

Fourier Tauberian Theorems and Applications

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The main aim of the paper is to present a general version of the Fourier Tauberian theorem for monotone functions. This result, together with Berezin's inequality, allows us to obtain a refined version of the Li–Yau estimate for the counting function of the Dirichlet Laplacian which implies the Li–Yau estimate itself and, at the same time, the asymptotic results obtained by the variational



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INTRODUCTION

Let F be a non-decreasing function and ρ is an appropriate test function on the real line \mathbb{R} . Then, under certain conditions on the Fourier transform of the convolution $\rho * F$, one can estimate the difference $F - \rho * F$. Results of this type are called Fourier Tauberian theorems.

The Fourier Tauberian theorems have been used by many authors for the study of spectral asymptotics of elliptic differential operators, with F being either the counting function or the spectral function (see, for example, [L], [H1], [H2], [DG], [I1], [I2], [S], [SV]). The required estimate for $F - \rho * F$ was obtained under the assumption that the derivative $\rho * F'$ admits a sufficiently good estimate.

In applications F often depends on additional parameters and we are interested in estimates which are uniform with respect to these parameters. Then one has to assume that the estimate for $\rho * F'$ holds uniformly and to take this into account when estimating $F - \rho * F$. As a result, there have been produced a number of Fourier Tauberian theorems designed for the study of various parameter dependent problems. This has been done, in particular, for semi-classical asymptotics (see, for example, [PP]). Note that all the authors used the same idea of proof which goes back to the papers [L] and [H1].



The main aim of this paper is to present a general version of Fourier Tauberian theorem which does not require any a priori estimates of $\rho * F'$. Our estimates contain only convolutions of F and test functions (see Section 1). This enables one to obtain results which are uniform with respect to any parameters involved.

Our proof is very different from the usual one. It leads to more general results and, at the same time, allows one to evaluate constants appearing in the estimates (Section 2). Therefore our Tauberian theorems can be used not only for the study of asymptotics but also for obtaining explicit estimates of the spectral and counting functions.

In particular, in Section 3, applying our Tauberian theorems and Berezin's inequality, we prove a refined version of the Li–Yau estimate for the counting function of the Dirichlet Laplacian in an arbitrary domain of finite volume. Our inequality implies the Li–Yau estimate itself and, along with that, the results on the asymptotic behaviour of the counting function which are obtained by the variational method. Note that the proof of the Berezin inequality does not use variational techniques. This implies that, even in the non-smooth case, the classical asymptotic formulae can be proved without referring to the Whitney decompositions and Dirichlet–Neumann bracketing.

Throughout the paper χ_+ , χ_- denote the characteristic functions of the positive and negative semi-axes, $\hat{f}(t) := (2\pi)^{-1/2} \int e^{-it\tau} f(\tau) d\tau$ is the Fourier transform of f , and $\langle \tau \rangle := \sqrt{1 + \tau^2}$.

1. TAUBERIAN THEOREMS I: BASIC ESTIMATES

Let F be a non-decreasing function on \mathbb{R} . For the sake of definiteness, we shall always be assuming that

$$(1.1) \quad F(\tau) = \frac{1}{2} [F(\tau+0) + F(\tau-0)], \quad \forall \tau \in \mathbb{R}.$$

1.1. *Auxiliary functions.* We shall deal with continuous functions ρ on \mathbb{R} satisfying the following conditions:

- (1_m) $|\rho(\tau)| \leq \text{const} \langle \tau \rangle^{-2m-2}$, where $m > -\frac{1}{2}$;
- (2) $c_{\rho,0} := \int \rho(\tau) d\tau = 1$;
- (3) ρ is even;
- (4) $\rho \geq 0$;
- (5) $\text{supp } \hat{\rho} \subset [-1, 1]$.

For every m the functions ρ satisfying (1_m) –(5) do exist (see, for example, [H2], Section 17.5, or Example 1.1 below).

EXAMPLE 1.1. Let l be a positive integer and

$$(1.2) \quad \gamma(\tau) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\tau}{2l} + s \right)^{-2l} \sin^{2l} \left(\frac{\tau}{2l} + s \right) ds.$$

The function γ satisfies (3), (5), and

$$(1.3) \quad c_\gamma^- \langle \tau \rangle^{-2l} \leq \gamma(\tau) \leq c_\gamma^+ \langle \tau \rangle^{-2l}$$

with some positive constants c_γ^\pm . Indeed, (3) and (1.3) are obvious, and (5) follows from the fact that $-2(2\pi)^{-1/2} \tau^{-1} \sin \tau$ is the Fourier transform of the characteristic function of the interval $[-1, 1]$. If $\rho(\tau) := c_{\gamma,0}^{-1} \gamma(\tau)$ then the conditions (2)–(5) are fulfilled and (1_m) holds with $m = l - 1$.

We shall always be assuming (1_m) . Let

$$\rho_{1,1}(\tau) := \begin{cases} \int_\tau^\infty \rho(\mu) d\mu, & \tau > 0, \\ 0, & \tau = 0, \\ -\int_{-\infty}^\tau \rho(\mu) d\mu, & \tau < 0, \end{cases}$$

and, if (1_m) holds with $m > 0$,

$$\rho_{1,0}(\tau) := \int_\tau^\infty \mu \rho(\mu) d\mu, \quad \rho_{1,2}(\tau) := \begin{cases} \int_\tau^\infty \int_\mu^\infty \rho(\lambda) d\lambda d\mu, & \tau \geq 0, \\ \int_{-\infty}^\tau \int_{-\infty}^\mu \rho(\lambda) d\lambda d\mu, & \tau \leq 0. \end{cases}$$

One can easily see that

$$\begin{aligned} \rho_{1,0}(\tau) &\leq \text{const} \langle \tau \rangle^{-2m}, & \rho_{1,1}(\tau) &\leq \text{const} \langle \tau \rangle^{-2m-1}, \\ \rho_{1,2}(\tau) &\leq \text{const} \langle \tau \rangle^{-2m} \end{aligned}$$

for all $\tau \geq 0$. Integrating by parts, we obtain

$$(1.4) \quad \rho_{1,0}(\tau) = -\int_\tau^\infty \mu \rho'_{1,1}(\mu) d\mu = \rho_{1,2}(\tau) + \tau \rho_{1,1}(\tau), \quad \forall \tau \geq 0.$$

Denote

$$c_{\rho, \kappa} := \int |\mu|^\kappa \rho(\mu) \, d\mu, \quad \forall \kappa \in (-1, 2m+1).$$

Under condition (2), by Jensen's inequality, we have

$$(1.5) \quad c_{\rho, r}^\kappa \leq c_{\rho, \kappa}^r, \quad \forall \kappa \geq r \geq 0.$$

If the condition (3) is fulfilled then $\rho_{1,0}$ and $\rho_{1,2}$ are even continuous functions, $\rho_{1,1}$ is an odd function continuous outside the origin and

$$(1.6) \quad \rho_{1,1}(\pm 0) = \pm \frac{1}{2} c_{\rho, 0}, \quad \rho_{1,0}(0) = \rho_{1,2}(0) = \frac{1}{2} c_{\rho, 1}.$$

Indeed, the first two equalities in (1.6) are obvious, and the last follows from (1.4).

The condition (4) and (1.4) imply that

$$(1.7) \quad \begin{aligned} 0 \leq \rho_{1,2}(\tau) \leq \rho_{1,0}(\tau), & \quad \forall \tau \geq 0, \\ 0 \leq \rho_{1,k}(\mu) \leq \rho_{1,k}(\tau), & \quad k = 0, 1, 2, \quad \forall \mu \geq \tau \geq 0. \end{aligned}$$

Let

$$(1.8) \quad \rho_\delta(\tau) := \delta \rho(\delta\tau), \quad \rho_{\delta,k}(\tau) := \delta^{1-k} \rho_{1,k}(\delta\tau), \quad k = 0, 1, 2,$$

where δ is an arbitrary positive number. If (5) is fulfilled then

$$(1.9) \quad \text{supp } \hat{\rho}_{\delta,0} \subset \text{supp } \hat{\rho}_\delta \subset [-\delta, \delta].$$

Indeed, these inclusions follow from (1.8) and the fact that $\rho_{1,0}$ is the convolution of the functions $\mu\rho(\mu)$ and $\chi_-(\mu)$.

1.2. Main estimates. If f is a piecewise continuous function on \mathbb{R}^1 , we denote

$$\begin{aligned} f * F(\tau) &:= \lim_{R \rightarrow \infty} \int_{-R}^R f(\tau - \mu) F(\mu) \, d\mu, \\ f * F'(\tau) &:= \lim_{R \rightarrow \infty} \int_{(-R, R)} f(\tau - \mu) \, dF(\mu), \end{aligned}$$

whenever the limits exist. We shall deduce the estimates for $F(\tau)$ from the following simple lemma.

LEMMA 1.2. *Let ρ satisfy the conditions (1_m)–(3) and $\rho_{T,1}(\tau - s) F(s) \rightarrow 0$ as $s \rightarrow \pm\infty$ for some $T > 0$ and $\tau \in \mathbb{R}$. Then $\rho_{T,1} * F'(\tau)$ is well defined if and only if $\rho_T * F(\tau)$ is well defined, and*

$$(1.10) \quad F(\tau) - \rho_T * F(\tau) = \rho_{T,1} * F'(\tau).$$

Proof. Integrating by parts, we obtain

$$\begin{aligned} \int_{(-R, R)} \rho_{T,1}(\tau - \mu) dF(\mu) &= \int_{(-R, \tau)} \rho_{T,1}(\tau - \mu) dF(\mu) + \int_{(\tau, R)} \rho_{T,1}(\tau - \mu) dF(\mu) \\ &= - \int_{-R}^R \rho_T(\tau - \mu) F(\mu) d\mu + \rho_{T,1}(+0) F(\tau - 0) \\ &\quad - \rho_{T,1}(-0) F(\tau + 0) - \rho_{T,1}(\tau + R) F(-R + 0) \\ &\quad + \rho_{T,1}(\tau - R) F(R - 0). \end{aligned}$$

In view of (1.1), (1.6) and (2), we have

$$\rho_{T,1}(+0) F(\tau - 0) - \rho_{T,1}(-0) F(\tau + 0) = F(\tau).$$

Now the lemma is proved by passing to the limit as $R \rightarrow \infty$. ■

THEOREM 1.3. *Let ρ satisfy the conditions (1_m)–(4) with $m > 0$. Assume that $\rho_{\delta,0}(\tau - s) F(s) \rightarrow 0$ as $s \rightarrow \pm \infty$ and $\rho_{\delta,0} * F'(\tau) < \infty$ for some $\delta > 0$ and $\tau \in \mathbb{R}$. Then $\rho_T * F(\tau) < \infty$ and*

$$(1.11) \quad |F(\tau) - \rho_T * F(\tau)| \leq c_{\rho,1}^{-1} \delta^{-1} \rho_{\delta,0} * F'(\tau)$$

for all $T \geq \delta$.

Proof. The identity (1.4) and (4) imply that

$$\frac{d}{d\tau} \left(\frac{\rho_{1,1}(\tau)}{\rho_{1,0}(\tau)} \right) = \frac{\rho(\tau)(\tau \rho_{1,1}(\tau) - \rho_{1,0}(\tau))}{(\rho_{1,0}(\tau))^2} \leq 0, \quad \forall \tau > 0.$$

Therefore, in view of (2) and (1.6),

$$\frac{|\rho_{1,1}(\tau)|}{\rho_{1,0}(\tau)} \leq \frac{|\rho_{1,1}(+0)|}{\rho_{1,0}(0)} = c_{\rho,1}^{-1}, \quad \forall \tau > 0.$$

Taking into account (3), (1.8) and the second inequality (1.7), we obtain

$$(1.12) \quad |\rho_{T,1}(\tau)| \leq \frac{\rho_{T,0}(\tau)}{c_{\rho,1} T} \leq \frac{\rho_{\delta,0}(\tau)}{c_{\rho,1} \delta}, \quad \forall T \geq \delta > 0, \quad \forall \tau \in \mathbb{R}.$$

The inequality (1.12) implies that ρ and F satisfy the conditions of Lemma 1.2 and that $\rho_T * F(\tau) < \infty$. Obviously, (1.11) follows from (1.10) and (1.12). ■

Remark 1.4. If $T = \delta$ then the estimate (1.11) can be rewritten in the form

$$(1.13) \quad \rho_{\delta}^{+} * F(\tau) \leq F(\tau) \leq \rho_{\delta}^{-} * F(\tau),$$

where $\rho_{\delta}^{\pm}(\tau) := \rho_{\delta}(\tau) \pm c_{\rho,1}^{-1} \delta \tau \rho_{\delta}(\tau)$.

Remark 1.5. The inequality (1.11) is not precise in the sense that, apart from some degenerate situations, it never turns into an equality. The crucial point in our proof is the estimate $|\rho_{T,1}| \leq c_{\rho,1}^{-1} \delta^{-1} \rho_{\delta,0}$ which implies that $|\rho_{T,1} * F'| \leq c_{\rho,1}^{-1} \delta^{-1} \rho_{\delta,0} * F'(\tau)$. However, the function $\rho_{T,1}$ is negative on one half-line and positive on another, so $|\rho_{T,1} * F'|$ may well admit much a better estimate. Using this observation, one can try to improve our results under additional conditions on the function F .

THEOREM 1.6. *Let $[a, b]$ be a bounded interval. Assume that the conditions of Theorem 1.3 are fulfilled for every $\tau \in [a, b]$ and that $\rho_{\delta,0} * F'(\tau)$ is uniformly bounded on $[a, b]$. Then*

$$(1.14) \quad -T^{-1} \delta^{-1} f(b) \rho_{\delta,0} * F'(b) \\ \leq \int_a^b f(\tau) [F(\tau) - \rho_T * F(\tau)] d\tau \\ \leq T^{-1} \delta^{-1} f(a) \rho_{\delta,0} * F'(a) + T^{-1} \delta^{-1} \int_a^b f'(\tau) \rho_{\delta,0} * F'(\tau) d\tau$$

for every non-negative non-decreasing function $f \in C^1[a, b]$ and all $T \geq \delta$.

Proof. In view of (1.7) and (1.8) we have

$$(1.15) \quad T \rho_{T,2}(\tau) \leq T^{-1} \rho_{T,0}(\tau) \leq \delta^{-1} \rho_{\delta,0}(\tau), \quad \forall T \geq \delta > 0, \quad \forall \tau \in \mathbb{R}.$$

This estimates, (1.12) and Lemma 1.2 imply that the functions $\rho_{T,2} * F'(\tau)$, $|\rho_{T,1}| * F'(\tau)$ and $\rho_T * F(\tau)$ are uniformly bounded on $[a, b]$. Since $\rho'_{T,2}(s) = -\rho_{T,1}(s)$ whenever $s \neq 0$, integrating by parts with respect to τ we obtain

$$\int_a^b f(\tau) \int \rho_{T,1}(\tau - \mu) dF(\mu) d\tau \\ = f(a) \int \rho_{T,2}(a - \mu) dF(\mu) - f(b) \int \rho_{T,2}(b - \mu) dF(\mu) \\ + \int_a^b f'(\tau) \left(\int \rho_{T,2}(\tau - \mu) dF(\mu) \right) d\tau.$$

Now (1.14) follows from Lemma 1.2 and (1.15). \blacksquare

If $f \equiv 1$ then (1.14) turns into

$$(1.16) \quad -\rho_{\delta,0} * F'(b) \leq T\delta \int_a^b [F(\mu) - \rho_T * F(\mu)] d\mu \leq \rho_{\delta,0} * F'(a).$$

This estimate and the obvious inequalities

$$(1.17) \quad \varepsilon^{-1} \int_{\tau-\varepsilon}^{\tau} F(\mu) d\mu \leq F(\tau) \leq \varepsilon^{-1} \int_{\tau}^{\tau+\varepsilon} F(\mu) d\mu, \quad \forall \varepsilon > 0$$

imply the following corollary.

COROLLARY 1.7. *Under conditions of Theorem 1.6*

$$(1.18) \quad F(b) \geq \varepsilon^{-1} \int_{b-\varepsilon}^b \rho_T * F(\mu) d\mu - \varepsilon^{-1} T^{-1} \delta^{-1} \rho_{\delta,0} * F'(b),$$

$$(1.19) \quad F(a) \leq \varepsilon^{-1} \int_a^{a+\varepsilon} \rho_T * F(\mu) d\mu + \varepsilon^{-1} T^{-1} \delta^{-1} \rho_{\delta,0} * F'(a)$$

for all $\varepsilon \in (0, b-a]$ and $T \geq \delta$.

If (4) is fulfilled then $\rho_T * F$ is a non-decreasing function. Therefore (1.18) and (1.19) imply that

$$(1.20) \quad F(b) \geq \rho_T * F(b-\varepsilon) - \varepsilon^{-1} T^{-1} \delta^{-1} \rho_{\delta,0} * F'(b),$$

$$(1.21) \quad F(a) \leq \rho_T * F(a+\varepsilon) + \varepsilon^{-1} T^{-1} \delta^{-1} \rho_{\delta,0} * F'(a).$$

Remark 1.8. It is clear from the proof that Theorems 1.3 and 1.6 remain valid (with some other constants independent of δ and T) if we drop the condition (4) and replace $\rho_{\delta,0}(\tau)$ with an arbitrary non-negative function γ_δ such that $|\rho_{T,1}(\tau)| \leq \text{const } \delta^{-1} \gamma_\delta(\tau)$ and $|\rho_{T,2}(\tau)| \leq \text{const } T^{-1} \delta^{-1} \gamma_\delta(\tau)$. In particular, one can take $\gamma_\delta(\tau) = \delta \gamma(\delta\tau)$, where γ is the function defined by (1.3) with $l = m$.

2. TAUBERIAN THEOREMS II: APPLICATIONS

2.1. General remarks. From now on we shall be assuming that the function F is polynomially bounded. Then the conditions of Theorems 1.3 and 1.6 are fulfilled for all $\tau, a, b \in \mathbb{R}^1$ and $T \geq \delta > 0$ whenever ρ satisfies (1_m) with a sufficiently large m .

So far we have not used the condition (5), which is not needed to prove the estimates. However, this condition often appears in applications. It implies that the convolutions $\rho_T * F$ and $\rho_{T,0} * F'$ are determined by the restrictions of \hat{F} to the interval $(-T, T)$. If

$$(2.1) \quad \hat{F}_0(t)|_{(-T, T)} = \hat{F}(t)|_{(-T, T)}$$

then, under condition (5), $\rho_T * F = \rho_T * F_0$ and $\rho_{\delta,0} * F' = \rho_{\delta,0} * F'_0$ for all $\delta \leq T$. If $F_0(\tau)$ behaves like a linear combination of homogeneous functions for large τ then $\rho_{\delta,0} * F'_0$ is of lower order than $\rho_T * F_0$, so it plays the role of an error term in asymptotic formulae.

It is not always possible to find a model function F_0 satisfying (2.1). However, one can often construct \tilde{F}_0 in such a way that the convolutions $\rho_T * (F - \tilde{F}_0)(\tau)$ and $\rho_{\delta,0} * (F' - \tilde{F}'_0)(\tau)$ admit good estimates for large τ (roughly speaking, it happens if the Fourier transforms of F and \tilde{F}_0 have similar singularities on the corresponding interval). Then the Tauberian theorems imply estimates with the error term

$$\pm (|\rho_T * (F - \tilde{F}_0)(\tau)| + |\rho_{\delta,0} * (F' - \tilde{F}'_0)(\tau)|).$$

In particular, if F is the spectral or counting function of an elliptic partial differential operator with smooth coefficients then (1.11) gives a precise remainder estimate in the Weyl asymptotic formula, and the refined estimates (1.20), (1.21) allow one to obtain the second asymptotic term by letting $T \rightarrow \infty$ (see [SV] for details).

In applications to the second order differential operators it is usually more convenient to deal with the cosine Fourier transform of F' . The following elementary observation enables one to apply our results in the case where information on the sine Fourier transform of F' is not available.

PROPOSITION 2.1. *If the cosine Fourier transforms of the derivatives F' and F'_0 coincide on an interval $(-\delta, \delta)$ then the Fourier transforms of the functions $F(\tau) - F(-\tau)$ and $F_0(\tau) - F_0(-\tau)$ coincide on the same interval.*

2.2. Test functions ρ . In this subsection we consider a class of functions ρ satisfying (1_m) –(5) and estimate the constants $c_{\rho, \kappa}$.

LEMMA 2.2. *Let $\zeta \in C^{m+1}[-\frac{1}{2}, \frac{1}{2}]$ be a real-valued even function such that $\|\zeta\|_{L_2} = 1$ and $\zeta^{(k)}(\pm\frac{1}{2}) = 0$ for $k = 0, 1, \dots, m-1$, where $\zeta^{(k)}$ denotes the k th derivative. If we extend ζ to \mathbb{R} by zero then $\rho := (\hat{\zeta})^2$ satisfies (1_m) –(5) and*

$$(2.2) \quad c_{\rho, 2k} = \|\zeta^{(k)}\|_{L_2}^2, \quad k = 0, 1, \dots, m.$$

Proof. The conditions (3) and (4) are obviously fulfilled; (2), (5) and (2.2) follow from the fact that $\hat{\rho} = (2\pi)^{-1/2} \zeta * \zeta$. Finally, (1_m) holds true because the $(m+1)$ th derivative of the extended function ζ coincides with a linear combination of an L_1 -function and two δ -functions. ■

The following lemma is a consequence of the uncertainty principle.

LEMMA 2.3. *If ρ is defined as in Lemma 2.2 then*

$$(2.3) \quad c_{\rho,1} \geq \frac{\pi}{2}.$$

Proof. Let Π_a be the multiplication operator and $\hat{\Pi}_a$ be the Fourier multiplier generated by the characteristic function of the interval $[-a, a]$. Then the Hilbert–Schmidt norm of the operator $\hat{\Pi}_{a_1} \Pi_{a_2}$ acting in $L_2(\mathbb{R})$ is equal to $\sqrt{2\pi^{-1}a_1a_2}$. Therefore

$$2 \int_0^\mu \hat{\zeta}^2(\tau) d\tau = \|\hat{\Pi}_\mu \Pi_{1/2} \zeta\|_{L_2}^2 \leq \pi^{-1}\mu \|\zeta\|_{L_2}^2 = \pi^{-1}\mu,$$

which implies that

$$\begin{aligned} c_{\rho,1} &= 2 \int_0^\infty \mu \hat{\zeta}^2(\mu) d\mu = 2 \int_0^\infty \int_\mu^\infty \hat{\zeta}^2(\tau) d\tau d\mu \\ &\geq 2 \int_0^\pi \int_\mu^\infty \hat{\zeta}^2(\tau) d\tau d\mu \\ &\geq \int_0^\pi (1 - \pi^{-1}\mu) d\mu = \frac{\pi}{2}. \quad \blacksquare \end{aligned}$$

Remark 2.4. As follows from Nazarov's theorem (see [Na] or [HJ]),

$$\int_\mu^\infty \hat{\phi}^2(\tau) d\tau \geq b_1 e^{-b_2\mu}, \quad \forall \phi \in C_0^\infty(-\frac{1}{2}, \frac{1}{2}), \quad \forall \mu \geq 0,$$

where $b_1, b_2 > 0$ are some absolute constants. Using the estimates for b_1, b_2 obtained in [Na], one can slightly improve the estimate (2.3).

EXAMPLE 2.5. Let $\tilde{\nu}_m$ be the first eigenvalue of the operator $\frac{d^{2m}}{dt^{2m}}$ on the interval $(-\frac{1}{2}, \frac{1}{2})$ subject to Dirichlet boundary condition, and let ζ_m be the corresponding real even normalized eigenfunction. Denote $\nu_m := (\tilde{\nu}_m)^{\frac{1}{2m}}$. If we define ρ as in Lemma 2.2 then, in view of (2.2) and (1.5),

$$(2.4) \quad c_{\rho,2m} = \nu_m^{2m}, \quad c_{\rho,\kappa} \leq \nu_m^\kappa, \quad \forall \kappa < 2m.$$

The eigenvalues $\tilde{v}_m = v_m^{2m}$ grow very fast as $m \rightarrow \infty$. The following lemma gives a rough estimate for v_m .

LEMMA 2.6. *We have $v_m \leq 2m \sqrt[2m]{3}$ for all $m \geq 2$.*

Proof. If $\phi(t) = (\frac{1}{4} - t^2)^m$ and $\|\cdot\|_{L_2}$ is the norm in $L_2(-\frac{1}{2}, \frac{1}{2})$ then

$$(2.5) \quad \tilde{v}_m \leq \frac{\|\phi^{(m)}\|_{L_2}^2}{\|\phi\|_{L_2}^2} = \frac{(4m+1)!(m!)^2}{(2m+1)!(2m)!} \leq 2^{2m+1}(2m)!.$$

One can easily see that

$$\frac{2^{2m}(2m)!}{(2m)^{2m}} = \frac{2(m^2-1) \cdots (m^2-(m-1)^2)}{m^{2m-2}} \leq \frac{2(m^2-(m-1)^2)}{m^2} \leq \frac{3}{2}.$$

Therefore (2.5) implies the required estimate. ■

2.3. *Power like singularities.* Assume that $|F(\tau)| \leq \text{const}(|\tau|+1)^n$ with a non-negative integer n and define

$$\sigma_n := \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even,} \end{cases} \quad m_n := \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd,} \\ \frac{n+2}{2}, & \text{if } n \text{ is even,} \end{cases}$$

$$P_n^+(\tau, \mu) := \frac{(\tau + \mu)^n + (\tau - \mu)^n}{2}, \quad P_n^-(\tau, \mu) := \frac{\mu(\tau + \mu)^n - \mu(\tau - \mu)^n}{2}.$$

Clearly, P_n^\pm are homogeneous polynomials in (τ, μ) with positive coefficients, which contain only even powers of μ .

LEMMA 2.7. *Let ρ be a function satisfying (3), (5) and (1_m) with $m > \frac{n}{2}$. If $\text{supp } F \subset (0, +\infty)$ and the cosine Fourier transform of $F'(\tau)$ coincides on the interval $(-\delta, \delta)$ with the cosine Fourier transform of the function $n\tau_+^{n-1}$ then*

$$(2.6) \quad \rho_\delta * F(\tau) \geq \int [P_n^+(\tau, \delta^{-1}\mu) - \sigma_n \delta^{-n} |\mu|^n] \rho(\mu) d\mu,$$

$$(2.7) \quad \rho_\delta * F(\tau) \leq \int P_n^+(\tau, \delta^{-1}\mu) \rho(\mu) d\mu,$$

$$(2.8) \quad \rho_{\delta,0} * F'(\tau) \leq \delta^2 \int [P_n^-(\tau, \delta^{-1}\mu) + \sigma_n \delta^{-n-1} |\mu|^{n+1}] \rho(\mu) d\mu$$

for all $\tau > 0$.

Proof. According to Proposition 2.1, the Fourier transform of $F(\tau) - F(-\tau)$ coincides on the interval $(-\delta, \delta)$ with the Fourier transform of

$$\text{sign } \tau |\tau|^n = (1 - 2\sigma_n \chi_-(\tau)) \tau^n.$$

Since ρ is even, this implies that

$$\begin{aligned} \rho_\delta * F(\tau) &= \delta \int (1 - 2\sigma_n \chi_-(\tau - \mu)) (\tau - \mu)^n \rho(\delta\mu) d\mu \\ &= \int P_n^+(\tau, \delta^{-1}\mu) \rho(\mu) d\mu - 2\sigma_n \int_{\delta\tau}^{\infty} (\delta^{-1}\mu - \tau)^n \rho(\mu) d\mu, \end{aligned}$$

$$\begin{aligned} \rho_{\delta,0} * F'(\tau) &= \rho'_{\delta,0} * F(\tau) = -\delta^3 \int (1 - 2\sigma_n \chi_-(\tau - \mu)) (\tau - \mu)^n \mu \rho(\delta\mu) d\mu \\ &= \delta^2 \int P_n^-(\tau, \delta^{-1}\mu) \rho(\mu) d\mu + 2\sigma_n \delta \int_{\delta\tau}^{\infty} (\delta^{-1}\mu - \tau)^n \mu \rho(\mu) d\mu \end{aligned}$$

for all $\tau > 0$. Estimating $0 \leq (\delta^{-1}\mu - \tau) \leq \delta^{-1}\mu$ in the integrals on the right hand sides, we arrive at (2.6)–(2.8). ■

The obvious inequalities

$$\begin{aligned} \tau^n + \sigma_n |v|^n &\leq P_n^+(\tau, v) \leq \tau^n + n |v| (\tau + |v|)^{n-1}, \\ P_n^-(\tau, v) + \sigma_n |v|^{n+1} &\leq n v^2 (\tau + |v|)^{n-1} \end{aligned}$$

and (2.6)–(2.8) imply that, for all $\tau > 0$,

$$(2.9) \quad 0 \leq \rho_\delta * F(\tau) - \tau^n \leq n\delta^{-1} \int |\mu| (\tau + \delta^{-1} |\mu|)^{n-1} \rho(\mu) d\mu,$$

$$(2.10) \quad \rho_{\delta,0} * F'(\tau) \leq n \int \mu^2 (\tau + \delta^{-1} |\mu|)^{n-1} \rho(\mu) d\mu.$$

Note that m_n is the minimal positive integer which is greater than $\frac{n}{2}$. If ρ is defined as in Lemma 2.2 with $m = m_n$ then, by (2.2),

$$(2.11) \quad \int P_n^\pm(\tau, \delta^{-1}\mu) \rho(\mu) d\mu = (P_n^\pm(\tau, \delta^{-1}D_t) \zeta, \zeta)_{L_2}.$$

Applying (2.6)–(2.11) and (1.11) or (1.18), (1.19), one can obtain various estimates for $F(\tau)$.

EXAMPLE 2.8. Let $n = 3$ and ζ be an arbitrary function satisfying conditions of Lemma 2.2 with $m = m_n = 2$. If the conditions of Lemma 2.7 are fulfilled then (2.6)–(2.8), (2.11) and (1.19), (1.20) with $T = \delta$ imply that

$$F(\tau) \geq \tau^3 - \frac{3\varepsilon\tau^2}{2} + \varepsilon^2\tau - \frac{\varepsilon^3}{4} + \frac{3}{2\delta^2} \left(\tau - \frac{\tau^2}{\varepsilon} - \frac{\varepsilon}{2} \right) \|\zeta'\|_{L_2}^2 - \frac{1}{\varepsilon\delta^4} \|\zeta''\|_{L_2}^2,$$

$$F(\tau) \leq \tau^3 + \frac{3\varepsilon\tau^2}{2} + \varepsilon^2\tau + \frac{\varepsilon^3}{4} + \frac{3}{2\delta^2} \left(\tau + \frac{\tau^2}{\varepsilon} + \frac{\varepsilon}{2} \right) \|\zeta'\|_{L_2}^2 + \frac{1}{\varepsilon\delta^4} \|\zeta''\|_{L_2}^2$$

for all $\varepsilon > 0$ and $\tau > 0$. Thus, $F(\tau)$ lies between the first Dirichlet eigenvalues of ordinary differential operators generated by the quadratic forms on the right hand sides of the above inequalities.

COROLLARY 2.9. Under conditions of Lemma 2.7

$$(2.12) \quad F(\tau) \geq \tau^n - 2\pi^{-1}v_{m_n}^2 n\delta^{-1}(\tau + \delta^{-1}v_{m_n})^{n-1},$$

$$(2.13) \quad F(\tau) \leq \tau^n + (2\pi^{-1}v_{m_n}^2 + v_{m_n}) n\delta^{-1}(\tau + \delta^{-1}v_{m_n})^{n-1}$$

for all $\tau > 0$.

Proof. If we define ρ as in Lemma 2.2 with $\zeta = \zeta_m$ (see Example 2.5) then (2.12), (2.13) follow from (1.11) with $T = \delta$, (2.9), (2.10), (2.3) and (2.4). ■

COROLLARY 2.10. Under conditions of Lemma 2.7

$$(2.14) \quad \int_0^{\lambda^2} F(\sqrt{\mu}) d\mu \geq \frac{2\lambda^{n+2}}{n+2} - 2nv_{m_n}^2 \delta^{-2}\lambda(\lambda + \delta^{-1}v_{m_n})^{n-1},$$

$$(2.15) \quad \int_0^{\lambda^2} F(\sqrt{\mu}) d\mu \leq \frac{2\lambda^{n+2}}{n+2} + (n+1)v_{m_n}^2 \delta^{-2}(\lambda + \delta^{-1}v_{m_n})^n$$

for all $\lambda > 0$.

Proof. Since $\int_0^{\lambda^2} F(\sqrt{\mu}) d\mu = 2 \int_0^\lambda F(\tau) \tau d\tau$, Theorem 1.6 with $T = \delta$, $a = 0$, $b = \lambda$ and $f(\tau) = \tau$ implies

$$(2.16) \quad \int_0^{\lambda^2} F(\sqrt{\mu}) d\mu \geq 2 \int_0^\lambda \tau \rho_\delta * F(\tau) d\tau - 2\delta^{-2} \lambda \rho_{\delta,0} * F'(\lambda),$$

$$(2.17) \quad \int_0^{\lambda^2} F(\sqrt{\mu}) d\mu \leq 2 \int_0^\lambda (\tau \rho_\delta * F(\tau) + \delta^{-2} \rho_{\delta,0} * F'(\tau)) d\tau.$$

Let ρ be defined as in Lemma 2.2 with $\zeta = \zeta_m$. Then (2.14) follows from (2.16), (2.9), (2.10) and (2.4). Since $\tau P_n^+(\tau, \nu) + P_n^-(\tau, \nu) = P_{n+1}^+(\tau, \nu)$, the inequality (2.17) and (2.7), (2.8) imply that

$$\int_0^{\lambda^2} F(\sqrt{\mu}) \, d\mu \leq 2 \int_0^\lambda \int (P_{n+1}^+(\tau, \delta^{-1}\mu) + \sigma_n |\delta^{-1}\mu|^{n+1}) \rho(\mu) \, d\mu \, d\tau.$$

Estimating

$$\int_0^\lambda [P_{n+1}^+(\tau, \nu) + \sigma_n |\nu|^{n+1}] \, d\tau \leq \frac{\lambda^{n+2}}{n+2} + \frac{n+1}{2} \nu^2 (\lambda + |\nu|)^n$$

with $\nu = \delta^{-1}\mu$ and applying (2.4), we obtain (2.15). ■

3. APPLICATIONS TO THE LAPLACE OPERATOR

Let $\Omega \subset \mathbb{R}^n$ be an open domain and $d(x)$ be the distance from $x \in \Omega$ to the boundary $\partial\Omega$.

3.1. *Estimates of the spectral function.* Consider the Laplacian Δ_B in Ω subject to a self-adjoint boundary condition $B(x, D_x) u|_{\partial\Omega} = 0$, where B is a differential operator. Assume that the operator $-\Delta_B$ is non-negative and denote by $\Pi(\lambda)$ its spectral projection corresponding to the interval $[0, \lambda)$. Let $e(x, y; \lambda)$ be the integral kernel of the operator $\frac{\Pi(\lambda-0) + \Pi(\lambda+0)}{2}$ (the so-called *spectral function*). The Sobolev embedding theorem implies that $e(x, y; \lambda)$ is a smooth function on $\Omega \times \Omega$ for each fixed λ and that $e(x, x; \lambda)$ is a non-decreasing polynomially bounded function of λ for each fixed $x \in \Omega$.

Let Δ_0 be the Laplacian on \mathbb{R}^n , and $e_0(x, y; \lambda)$, $\tilde{e}_0(x, y; \lambda)$, $\tilde{e}(x, y; \lambda)$ be the spectral functions of the operators Δ_0 , $\sqrt{\Delta_0}$, $\sqrt{\Delta_B}$ respectively. Then

$$\begin{aligned} \chi_+(\tau) e(x, x; \tau^2) &= \tilde{e}(x, x; \tau), \\ \chi_+(\tau) e_0(x, x; \tau^2) &= \tilde{e}_0(x, x; \tau) = C_n \tau_+^n, \end{aligned}$$

where

$$(3.1) \quad C_n := (2\pi)^{-n} \text{meas}\{\xi \in \mathbb{R}^n : |\xi| < 1\}.$$

By the spectral theorem, the cosine Fourier transform of $\frac{d}{dt} \tilde{e}(x, y; \tau)$ coincides with the fundamental solution $u(x, y; t)$ of the wave equation in Ω ,

$$u_{tt} = \Delta u, \quad Bu|_{\partial\Omega} = 0, \quad u|_{t=0} = \delta(x-y), \quad u_t|_{t=0} = 0.$$

Due to the finite speed of propagation, $u(x, x; t)$ is equal to $u_0(x, x; t)$ whenever $t \in (-d(x), d(x))$, where $u_0(x, y; t)$ is the fundamental solution of the wave equation in \mathbb{R}^n . Thus, the cosine Fourier transforms of the derivatives $\frac{d}{d\tau} \tilde{e}_0(x, x; \tau)$ and $\frac{d}{d\tau} \tilde{e}(x, x; \tau)$ coincide on the time interval $(-d(x), d(x))$. Applying (2.12)–(2.15) to $F(\tau) = C_n^{-1} \tilde{e}(x, x; \tau)$ we obtain the following corollary.

COROLLARY 3.1. *For every $x \in \Omega$ and all $\lambda > 0$ we have*

$$(3.2) \quad e(x, x; \lambda) \geq C_n \lambda^{n/2} - \frac{n C_n 2\pi^{-1} v_{m_n}^2}{d(x)} \left(\lambda^{1/2} + \frac{v_{m_n}}{d(x)} \right)^{n-1},$$

$$(3.3) \quad e(x, x; \lambda) \leq C_n \lambda^{n/2} + \frac{n C_n (2\pi^{-1} v_{m_n}^2 + v_{m_n})}{d(x)} \left(\lambda^{1/2} + \frac{v_{m_n}}{d(x)} \right)^{n-1},$$

$$(3.4) \quad \int_0^\lambda e(x, x; \mu) d\mu \geq \frac{2C_n \lambda^{n/2+1}}{n+2} - \frac{2n C_n v_{m_n}^2 \lambda^{1/2}}{(d(x))^2} \left(\lambda^{1/2} + \frac{v_{m_n}}{d(x)} \right)^{n-1}$$

$$(3.5) \quad \int_0^\lambda e(x, x; \mu) d\mu \leq \frac{2C_n \lambda^{n/2+1}}{n+2} + \frac{(n+1) C_n v_{m_n}^2}{(d(x))^2} \left(\lambda^{1/2} + \frac{v_{m_n}}{d(x)} \right)^n.$$

3.2. Estimates of the counting function of the Dirichlet Laplacian. In this subsection we shall be assuming that $|\Omega| < \infty$, where $|\cdot|$ denotes the n -dimensional Lebesgue measure.

Consider the positive operator $-\Delta_D$, where Δ_D is the Dirichlet Laplacian in Ω . Let $N(\lambda)$ be the number of its eigenvalues lying below λ . The following theorem is due to F. Berezin [B].

THEOREM 3.2. *For all $\lambda \geq 0$ we have*

$$(3.6) \quad \int_0^\lambda N(\mu) d\mu \leq \frac{2}{n+2} C_n |\Omega| \lambda^{n/2+1}.$$

This results was reproduced in [La]. A. Laptev also noticed that the famous Li–Yau estimate

$$(3.7) \quad N(\lambda) \leq (1 + 2/n)^{n/2} C_n |\Omega| \lambda^{n/2}, \quad \forall \lambda \geq 0,$$

(see [LY]) is a one line consequence of (3.6). Indeed, (3.7) can be proved by estimating

$$(3.8) \quad N(\lambda) \leq (\theta\lambda)^{-1} \int_0^{\lambda+\theta\lambda} N(\mu) d\mu \leq \frac{2(1+\theta)^{n/2+1}}{(n+2)\theta} C_n |\Omega| \lambda^{n/2}$$

and optimizing the choice of $\theta > 0$.

Remark 3.3. In [B] F. Berezin proved an analogue of (3.6) for general operators with constant coefficients subject to Dirichlet boundary condition. In the same way as above, applying the first inequality (3.8) and Berezin's estimates, one can easily obtain upper bounds for the corresponding counting functions (see [La]).

According to the Weyl asymptotic formula

$$(3.9) \quad N(\lambda) = C_n |\Omega| \lambda^{n/2} + o(\lambda^{n/2}), \quad \lambda \rightarrow +\infty,$$

(in the general case (3.9) was proved in [BS]). The coefficient in the right hand side of (3.7) contains an extra factor $(1+2/n)^{n/2}$. G. Pólya conjectured [P] that (3.7) holds without this factor. However, this remains an open problem.

Given a positive ε , denote

$$\Omega_\varepsilon^b := \{x \in \Omega : d(x) \leq \varepsilon\}, \quad \Omega_\varepsilon^i := \{x \in \Omega : d(x) > \varepsilon\}.$$

If

$$(3.10) \quad |\Omega_\varepsilon^b| \leq \text{const } \varepsilon^r, \quad r \in (0, 1],$$

then, using the variational method [CH], one can prove that

$$(3.11) \quad |N(\lambda) - C_n |\Omega| \lambda^{n/2}| \leq \begin{cases} \text{const } \lambda^{(n-1)/2} \ln \lambda, & r = 1, \\ \text{const } \lambda^{(n-r)/2}, & r < 1. \end{cases}$$

It is well known that in the smooth case

$$N(\lambda) - C_n |\Omega| \lambda^{n/2} = O(\lambda^{(n-1)/2})$$

(see, for example, [I1] or [SV]), but it is not clear whether this estimate remains valid for an arbitrary domain satisfying (3.10) with $r = 1$.

There is a number of papers devoted to estimates of the remainder term in the Weyl formula. In [BL] the authors, applying the variational technique, obtain explicit estimates for the constants in (3.11). In order to prove the estimate of $N(\lambda)$ from above, they imposed an additional condition on the outer neighbourhood of the boundary $\partial\Omega$, but this condition can probably be removed [Ne]. In [Kr] the author estimated the remainder term with the use of a different technique (similar to that in [LY]); his results seem to be less precise than those obtained in [BL].

Let

$$N_\varepsilon^b(\lambda) := \int_{\Omega_\varepsilon^b} e(x, x; \lambda) dx, \quad N_\varepsilon^i(\lambda) := \int_{\Omega_\varepsilon^i} e(x, x; \lambda) dx.$$

Then $N(\lambda) = N_\varepsilon^b(\lambda) + N_\varepsilon^i(\lambda)$ for every $\varepsilon > 0$.

COROLLARY 3.4. *For all $\lambda > 0$ and $\varepsilon > 0$ we have*

$$(3.12) \quad N_\varepsilon^i(\lambda) \geq C_n |\Omega_\varepsilon^i| \lambda^{n/2} - C_{n,1} (\lambda^{1/2} + \varepsilon^{-1} v_{m_n})^{n-1} \int_{\Omega_\varepsilon^i} \frac{dx}{d(x)},$$

$$(3.13) \quad N_\varepsilon^i(\lambda) \leq C_n |\Omega_\varepsilon^i| \lambda^{n/2} + C_{n,2} (\lambda^{1/2} + \varepsilon^{-1} v_{m_n})^{n-1} \int_{\Omega_\varepsilon^i} \frac{dx}{d(x)},$$

$$(3.14) \quad N_\varepsilon^b(\lambda) \leq C_{n,3} |\Omega_\varepsilon^b| \lambda^{n/2} + C_{n,4} \lambda^{-1/2} (\lambda^{1/2} + \varepsilon^{-1} v_{m_n})^{n-1} \int_{\Omega_\varepsilon^i} \frac{dx}{(d(x))^2},$$

where

$$\begin{aligned} C_{n,1} &= n C_n 2\pi^{-1} v_{m_n}^2, & C_{n,2} &= n C_n (2\pi^{-1} v_{m_n}^2 + v_{m_n}), \\ C_{n,3} &= (1 + 2/n)^{n/2} C_n, & C_{n,4} &= (1 + 2/n)^{n/2} n^2 C_n v_{m_n}^2. \end{aligned}$$

Proof. The inequalities (3.12), (3.13) are proved by straightforward integration of (3.2), (3.3). Theorem 3.2 and (3.4) imply that

$$\begin{aligned} (3.15) \quad \int_0^\lambda N_\varepsilon^b(\mu) d\mu &= \int_0^\lambda N(\lambda) d\mu - \int_0^\lambda N_\varepsilon^i(\mu) d\mu \\ &\leq \frac{2}{n+2} C_n |\Omega_\varepsilon^b| \lambda^{n/2+1} \\ &\quad + 2n C_n v_{m_n}^2 \lambda^{1/2} (\lambda^{1/2} + \varepsilon^{-1} v_{m_n})^{n-1} \int_{\Omega_\varepsilon^i} \frac{dx}{(d(x))^2}. \end{aligned}$$

Now, applying the first inequality (3.8) with $\theta = 2/n$, we arrive at (3.14). ■

Adding up the inequalities (3.13) and (3.14) we obtain

$$\begin{aligned} (3.16) \quad N(\lambda) &\leq C_n |\Omega_\varepsilon^i| \lambda^{n/2} + C_{n,3} |\Omega_\varepsilon^b| \lambda^{n/2} \\ &\quad + (\lambda^{1/2} + \varepsilon^{-1} v_{m_n})^{n-1} \int_{\Omega_\varepsilon^i} \frac{C_{n,2} d(x) + C_{n,4} \lambda^{-1/2}}{(d(x))^2} dx, \quad \forall \varepsilon > 0. \end{aligned}$$

Since

$$(3.17) \quad \int_{\Omega_\varepsilon^i} \frac{dx}{(d(x))^j} = \int_\varepsilon^\infty s^{-j} d(|\Omega_s^b|) = j \int_\varepsilon^\infty s^{-j-1} |\Omega_s^b| ds - \varepsilon^{-j} |\Omega_\varepsilon^b|,$$

(3.10) and the inequalities (3.12), (3.16) with $\varepsilon = \lambda^{-1/2}$ imply (3.11).

By (3.12) and (3.16) we have

$$(3.18) \quad -C_n |\Omega_\varepsilon^b| - |\Omega_\varepsilon^i| \frac{C_{n,1}}{\varepsilon \lambda^{1/2}} \left(1 + \frac{v_{m_n}}{\varepsilon \lambda^{1/2}}\right)^{n-1} \\ \leq \lambda^{-n/2} N(\lambda) - C_n |\Omega| \\ \leq (C_{n,3} - C_n) |\Omega_\varepsilon^b| + |\Omega_\varepsilon^i| \left(\frac{C_{n,2}}{\varepsilon \lambda^{1/2}} + \frac{C_{n,4}}{\varepsilon^2 \lambda}\right) \left(1 + \frac{v_{m_n}}{\varepsilon \lambda^{1/2}}\right)^{n-1}$$

for all $\varepsilon > 0$. If $\varepsilon \rightarrow \infty$ then the second inequality (3.18) turns into (3.7). Since $|\Omega_\varepsilon^b| \rightarrow 0$ as $\varepsilon \rightarrow 0$, (3.18) implies (3.9). Moreover, taking $\varepsilon = \lambda^{-\kappa}$ with an arbitrary $\kappa \in (0, \frac{1}{2})$, we obtain the Weyl formula with a remainder estimate

$$\lambda^{-n/2} N(\lambda) - C_n |\Omega| = O(|\Omega_\lambda^{b-\kappa}| + \lambda^{\kappa-1/2}), \quad \lambda \rightarrow +\infty.$$

Remark 3.5. If the condition (3.10) is fulfilled then integrating (3.4) over $\Omega_\lambda^{i-1/2}$, applying (3.17) and taking into account (3.6), we see that

$$(3.19) \quad \lambda^{-1} \int_0^\lambda N(\mu) d\mu = \frac{2}{n+2} C_n |\Omega| \lambda^{n/2} + O(\lambda^{(n-r)/2})$$

for all $r \in (0, 1]$.

REFERENCES

- [B] F. A. Berezin, Covariant and contravariant symbols of operators, *Izv. Akad. Nauk SSSR* **37** (1972), 1134–1167. [In Russian, English transl. in *Math. USSR Izv.* **6** (1972), 1117–1151]
- [BL] M. van den Berg and M. Lianantonakis, Asymptotics for the spectrum of the Dirichlet Laplacian on horn-shaped regions and ζ functions on cross-sections, *J. Funct. Anal.*, in press.
- [BS] M. S. Birman and M. Z. Solomyak, The principal term of spectral asymptotics for “non-smooth” elliptic problems, *Funkt. Anal. Prilozhen.* **4**, 4 (1970), 1–13. [Russian, English transl. in *Funct. Anal. Appl.* **4** (1971).]
- [CH] R. Courant and D. Hilbert, “Methods of Mathematical Physics,” Wiley, New York, 1989.
- [DG] J. J. Duistermaat and V. W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.* **25** (1975), 39–79.

- [I1] V. Ya. Ivrii, On the second term of the spectral asymptotics for the Laplace-Beltrami operator on manifolds with boundary, *Funkt. Anal. Prilozhen.* **14**, 2 (1980), 25–34. [In Russian, English transl. in *Funct. Anal. Appl.* **14** (1980).]
- [I2] V. Ya. Ivrii, The asymptotic Weyl formula for the Laplace-Beltrami operator in Riemannian polyhedra and domains with conical singularities of the boundary, *Dokl. Akad. Nauk SSSR* **288** (1986), 35–38. [In Russian]
- [H1] L. Hörmander, The spectral function of an elliptic operator, *Acta Math.* **121** (1968), 193–218.
- [H2] L. Hörmander, “Theory of Linear Partial Differential Operators I–IV,” Springer-Verlag, Berlin/New York, 1985.
- [HJ] V. Havin and B. Jöricke, “The uncertainty principle in harmonic analysis,” Springer-Verlag, Berlin/New York, 1994.
- [Kr] P. Kröger, Estimates for sums of eigenvalues of the Laplacian, *J. Funct. Anal.* **126** (1994), 217–227.
- [L] B. M. Levitan, On the asymptotic behavior of the spectral function of a self-adjoint differential second order equation, *Izv. Akad. Nauk SSSR Ser. Mat.* **16** (1952), 325–352. [In Russian]
- [LY] P. Li and S. T. Yau, On the Schrödinger equation and the eigenvalue problem, *Comm. Math. Phys.* **88** (1983), 309–318.
- [La] A. Laptev, Dirichlet and Neumann eigenvalue problems on domains in Euclidean space, *J. Funct. Anal.* **151** (1997), 531–545.
- [Na] F. L. Nazarov, Local estimates for exponential polynomials, *Algebra Anal.* **5**, 4 (1993), 3–66. [In Russian]
- [Ne] Yu. Netrusov, private communication, 1999.
- [P] G. Pólya, On the eigenvalues of vibrating membranes, *Proc. London Math. Soc.* **11** (1961), 419–433.
- [PP] V. Petkov and G. Popov, Semi-classical trace formula and clustering of eigenvalues for Schrödinger operators, *Ann. Inst. H. Poincaré* **68** (1998), 17–83.
- [S] Yu. Safarov, Asymptotics of the spectral function of a positive elliptic operator without non-trapping condition, *Funkt. Anal. Prilozhen.* **22**, 3 (1988), 53–65. [In Russian, English transl. in *Functional Anal. Appl.* **22** (1988).]
- [SV] Yu. Safarov and D. Vassiliev, “The Asymptotic Distribution of Eigenvalues of Partial Differential Operators,” American Mathematical Society, Providence, RI, 1996.