the coefficient of $z_1^a z_2^b$.

Let us still use the English convention for the partition $\mu$ but change the last index to the height of $\mu$. More precisely,

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{h(\mu)} > 0).$$

For a given set of indices

$$I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)\},$$

let $p(\mu, I)$ denote the composition obtained from $\mu$ upon decreasing by one each of the parts $\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_k}$ and let $\mu^{(I)}$ denote the partition obtained by rearranging $p(\mu, I)$ back again into a partition. To make sure we get across
this construction, we illustrate in Fig. 5 the diagrams of \( \mu, p(\mu, I) \), and \( \mu^{(I)} \) when \( \mu = (1^2 2^2 4^5 6^3 7^3) \) and \( I = (1, 2, 4, 7, 9, 12, 14) \). We clearly see that the operation \( \mu \to \mu^{(I)} \) generalizes the operation \( \mu \to \mu^{(I)} \) defined in Section 1. With this notation we can prove the following remarkable generalization of the recursion in (5.22).

**Proposition 5.3.** For any \( 1 < k < h(\mu) \) we have

\[
\Gamma_k \tilde{H}_\mu(X; q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k < h(\mu)} q^{i_1-1 + i_2-1 + \cdots + i_k-1} \tilde{H}_\mu(X; q). \tag{5.36}
\]

**Proof.** Our identity is an immediate consequence of the following beautiful commuting properties of the operators \( \Gamma_k \) and \( H_m \). Namely,

\[
\Gamma_k H_m = H_m \Gamma_k + H_{m-1} \Gamma_k \Gamma_{k+1}. \tag{5.37}
\]

To prove this we resort to a generating function argument. We set \( \Gamma(t) = \sum_k (-t)^k \Gamma_k \) and note that (5.16) gives

\[
\Gamma(t) \Omega(XY) = \Omega(XY) \sum_k (-t)^k e_k(Y) = \Omega(XY) \Omega(-Yt) = \Omega(X-t) Y).
\]

This implies that for any symmetric polynomial \( P(X) \) we must have

\[
\Gamma(t) P(X) = P(X-t). \tag{5.38}
\]

Thus, using the defining relation (5.30), we deduce that

\[
H(z) \Gamma(t) P(X) = \Omega(XY) P(X-t - \frac{1-q}{z}).
\]

---

**Figure 5**
as well as

\[ \Gamma(t) H(z) P(X) = \Omega((X-t) Y) P \left( X-t - \frac{1-q}{z} \right) \]

\[ = (1-tz) \Omega(X Y) P \left( X-t - \frac{1-q}{z} \right). \]

In other words,

\[ \Gamma(t) H(z) = (1-tz) H(z) \Gamma(t), \]

and (5.37) follows by equating coefficients of the monomial \( tz^m \). This given, we see that we must have

\[ \Gamma_{t^k} H_{\mu_1} H_{\mu_2} \cdots H_{\mu_s} 1 = H_{\mu_1} \Gamma_{t^k} H_{\mu_2} \cdots H_{\mu_s} 1 + H_{\mu_1-1} \Gamma_{t^{k-1}} H_{\mu_2} \cdots H_{\mu_s} 1, \]

and an obvious induction argument, based on the fact that \( \Gamma_{t^k} 1 = 0 \), (when \( s \geq 1 \)) immediately yields that

\[ \Gamma_{t^k} H_{\mu}(X; q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} H_{p(\mu, I)}(X; q), \quad (5.39) \]

where the meaning of \( H_{p(\mu, I)}(X; q) \) is as indicated in Remark 5.2. To obtain (5.36) we first rearrange \( p(\mu, I) \) to a partition using the commutativity property in (5.35), then get the relation involving the \( \tilde{H} \)'s, by means of the identities

\[ \tilde{H}_{\mu}(X; q) = H_{\mu}(X; 1/q) q^{n(\mu)}. \quad (5.40) \]

Now the first step changes (5.39) to the form

\[ \Gamma_{t^k} H_{\mu}(X; q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} q^{f(\mu, I)} H_{\mu(\nu)}(X; q), \quad (5.41) \]

and the second step reduces this to

\[ \Gamma_{t^k} \tilde{H}_{\mu}(X; q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} q^{-f(\mu, I)} \tilde{H}_{\mu(\nu)}(X; q) q^{n(\mu) - n(\mu(\nu))}. \quad (5.42) \]

We evaluate the expression \(-f(\mu, I) + n(\mu) - n(\mu(\nu))\) by separately working out the contribution of each vertical segment of the diagram of \( \mu \). We illustrate in Fig. 6 a typical portion of the diagrams of \( \mu, p(\mu, I), \) and \( \mu(\nu) \) corresponding to such a segment. The indices \( j_1 < \cdots < j_2 < j_1 \) are to indicate here the successive heights of the squares that are removed from that portion to obtain the diagram of \( p(\mu, I) \). The index \( h \) denotes the
height of the outer corner square of the diagram of $\mu$ which lies in that particular portion. This given, we see that to pass from $p(\mu, I)$ to $\mu^{(\ell)}$ we must move the first of these holes to height, $h$ the second to height $h-1$, etc., the last to height $h-s+1$. The commutativity relation in (5.35) then yields that the contribution to $f(\mu, I)$ coming from this portion must be

$$h - j_1 + h - 1 - j_2 + \cdots + h - s + 1 - j_s.$$  

On the other hand, in passing to $\mu^{(\ell)}$, the loss to $n(\mu)$ coming from this portion is given by

$$h - 1 + h - 2 + \cdots + h_s - s.$$  

In summary, we must conclude that the contribution of this portion to the exponent $-f(\mu, I) + n(\mu) - n(\mu^{(\ell)})$ reduces to

$$j_1 - 1 + j_2 - 1 + \cdots + j_s - 1.$$  

Summing all these contributions gives

$$-f(\mu, I) + n(\mu) - n(\mu^{(\ell)}) = i_1 - 1 + i_2 - 1 + \cdots + i_k - 1.$$  

Thus formula (5.36) must hold true precisely as asserted.

To complete our work we are left to establish that the symmetric polynomials $C_\mu(X; q)$ do also satisfy the same relation. This is the goal of our next and final section.
6. Identification of the Graded Character

Our main goal here is to show the recurrence

\[ \Gamma_{\nu} C_{\mu}(X; q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} q^{i_1 - 1 + i_2 - 1 + \cdots + i_k - 1} C_{\mu(i)}(X; q). \]  

(6.1)

More precisely, we show that

\[ \Gamma_{\nu} p^\mu(q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} q^{i_1 - 1 + i_2 - 1 + \cdots + i_k - 1} p_\mu(i)(q), \]  

(6.2)

where, with some abuse of notation, we are extending the definition of the operator \( \Gamma_{\nu} \) to act on class functions by setting

\[ \Gamma_{\nu} \sum_{\lambda} c_\lambda \chi^\lambda = \sum_{\lambda} c_\lambda \chi^{\lambda/^\nu}. \]

With this convention, we can easily see that (6.1) is simply the Frobenius image of (6.2).

Before we can even try to establish (6.2) we must find out what kind of operation on \( R_\mu \) produces a module with character \( \Gamma_{\nu} p^\mu(q) \). Now there is a beautiful answer to this question, but to state it we need some notation.

For a given alphabet \( A \) let us denote by \( S_{[A]} \) the group of permutations of the letters of \( A \). If \( A \) and \( B \) are disjoint alphabets, we also denote by \( S_{[A]} \otimes S_{[B]} \) the subgroup of \( S_{[A + B]} \) which leaves \( A \) and \( B \) invariant (as sets). We can assume here, without loss, that \( A = 1, 2, \ldots, h \) and \( B = h + 1, h + 2, \ldots, h + k \) with \( h + k = n \). If \( \sigma_A \in S_{[A]} \) and \( \sigma_B \in S_{[B]} \) with

\[ \sigma_A = \begin{pmatrix} 1 & 2 & \cdots & h \\ \sigma_1 & \sigma_2 & \cdots & \sigma_h \end{pmatrix} \quad \text{and} \quad \sigma_B = \begin{pmatrix} h + 1 & h + 2 & \cdots & h + k \\ \sigma_{h + 1} & \sigma_{h + 2} & \cdots & \sigma_{h + k} \end{pmatrix}, \]

then we set

\[ \sigma_A \cdot \sigma_B = \begin{pmatrix} 1 & 2 & \cdots & h & h + 1 & h + 2 & \cdots & h + k \\ \sigma_1 & \sigma_2 & \cdots & \sigma_h & \sigma_{h + 1} & \sigma_{h + 2} & \cdots & \sigma_{h + k} \end{pmatrix}. \]

Finally, for an alphabet \( B \) we let \( N[B] \) denote the sum of the signed elements of \( S_{[B]} \). More precisely,

\[ N[B] = \sum_{\sigma_B \in S_{[B]}} \text{sign}(\sigma_B) \sigma_B. \]

The idempotent \( N[B] \) is what Young refers to as the negative symmetric group of \( B \). With some abuse of notation we also view \( N[B] \) as an idempotent in \( S_{[A]} \otimes S_{[B]} \). In fact, we systematically identify each element of
\[ \sigma_A \in S_{[A]} \] with the element \( \sigma_A \cdot e_B \) and each element \( \sigma_B \in S_{[B]} \) with the element \( e_A \cdot \sigma_B \). Here of course \( e_A \) and \( e_B \) denote the identity elements of \( S_{[A]} \) and \( S_{[B]} \), respectively.

This given, we can show that if \( M \) is an \( S_{[A+B]} \)-module with character \( \chi \) then the restriction of \( N[B]M \) to the subgroup \( S_{[A]} \otimes e_B \) is an \( S_{[A]} \)-module with character \( \Gamma_{\chi} \). More generally, we have:

**Proposition 6.1.** Let \( \mathcal{M} = \{m_1, m_2, ..., m_M\} \) be a basis for an \( S_{[A+B]} \)-module \( M \) with character \( \chi \) and let \( \gamma_B \) be the central idempotent of \( S_{[B]} \), which corresponds to the partition \( \nu \). Then the action of \( S_{[A]} \) on the linear span of \( \gamma_B \cdot \mathcal{M} = \{\gamma_B m_1, \gamma_B m_2, ..., \gamma_B m_M\} \) induces a representation with character \( \Gamma_{\chi} \).

**Proof.** Note first that the linear span of \( \gamma_B \cdot \mathcal{M} \) is an \( S_{[A]} \otimes S_{[B]} \)-invariant subspace. This is simply due to the fact that since \( \gamma_B \) is central in the group algebra of \( S_{[B]} \) we necessarily have for all \( \sigma_A \in S_{[A]} \) and \( \sigma_B \in S_{[B]} \)

\[
\sigma_A \cdot \sigma_B \cdot \gamma_B = \gamma_B \cdot \sigma_A \cdot \sigma_B.
\]

Secondly, it is clear that we need only establish the result in the case that \( M \) is an irreducible \( S_{[A+B]} \)-module. So let the \( \chi = \chi^\lambda \) for a given \( \lambda \leftarrow h + k = n \). Now, it is well known (see [4] for a combinatorial proof) that the restriction of \( \chi^\lambda \) to \( S_{[A]} \otimes S_{[B]} \) decomposes into the sum

\[
\chi^\lambda |_{S_{[A]} \otimes S_{[B]}} = \sum_{\mu \leftarrow k} \chi^{\lambda/\mu} \chi^\mu.
\]

This means that the matrix corresponding to the action of an element \( \sigma_A \cdot \sigma_B \) on the basis \( \mathcal{M} \) is similar to the matrix

\[
\bigoplus_{\rho \leftarrow k} A^{\lambda/\rho}(\sigma_A) \otimes A^\rho(\sigma_B),
\]

where \( A^{\lambda/\rho} \) and \( A^\rho \) are representations of \( S_{[A]} \) and \( S_{[B]} \) with characters \( \chi^{\lambda/\rho} \) and \( \chi^\rho \), respectively. In particular, the action of the element \( \sigma_A \gamma_B \) on the basis \( \mathcal{M} \) is expressed by right multiplication with a matrix similar to

\[
\bigoplus_{\rho \leftarrow k} A^{\lambda/\rho}(\sigma_A) \otimes A^\rho(\gamma_B).
\]

On the other hand, since \( \gamma_B \) is the central idempotent corresponding to \( \nu \), all of the matrices \( A^\rho(\gamma_B) \) with \( \rho \neq \nu \) vanish, while \( A^\nu(\gamma_B) \) reduces to the identity matrix of the appropriate dimension. Thus we can plainly see that the action of an element \( \sigma_A \cdot e_B \) on the range of \( e_A \cdot \gamma_B \) in \( M \) can be expressed by a matrix similar to \( A^{\lambda/\nu}(\sigma_A) \). This establishes our assertion.
We aim to apply this proposition to the case when $M = R_\mu$ and $\mathcal{M} = B(\mu)$. But before we do this, it is instructive to give a separate treatment to the case $\mu = 1^n$. It is good here and in the following to set $A = (x_1, x_2, \ldots, x_h)$ and $B = (x_{h+1}, x_{h+2}, \ldots, x_{h+k})$. Set also

$$\alpha_\epsilon(A) = x_1^{a_1} x_2^{a_2} \cdots x_h^{a_h} \quad \text{with} \quad 0 \leq a_i \leq i \quad (\text{for } i = 1 \cdots h) \quad (6.3)$$

and

$$\beta_\eta(B) = x_{h+1}^{\eta_1} x_{h+2}^{\eta_2} \cdots x_{h+k}^{\eta_k} \quad \text{with} \quad 0 \leq \eta_i \leq i - 1 \quad (\text{for } i = 1 \cdots k). \quad (6.4)$$

Note that, in the notation of Section 2, the set of monomials $\alpha_\epsilon$ constitute precisely the basis $\mathcal{B}(1^n)$. It is well known (and easy to show) that every polynomial in the alphabet $A$ can be uniquely written as a linear combination of the polynomials

$$\alpha_\epsilon(A) e_1^{a_1}(A) e_2^{a_2}(A) \cdots e_h^{a_h}(A) \quad \text{(with the } a_i\text{'s } \geq 0 \text{ integers).} \quad (6.5)$$

The analogous result holds true for the polynomials in the alphabet $B$. This implies that every polynomial in the alphabet $X = A + B$ is a unique linear combination of the polynomials

$$\alpha_\epsilon(A) \beta_\eta(B) e_1^{a_1}(A) e_2^{a_2}(A) \cdots e_h^{a_h}(A) e_1^{b_1}(B) e_2^{b_2}(B) \cdots e_k^{b_k}(B). \quad (6.6)$$

Let us now recall that in the ring $R_{1^n}$ we have the identity

$$(1 - tx_1)(1 - tx_2) \cdots (1 - tx_n) \cong 1_n 1.$$

This implies in particular that

$$\frac{1}{(1 - tx_1)(1 - tx_2) \cdots (1 - tx_h)} \cong 1_n \frac{1}{(1 - tx_{h+1})(1 - tx_{h+2}) \cdots (1 - tx_{h+k})}.$$

In other words we have

$$e_i(A) \cong 1_n (-1)^i h_i(B) \quad (\text{for } i = 1 \cdots h). \quad (6.7)$$

Substituting these identities in (6.6) yields that every element of $R_{1^n}$ is a linear combination of polynomials of the form

$$\alpha_\epsilon(A) \beta_\eta(B) Q(B), \quad (6.8)$$

with $Q(B)$ a generic homogeneous polynomial symmetric in the alphabet $B$. However, we can say something more precise than that. Namely,
Theorem 6.1. The polynomials

$$\alpha_\lambda(A) \beta_{\eta}(B) S_{\lambda}(B) \tag{6.9}$$

with $S_{\lambda}(B)$ a Schur function indexed by a partition with diagram contained in the $k \times h$ rectangle form a basis for $R_{1^n}$.

Proof. Since the Schur functions $S_{\lambda}(B)$ are a basis for the symmetric polynomials we see that we can restrict $Q(B)$ in (6.8) to be of a Schur function. Note further that, because of (6.7), we do have the identity

$$S_{\lambda}(B) \cong \frac{1}{h!} (-1)^{|\lambda|} S_{\lambda'}(A), \tag{6.10}$$

where, as before, $\lambda'$ denotes the partition conjugate to $\lambda$. Now since $B$ has $k$ letters, the left-hand side of (6.10) vanishes if $\lambda$ has more than $k$ parts. Similarly, since $A$ has only $h$ letters, the right-hand side must vanish if $\lambda'$ has more than $h$ parts. This shows we can further restrict the polynomials in (6.9) by the stated condition on the index $\lambda$. Finally, to show that, thus restricted, they form a basis we resort to a simple counting argument. There are $h!$ of the $\alpha_\lambda(A)$'s and $k!$ of the $\beta_{\eta}(B)$'s, while there are $\left(\begin{array}{c} h+k \\ k \end{array}\right)$ partitions in the $h \times k$ rectangle. Thus in total, our spanning set has $n!$ elements. But we know that the dimension of $R_{1^n}$ is also $n!$. Thus the theorem holds true precisely as asserted.

Remark 6.1. Before we can proceed we need to make a crucial observation here. Note that although Eqs. (6.1) and (6.2) are yet to be demonstrated, we should point out that we have already established them for $q = 1$. Indeed, from (5.19) and (3.11) we get that

$$C_{\mu}(X; 1) = F_{\mu} = h_{\mu}(X) = \tilde{F}_{\mu}(X; 1). \tag{6.11}$$

Thus, (5.36) gives that

$$\Gamma_{1^n} C_{\mu}(X; 1) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} h_{\mu(i)}(X)$$

$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} C_{\mu(i)}(X; 1). \tag{6.12}$$

In particular we deduce the important identity

$$\dim \Gamma_{1^n} R_{\mu} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} \dim R_{\mu(i)}. \tag{6.13}$$
Indeed, this is simply obtained by equating coefficients of the identity in
\[ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} p^{\mu(i)}(1) = \prod_{1 \leq i < j \leq k} (1 - \lambda_i / \lambda_j), \]
which is the inverse image of (6.12) by the Frobenius map.

This remark applied to the case \( \mu = 1^n \) immediately gives

**Proposition 6.2.** The polynomials
\[ \{ \alpha_\lambda(A) \Delta(B) S_\lambda(B) \}, \quad (6.14) \]
with \( \Delta(B) \) the Vandermonde determinant in the alphabet \( B \), \( \alpha_\lambda \in \mathcal{B}(1^k) \) and \( \lambda \subseteq k^{n-k} \) form a basis for the \( S_{[A]} \)-module \( \Gamma_{1^k} R_{1^n} \).

**Proof.** We simply note that
\[ N(B) \beta_n(B) = \begin{cases} \Delta(B) & \text{if } \eta = (0, 1, \ldots, k-1) \\ 0 & \text{otherwise} \end{cases}. \]

Thus upon the action of \( N(B) \) the collection in (6.9) reduces to that in (6.14). So at least we know from Proposition 6.1 that the latter spans \( \Gamma_{1^k} R_{1^n} \). But, on the other hand (6.13) gives
\[ \dim \Gamma_{1^k} R_{1^n} = \binom{n}{k} \dim R_{1^{n-k}} = \binom{n}{k} (n-k)! , \]
and since the spanning set in (6.14) has cardinality \( \binom{n}{k}(n-k)! \) as well, we must conclude that it is also a basis as asserted.

Let us now set
\[ \mathcal{H}_\lambda^B(R_{1^n}) = \mathcal{L} [ \alpha_\lambda(A) \Delta(B) S_\lambda(B) : 0 \leq i \leq n - 1 ], \quad (6.15) \]
where the symbol \( \mathcal{L} \) denotes the operation of taking a linear span. Moreover, for any integer \( m \) let
\[ \mathcal{H}_\lambda^B(R_{1^n}) = \bigoplus_{|\lambda| > m} \mathcal{H}_\lambda^B(R_{1^n}). \quad (6.16) \]

Note that we necessarily have
\[ \mathcal{H}_\lambda^B(R_{1^n}) = \{ 0 \} \quad \text{whenever } \lambda \not\subseteq k^{n-k}. \quad (6.17) \]

It will also be convenient to denote by \( R_{(1^n)}(A) \) the ring \( R_{(1^n)} \) in the alphabet \( A \).
This given, the computation of the character of $\Gamma_{1^n} R_{1^n}$ as an $S_{[\lambda]}$-module is facilitated by the following basic property of the basis in (6.14).

**Proposition 6.3.** Let $|\lambda| = m$ and let $Q(A)$ be a polynomial which, as an element of $R_{1^n}(A)$, has the expansion

$$Q(A) \cong \sum c_\mu \chi_\mu(A).$$

(6.18)

Then in $R_{1^n}$ we have the relation

$$Q(A) A(B) S_{\lambda}(B) \cong \sum c_\mu \chi_\mu(A) A(B) S_{\lambda}(B) + E,$$

(6.19)

with

$$E \in \mathcal{H}^B_{> m}(R_{1^n}).$$

(6.20)

**Proof.** Equation (6.18) means that we must have the equality

$$Q(A) = \sum c_\mu \chi_\mu(A) + \sum_{i=1}^h Q_i(A) e_i(A)$$

for some suitable polynomials $Q_i(A)$. Thus, using (6.7), we derive that

$$Q(A) A(B) S_{\lambda}(B) \cong \sum c_\mu \chi_\mu(A) A(B) S_{\lambda}(B)$$

$$+ \sum_{i=1}^h Q_i(A)(-1)^i h_i(B) A(B) S_{\lambda}(B).$$

(6.21)

But Pieri’s rule gives

$$S_{\lambda}(B) h_i(B) = \sum_{\mu}^{(*)} S_\mu(B),$$

(6.22)

where the $(*)$ is to indicate that the sum is to be carried out over partitions $\mu$ for which $\mu/\lambda$ is a horizontal $i$-strip. In any case we see that (6.21) and (6.22) are precisely what we need to put together an algorithm for expanding in terms of the basis in (6.14). Indeed, from (6.17) we get that the second sum in (6.21) vanishes when $\lambda$ is largest, that is when $\lambda = k^h$. Proceeding by reverse induction on the size of $m = |\lambda|$, we see that in all other cases the second sum in (6.21) will necessarily lie in $\mathcal{H}^B_{> m}(R_{1^n})$. This completes the induction and our proof.

This result has the following immediate corollary.
THEOREM 6.2. The graded character of the $S_{[\lambda]}$-module $\Gamma_1 R_1$ decomposes into the product

$$\Gamma_1^p p^{(1^r)}(q) = q^{\frac{n}{2}}\left[\begin{array}{c} n \\ k \end{array}\right] q^{p^{(1^t)}(q)}. \quad (6.23)$$

Proof. We can express the action of $S_{[\lambda]}$ on $R_{\{1^\lambda\}}(A)$ in the form

$$\sigma_\lambda \alpha_e(A) \cong (1^\lambda) \sum_{e'} \alpha_{e'}(A) a_{e',e}(\sigma_\lambda). \quad (6.24)$$

Thus from the definition of our graded character we derive that

$$p^{(1^t)}(q) = \sum_e q^\text{degree}_e \alpha_e a_{e,e}. \quad (6.25)$$

On the other hand, Proposition 6.3 and Eq. (6.24) give that for $|\lambda| = m$

$$\sigma_\lambda \alpha_e(A) \Delta(B) S_\lambda(B) \cong (1^\lambda) \sum_{e'} \alpha_{e'}(A) \Delta(B) S_\lambda(B) a_{e',e}(\sigma_\lambda) + E,$$

with $E \in \mathcal{H}_{>m}(R_1)$. But this implies that in $R_1$

$$\sigma_\lambda \alpha_e(A) \Delta(B) S_\lambda(B) |_{\alpha_e(A) \Delta(B) S_\lambda(B) = a_{e,e}(\sigma_\lambda)}. \quad (6.26)$$

Thus (6.25) gives

$$\Gamma_1^p p^{(1^r)}(q) = \sum_{\lambda \equiv k^t} \sum_e q^\text{degree}_e \Delta S_i q^\text{degree}_e \alpha_e a_{e,e}$$

$$= q^{\frac{n}{2}} \left( \sum_{\lambda \equiv k^t} q^{\text{degree}_\lambda}\right) p^{(1^t)}(q).$$

But this is (6.23), since it is well known and easy to show that

$$\sum_{\lambda \equiv k^t} q^{\text{degree}_\lambda} = \left[\begin{array}{c} n \\ k \end{array}\right].$$

Note that we have thus established the identity in (6.2) for $\mu = 1^r$. Indeed, in this case (6.2) reduces to

$$\Gamma_1^p p^{(1^r)}(q) = \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} q^{i_1 - 1 + i_2 - 1 + \cdots + i_k - 1} \right) p^{(1^t)}(q),$$

which is easily seen to be another way of writing (6.23).

Establishing (6.2) for arbitrary $\mu$ requires essentially the same steps, but
the calculations are a bit more intricate. The crucial step consists in the construction of the basis that plays the role of (6.14) for the general case.

For a composition \( p = (p_1, p_2, \ldots, p_k) \) and an alphabet \( B = \{ y_1, y_2, \ldots, y_k \} \) it is convenient to let

\[
A_p(B) = \det \| y_i^p \|.
\]

For a given set of indices

\[
I = \{ 1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu) \},
\]

(6.26)

let us set

\[
\mathcal{B}(\mu(I)) = \{ x^{(I)}_\epsilon \}.
\]

More precisely, \( x^{(I)}_\epsilon \) denotes the generic monomial in the basis \( \mathcal{B}(\mu(I)) \), where \( \mathcal{B}(\mu(I)) \) is the collection of monomials produced by applying the tree algorithm of Section 2 to the diagram of \( \mu(I) \).

This given, we have the following extension of Proposition 6.2.

**Proposition 6.4.** The collection of polynomials

\[
\{ x^{(I)}_\epsilon(A) \Delta_{i_1-1,i_2-1,\ldots,i_k-1}(B) \}
\]

(6.27)

with \( I = \{ 1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu) \} \) and \( x^{(I)}_\epsilon \in \mathcal{B}(\mu(I)) \) is a basis for \( \Gamma_{\gamma,k}R_\mu \).

**Proof.** Note that in view of (6.13) the cardinality of this collection is equal to the dimension of \( \Gamma_{\gamma,k}R_\mu \). Thus we need only show that it spans \( \Gamma_{\gamma,k}R_\mu \). The idea of the argument is best illustrated by an example. Let us take \( \mu = (1, 1, 2, 2) \), \( n = 6 \) and \( k = 2 \). We depicted in Fig. 7 the first two
layers of the tree which give the basis $\mathcal{B}(1122)$. From this picture we can easily derive that

$$\mathcal{B}(1122) = (1 + x_6) \mathcal{B}(1111) + (x_5 + x_5^3 + x_5^2 + x_5x_6 + x_5^2x_6 + x_5^3x_6$$

$$+ x_6^2 + x_5x_6^2 + x_5x_6^3) \mathcal{B}(112) + (x_5^2x_6^2 + x_5^3x_6^2) \mathcal{B}(22), \quad (6.28)$$

where we are using the same conventions as in Section 2. That is, addition here means disjoint union and a product such as $x_5^2x_6^3 \mathcal{B}(22)$ denotes the collection of all the monomials obtained by multiplying each monomial of $\mathcal{B}(22)$ by $x_5^2x_6^3$. From Proposition 6.1 we get that a spanning set for $\Gamma_1, R_\mu$ may be obtained by applying $N(B) = N(x_5, x_6)$ to the monomials in (6.28). However, this operation yields zero whenever $x_5$ and $x_6$ have equal exponents. The remaining terms produce the following collections of polynomials:

$$x_6 \mathcal{B}(1111) \rightarrow A_{01}(B) \mathcal{B}(1111)$$
$$x_5 \mathcal{B}(112) \rightarrow A_{10}(B) \mathcal{B}(112)$$
$$x_5^2 \mathcal{B}(112) \rightarrow A_{20}(B) \mathcal{B}(112)$$
$$x_5^3 \mathcal{B}(112) \rightarrow A_{30}(B) \mathcal{B}(112)$$
$$x_5^2x_6 \mathcal{B}(112) \rightarrow A_{21}(B) \mathcal{B}(112)$$
$$x_5x_6 \mathcal{B}(112) \rightarrow A_{31}(B) \mathcal{B}(112)$$
$$x_6^2 \mathcal{B}(112) \rightarrow A_{20}(B) \mathcal{B}(112)$$
$$x_5x_6^2 \mathcal{B}(112) \rightarrow A_{12}(B) \mathcal{B}(112)$$
$$x_6^3 \mathcal{B}(112) \rightarrow A_{03}(B) \mathcal{B}(112)$$
$$x_5x_6^3 \mathcal{B}(112) \rightarrow A_{13}(B) \mathcal{B}(112)$$
$$x_5^2x_6^2 \mathcal{B}(112) \rightarrow A_{23}(B) \mathcal{B}(22).$$

Note now that since $A_{ab} = -A_{ba}$ we still remain with a spanning set if we only use the productions

$$x_6 \mathcal{B}(1111) \rightarrow A_{01}(B) \mathcal{B}(1111)$$
$$x_5 \mathcal{B}(112) \rightarrow -A_{01}(B) \mathcal{B}(112)$$
$$x_5^2 \mathcal{B}(112) \rightarrow A_{02}(B) \mathcal{B}(112)$$
$$x_5x_6 \mathcal{B}(112) \rightarrow A_{12}(B) \mathcal{B}(112)$$
$$x_6^2 \mathcal{B}(112) \rightarrow A_{03}(B) \mathcal{B}(112)$$
$$x_5x_6^2 \mathcal{B}(112) \rightarrow A_{13}(B) \mathcal{B}(112)$$
$$x_5^2x_6^2 \mathcal{B}(112) \rightarrow A_{23}(B) \mathcal{B}(22).$$
Note further that the collection $\Delta_{0i}(B)A(112)$ may be omitted from the above list since from Proposition 4.1 we derive that $A(112) \subseteq A(1111)$. This last step leaves us with the spanning set

$$\{A_{i_1-1, i_2-1}(B)A(1122(i_1, i_2))\}_{1 \leq i_1 < i_2 \leq 4},$$

which is precisely what 6.27 gives in this case.

This simple example exhibits all the essential features of the proof. For a given $\mu$, we begin by constructing the first $k$ layers of the tree which gives the basis $A(\mu)$. By reading the labels of the edges of a path leading to a leaf of our partially constructed tree, we put together a monomial

$$x^{p_1-1}x^{p_2-1} \cdots x^{p_k-1}r.$$  (6.29)

It will be convenient to say that this leaf as well as the path that leads to it correspond to the sequence $p = (p_1, p_2, \ldots, p_k)$. Using the imagery suggested by the example, from this leaf hangs the diagram of a partition $\mu(p)$, which, together with the monomial in 6.29 yields the subcollection of the basis $A(\mu)$.

The next step is to apply the antisymmetrizer $N(B) = N(x_{h+1}, \ldots, x_n)$ ($h = n - k$) to each of these subcollections. There are then three types of leaves, (a), (b), or (c), according to the nature of the corresponding sequence $p$:

Type (a) $p$ has two equal components.

Type (b) The components of $p$ are strictly decreasing.

Type (c) The components of $p$ are distinct but not in decreasing order.

The leaves of type (a) produce no contribution to our spanning set since antisymmetrization of the corresponding monomial yields zero. It is also easy to see that the leaves of type (b) yield each and every one of the polynomials in (6.27). To complete the proof we are left to show that the application of $N(B)$ to the collections corresponding to leaves of type (c) produces polynomials which can also be obtained from leaves of type (b).

So let $\mu^{(p)}$ be the diagram hanging from a leaf of type (c) and let $I_p = (i_1 < i_2 < \cdots < i_k)$ be the increasing rearrangement of the components of $p$. We assert that the partition $\mu^{(b)}$ (in the notation of Section 5) is dominated by $\mu^{(p)}$. To better visualize why this is so, in Fig. 8 we give the diagrams of $\mu$, $\mu^{(p)}$, and $\mu^{(b)}$ for $\mu = (2^3 3^5 4^6)$, $p = (1, 3, 5, 7, 10, 14, 8, 4)$, and $I_p = (1, 3, 4, 5, 7, 8, 10, 14)$. In the diagram of $\mu$, given on the left, we have labelled by $i_1, i_2, \ldots, i_k$ respectively the squares at the end of rows.
To obtain the partition $\mu^{(p)}$, (as was indicated in section 5) we remove the labelled squares then rearrange the resulting rows. This, in our case, results in the diagram on the right. Note that the same diagram is also obtained from the tree algorithm by following the path which corresponds to the sequence

$$(i_k, i_{k-1}, \ldots, i_1).$$

Indeed, according to our tree algorithm, if we follow the path which corresponds to a sequence $q = (q_1, q_2, \ldots, q_k)$ we must hit the diagram of $\mu$ at height $q_1$, climb up its vertical face to the nearest corner square, and then remove it. Next we hit the resulting diagram at height $q_2$ climb up its vertical face to the nearest corner square, and then remove it, etc. Let us imagine that while we do this we label by $q_i$ the $i$th square that is being removed. If we do this in our example for $q = (1, 3, 5, 7, 10, 14, 8, 4)$ we obtain the middle diagram given above. Similarly, if we do this with $q = (14, 10, 8, 7, 5, 4, 3, 1)$ we (trivially) must obtain again the diagram on the right. The reason $\mu^{(p)}$ must dominate $\mu^{(p)}$ should now appear quite clear. In this construction of $\mu^{(p)}$ the labelled squares in the diagram on the left climb up their columns to their resting place in the diagram on the right. On the other hand, at the $i$th step in the construction of $\mu^{(p)}$, the square which bears the label $p_i$ is either on the same column as can be found in the diagram on the left or, in the worst case, in a column west of it. The reason for this is simple. The removal of squares due to the previous $i-1$ steps could have so depleted the column which originally contained the label $p_i$ that there no longer is a square at level $p_i$ in that column. Thus
to reach the boundary of this new diagram, at that level, we are forced to go further west. This ultimately results in the label \( p_i \) landing westward of its position in the diagram of \( \mu \).

It follows then that, on and to the right of any column, the diagram of \( \mu^{(p)} \) contains as many squares as the diagram of \( \mu^{(I_p)} \). This proves that \( \mu^{(p)} \) dominates \( \mu^{(I_p)} \).

We can then apply Proposition 4.1 and conclude that \( \mathcal{R}(\mu^{(p)}) \subseteq \mathcal{R}(\mu^{(I_p)}) \).

On the other hand, since we also have

\[
\Delta_{i_1, i_2, \ldots, i_k - 1}(B) = \pm \Delta_{p_1 - 1, p_2 - 1, \ldots, p_k - 1}(B)
\]

with the sign depending on the permutation which rearranges \( p \) to \( I_p \), we see that the application of the antisymmetrizer \( N(B) \) to the subcollection

\[
x_n^{p_1 - 1} x_n^{p_2 - 1} \ldots x_n^{p_k - 1} \mathcal{R}(\mu^{(p)})
\]

yields only polynomials than can also be obtained from

\[
x_n^{i_1 - 1} x_n^{i_2 - 1} \ldots x_n^{i_k - 1} \mathcal{R}(\mu^{(I_p)}).
\]

This completes the proof of our proposition.

Remark 6.2. It is well known that for an alphabet \( B = (y_1, y_2, \ldots, y_k) \) and a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) we have

\[
\Delta_{\lambda_1, \lambda_2 + 1, \ldots, \lambda_k + k - 1}(B) = \Delta(B) S_{\lambda}(B).
\]

Note further that there is a bijection between the partitions \( \lambda \) whose diagram is contained in the \( k \times (h(\mu) - k) \) rectangle and the sets of indices \( I = \{ 1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu) \} \). We can go from one set to the other by the map

\[
\lambda \rightarrow I_\lambda = (\lambda_1 + 1, \lambda_2 + 2, \ldots, \lambda_k + k).
\]

Thus the basis in (6.27) may also be described as

\[
\{ x^{(I_\lambda)}(A) \Delta(B) S_{\lambda}(B) \}.
\]

We see then that Proposition 6.4 does indeed specialize to Proposition 6.2 when \( \mu = 1^n \).

Our next goal is the extension of Proposition 6.3 to the general case. To this end we need an auxiliary polynomial identity and some notation.

Given an alphabet \( Z = \{ z_1, z_2, \ldots, z_v \} \), a composition \( p = (p_1, p_2, \ldots, p_v) \) and a subset \( U = \{ j_1 < j_2 < \cdots < j_u \} \subseteq [1, v] \), by

\[
0 \leq q_U < p_U
\]
we mean that the composition \( q_U = (q_{j_1}, q_{j_2}, \ldots, q_{j_u}) \) satisfies the inequalities

\[
0 \leq q_{j_1} < p_{j_1}; \ 0 \leq q_{j_2} < p_{j_2}; \ldots; \ 0 \leq q_{j_u} < p_{j_u}.
\]

Let us also set

\[
z_{j_1}^{q_{j_1}} z_{j_2}^{q_{j_2}} \cdots z_{j_u}^{q_{j_u}} \quad \text{and} \quad |q_U| = q_{j_1} + q_{j_2} + \cdots + q_{j_u}.
\]

This given,

**Lemma 6.1.** For arbitrary alphabets \( C, Z = \{z_1, z_2, \ldots, z_v\} \), exponents \( p = (p_1, p_2, \ldots, p_u) \), and \( r \geq 1 \) we have

\[
(-z_1)^{p_1} (-z_2)^{p_2} \cdots (-z_v)^{p_v} e_r(C)
= \sum_{U \subseteq [1,v]} (-1)^{|U|} \sum_{0 \leq q_U < p_U} (-1)^{|q_U|} z_{j_1}^{|q_U|} z_{j_2}^{|q_U|} \cdots z_{j_u}^{|q_U|} e_{r+|p|-|q_U|}(C+Z_U), \tag{6.30}
\]

where, for \( U = \{j_1, j_2, \ldots, j_u\} \), \( Z_U \) denotes the alphabet \( \{z_{j_1}, z_{j_2}, \ldots, z_{j_u}\} \).

**Proof.** It is convenient to set, for a given alphabet \( D \) and variable \( t \),

\[
\alpha_i = 1 - (-t z_i)^{p_i} \quad \text{and} \quad \bar{Q}_i(D) = \prod_{d \in D} (1 + td).
\]

We can then rewrite the left-hand side of (6.30) in the form

\[
(1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_v) \bar{Q}_i(C)|_{r+|p|+|q_U|}.
\]

We then deduce that

\[
\text{LHS} = \sum_{U \subseteq [1,v]} (-1)^{|U|} \prod_{j \in U} (1 - (-t z_j)^{p_j}) \bar{Q}_i(C)|_{r+|p|}
= \sum_{U \subseteq [1,v]} (-1)^{|U|} \prod_{j \in U} \frac{1 - (-t z_j)^{p_j}}{1 + t z_j} \bar{Q}_i(C+Z_U)|_{r+|p|},
= \sum_{U \subseteq [1,v]} (-1)^{|U|} \sum_{0 \leq q_U < p_U} (-1)^{|q_U|} z_{j_1}^{|q_U|} z_{j_2}^{|q_U|} \cdots z_{j_u}^{|q_U|} \bar{Q}_i(C+Z_U)|_{r+|p|-|q_U|},
\]

and we see that this last expression is equal to the right-hand side of (6.30).

In analogy with what we did in the case \( \mu = 1^n \), for a given partition \( \mu \) and index set \( I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)\} \) let us set

\[
\mathcal{H}_i^\mu(B) = \mathcal{L}[\alpha_z^{(I)}(A) \mathcal{A}_{i_1 - 1, i_2 - 1, \ldots, i_k - 1}(B); \alpha_z^{(I)} \in \mathcal{B}(\mu^{(I)})].
\]
Moreover, for a given degree \( d \) let

\[
\mathcal{H}^B_{>d}(R_\mu) = \bigoplus_{i_1 - 1 + i_2 - 1 + \cdots + i_k - 1 > d} \mathcal{H}^B_{i}(R_\mu).
\]

We now have the following extension of Proposition 6.3.

**Proposition 6.5.** Let \( I = \{ 1 < i_1 < i_2 < \cdots < i_k \leq h(\mu) \} \) with \( i_1 - 1 + i_2 - 1 + \cdots + i_k - 1 = d \) and let \( Q(A) \) be a polynomial which, as an element of \( R_\mu^{(n)} \), has the expansion

\[
Q(A) \approx \sum_{\alpha^{(o)} \in \mathcal{A}(\mu^{(n)})} c_\alpha \alpha^{(I)}(A).
\] (6.31)

Then in \( R_\mu \) we have

\[
Q(A) A_{i_1 - 1, i_2 - 1, \ldots, i_k - 1}(B) \\
\approx \sum_{\alpha^{(o)} \in \mathcal{A}(\mu^{(n)})} c_\alpha \alpha^{(I)}(A) A_{i_1 - 1, i_2 - 1, \ldots, i_k - 1}(B) + E,
\] (6.32)

with

\[
E \in \mathcal{H}^B_{>d}(R_\mu).
\] (6.33)

**Proof.** Equation (6.31) means that we have the equality

\[
Q(A) - \sum_{\alpha^{(o)} \in \mathcal{A}(\mu^{(n)})} c_\alpha \alpha^{(I)}(A) = \sum_{e_r(S) \in G_{\mu^{(n)}}} P_{r,S}(A) e_r(S),
\]

for some suitable polynomials \( P_{r,S}(A) \). Thus, to prove (6.33), in view of the definition of the class \( G_{\mu^{(n)}} \), we need only show that if \( P(A) \) is an arbitrary polynomial, \( S \) is an arbitrary subset of \( A \), and

\[
r > |S| - d_{|S|}(\mu^{(n)}),
\] (6.34)

then

\[
P(A) e_r(S) A_{i_1 - 1, i_2 - 1, \ldots, i_k - 1}(B) \in \mathcal{H}^B_{>d}(R_\mu).
\] (6.35)

It is convenient here and after to let \( s \) denote the cardinality of \( S \). Note that according to our conventions, the partition \( \mu' \) conjugate of \( \mu \) has parts

\[
\mu'_1 \leq \mu'_2 \leq \cdots \leq \mu'_n,
\]

with \( \mu'_i \) giving the length of column \( n + 1 - i \) of the diagram of \( \mu \). In order to avoid convoluted expressions, we refer to column \( n + 1 - i \) of the diagram of \( \mu \) simply as bar\([i]\). It also streamlines our language if, here and
after, we stop repeating the expression *diagram of* \( \mu \) and simply identify \( \mu \) with its own diagram. Now, it is crucial in our dealing with (6.35) that we have a precise understanding of the position of \( \text{bar}[k+s] \) among the squares we remove from \( \mu \) to get \( \mu^{(i)} \). To this end note that, if we are able to remove \( k \) distinct squares from the boundary of \( \mu \), then \( \mu \) must have at least \( k \) distinct rows; this implies that

\[
\mu'_1 = \mu'_2 = \cdots = \mu'_k = 0.
\]

In other words the first \( k \) bars of \( \mu \) are necessarily empty. Let us keep in mind some of the imagery used in the previous proof and label by \( i_1, i_2, \ldots, i_k \) respectively the squares at the end of rows \( i_1, i_2, \ldots, i_k \). From our construction of \( \mu^{(i)} \) we see then that we must necessarily have

\[
d_j(\mu(I)) = \mu'_{k+1} + \mu'_{k+2} + \cdots + \mu'_{k+j} - m_j = d_{k+j}(\mu) - m_j,
\]

where \( m_j \) denotes the number of labelled squares on or to the right of \( \text{bar}[k+j] \). Here and in the rest of this proof, the number of labelled squares weakly to the right of \( \text{bar}[k+s] \) is briefly denoted by \( m \). With this notation, taking account of (6.36), condition (6.34) may be rewritten as

\[
r > s + m - d_{s+k}(\mu).
\]

Our first step in the proof of (6.35) is to show that under this condition we necessarily have

\[
x_{h+1}^{i_1} x_{h+2}^{i_2} \cdots x_{h+k}^{i_k} e_r(S + x_{h+1} + x_{h+2} + \cdots + x_{h+m}) \equiv \mu 0.
\]

Of course, when \( m = 0 \) we intend the sum \( x_{h+1} + x_{h+2} + \cdots + x_{h+m} \) to be interpreted as zero. Note also that the case \( m = k \) is immediate. Indeed, then (6.37) reduces to

\[
r > s + k - d_{s+k}(\mu),
\]

and since the alphabet \( S + x_{h+1} + x_{h+2} + \cdots + x_{h+k} \) has exactly \( s + k \) letters we see that we already have

\[
e_r(S + x_{h+1} + x_{h+2} + \cdots + x_{h+k}) \in \mathcal{C}_\mu.
\]

This shows that, in this case, (6.38) holds true even without the factor

\[
x_{h+1}^{i_1} x_{h+2}^{i_2} \cdots x_{h+k}^{i_k}.
\]

In fact, for any \( m > 0 \), we have a stronger relation than (6.38). This further relation is best stated under a slight change of notation.

We recall that, from the very beginning of this section, we have let
$A = \{x_1, x_2, \ldots, x_h\}$ and $B = \{x_{h+1}, x_{h+2}, \ldots, x_{h+k}\}$ (with $h + k = n$). Here, in order not to get overburdened with subscripts, we simply set $x_{h+i} = y_i$ and

$$B = \{y_1, y_2, \ldots, y_k\}.$$ 

Now it develops that (6.37) implies

$$y_{m+1}^{i+1} \cdot y_{m+2}^{i+2} \cdots y_k^{i+k} e_r(S + y_1 + y_2 + \cdots + y_m) \equiv 0. \quad (6.39)$$

Since we only need to consider the case $m < k$ we may assume that there are labelled squares strictly to the left of bar[$s + k$]. It is crucial then to note that the lowest of these squares, which is the one with label $i_{m+1}$, is also at a level strictly higher than bar[$s + k$]. In other words, we must have

$$\mu'_{s+k} < i_{m+1}. \quad (6.40)$$

But now we are almost finished with our argument. All we need to do is apply the identity (6.30) with

$$C = \{S + y_1 + y_2 + \cdots + y_m\} = \{S + Y_m\}$$

$$Z = \{y_{m+1}, y_{m+2}, \ldots, y_k\}$$

$$p = (i_{m+1} - 1, i_{m+2} - 1, \ldots, i_k - 1)$$

$$v = k - m.$$ 

We then get

$$\pm y_{m+1}^{i_{m+1}} \cdots y_k^{i_k} e_r(S + y_1 + y_2 + \cdots + y_m)$$

$$= \sum_{U \subseteq [m+1, k]} (-1)^{|U|} \sum_{0 \leq q_U \leq |U| - 1} (-1)^{|q_U|}$$

$$\times y_{m+1}^{q_U} e_r + y_{m+2}^{i_{m+2} - 1} + \cdots + y_k^{i_k - 1} - |q_U| (S + Y_m + Y_U),$$

where for

$$U = \{m + 1 \leq j_1 < j_2 < \cdots < j_u \leq k\} \quad (6.41)$$

we set

$$I_U - 1 = (j_1 - 1, j_2 - 1, \ldots, j_u - 1) \quad \text{and} \quad Y_U = \{y_{j_1}, y_{j_2}, \ldots, y_{j_u}\}. \quad (6.42)$$

Thus (6.39) follows if we can show that (6.37), (6.40), (6.41), (6.42), and

$$q_U < I_U - 1 \quad (6.43)$$
imply that
\[ e_r + i_{m+1} - 1 + \cdots + i_k - 1 - |q_U| (S + Y_m + Y_U) \in \mathbb{C}_\mu. \] (6.44)

To this end, note that, since the alphabet \( S + Y_m + Y_U \) has a total of \( s + m + u \) letters, to establish (6.44) we need only show that the subscript of the elementary symmetric function in (6.44) is sufficiently large. To be precise, we are left to prove that
\[ r + i_{m+1} - 1 + \cdots + i_k - 1 - |q_U| > s + m + u - d_{s+m+u}(\mu). \] (6.45)

Clearly, we can rewrite this subscript in the form
\[ r + \sum_{j \in U} (i_j - 1 - q_j) + \sum_{j \in [m+1, k] - U} (i_j - 1). \]

Now for the first sum, because of (6.43), we must have
\[ \sum_{j \in U} (i_j - 1 - q_j) \geq u. \]

Moreover, since the second sum involves labels of \( k - m - u \) squares all of which are strictly to the left and thus strictly higher than bar\([s+k]\), we may in any case estimate it by the sum of the lengths of the first \( k - m - u \) bars weakly to the right of bar\([s+k]\). That gives
\[ \sum_{j \in [m+1, k] - U} (i_j - 1) \geq \sum_{j = s + m + u + 1}^{s+k} \mu_j. \]

Combining these last two estimates with (6.37) we finally get the inequality
\[ r + i_{m+1} - 1 + \cdots + i_k - 1 - |q_U| > s + m - d_{s+k}(\mu) + u + \sum_{j = s + m + u + 1}^{s+k} \mu_j, \]

which is another way of writing (6.45).

To complete the proof of Proposition 6.5 we are left to show that (6.38) implies (6.35). But this is easy now. Indeed, the addition formula gives
\[ e_r(S) = e_r(S + x_{h+1} + x_{h+2} + \cdots + x_{h+m}) \]
\[ - \sum_{j=1}^{r} e_{r-j}(S) e_j(x_{h+1} + x_{h+2} + \cdots + x_{h+m}). \] (6.46)

Note that we may write
\[ e_j(x_{h+1} + x_{h+2} + \cdots + x_{h+m}) = \sum x_{h+1}^{e_1} x_{h+2}^{e_2} \cdots x_{h+m}^{e_m}, \]
the sum running over all indicators of subsets of \([1, m]\) of cardinality \(j\).
Thus substituting (6.38) in (6.46) we see that the expression
\[
x_h^{i_1} \cdots x_{h+k}^{i_k} e_r(S)
\]
is congruent modulo \(I_\mu\) to a sum of terms of the form
\[
e_{r_j}(S) x_h^{i_1} \cdots x_{h+m}^{i_m} x_{h+m+1}^{i_m+1} \cdots x_{h+k}^{i_k}.
\]
This in turn, upon application of the antisymmetrizer \(N[B]\), yields that the expression
\[
\Delta_{i_1-1, i_2-1, \ldots, i_k-1}(B) e_r(S)
\]
is also congruent modulo \(I_\mu\) to a sum of terms of the form
\[
e_{r_j}(S) \Delta_{i_1-1 + \varepsilon_1, \ldots, i_m-1 + \varepsilon_m, i_{m+1}-1, \ldots, i_k-1}(B),
\]
each of which, since \(j \geq 1\), must necessarily belong to
\[
H > d(R_\mu).
\]
From this fact it is easy to derive (6.35) and complete the proof by reverse induction on the size of \(d\).

We are finally in a position to establish the recurrence (6.2) in full generality. This last step is now completely routine, since we may follow almost verbatim the proof of Theorem 6.2.

**Theorem 6.3.** The graded character of the \(S_{[A]}\)-module \(\Gamma_1 R_\mu\) decomposes into the sum
\[
\Gamma_1 p^\mu(q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)} q^{i_1-1 + i_2-1 + \cdots + i_k-1} p^\mu(q). \quad (6.47)
\]

**Proof.** For a given \(I=\{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu)\}\) we express the action of \(S_{[A]}\) on \(R_\mu(q)\) in the form
\[
\sigma_A \varepsilon^{(l)}(A) \cong \mu^{(l)} \sum_{\varepsilon^{(l)}\in \mathcal{A}(\mu^{(l)})} \varepsilon^{(l)}(A) a^{(l)}_{\varepsilon, \varepsilon}(\sigma_A).
\]
This gives that
\[
p^{\mu(l)} = \sum_{\varepsilon^{(l)}\in \mathcal{A}(\mu^{(l)})} q^{\text{degree } \varepsilon^{(l)}} a^{(l)}_{\varepsilon, \varepsilon}. \quad (6.49)
\]
From Proposition 6.5 and Eq. (6.48) we then get that

\[ \sigma _{\alpha } \alpha _{e}^{(l)}(A) A_{i_1-1, \ldots , i_k-1}(B) \]

\[ \cong \sum_{\alpha _{e}^{(l)} \in \mathcal{H}(\mu )} \alpha _{e}^{(l)}(A) A_{i_1-1, \ldots , i_k-1}(B) a_{e, e}^{(l)}(\sigma _{\alpha }) + E, \quad (6.50) \]

with

\[ E \in \mathcal{H}_{> d}(R_{\mu }). \]

Thus in \( R_{\mu } \) we have

\[ \sigma _{\alpha } \alpha _{e}^{(l)}(A) A_{i_1-1, \ldots , i_k-1}(B) |_{\sigma _{\mu }^{(l)}(A) A_{i_1-1, \ldots , i_k-1}(R)} = a_{e, e}^{(l)}(\sigma _{\alpha }). \quad (6.51) \]

Now since

\[ \text{degree } A_{i_1-1, \ldots , i_k-1}(B) = i_1 - 1 + \cdots + i_k - 1, \]

Proposition 6.4 and Eqs. (6.50) and (6.51) yield that

\[ \Gamma _{\mu } p^\mu (q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq h(\mu )} q^{i_1 - 1 + i_2 - 1 + \cdots + i_k - 1} \sum_{\alpha _{e}^{(l)} \in \mathcal{H}(\mu )} q^{\text{degree } \alpha _{e}^{(l)}(A)} a_{e, e}^{(l)}. \]

Combining this with (6.49) the recursion in (6.47) follows as desired.

This completes our identification of the graded character \( p^\mu (q) \).

\[ \text{REFERENCES} \]


11. A. Lascaux, Cyclic permutations on words, Tableaux and Harmonic polynomials, to appear.


15. A. Lascaux and M. P. Schützenberger, Cyclic permutations, Tableaux and Harmonic Polynomials (preliminary version of Ref. [11]).


